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HIGH ACCURACY ANALYSIS OF THE WILSON ELEMENT*

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Abstract

In this paper, the Wilson nonconforming finite element is considered for solving a class of second-order elliptic boundary value problems. Based on an asymptotic error expansion for the Wilson finite element, the global superconvergences, the local superconvergences and the defect correction schemes are presented.

Key words: Finite elements, Defect correction, Global superconvergence, Wilson element.

1. Introduction

It is well known that superconvergence estimates and error expansions for the conforming finite elements are well studied in many papers. We refer to [16] for a survey on various results of superconvergence and to [10] for a fundamental work on asymptotic error expansions and to [1]–[3] for some techniques on high accuracy analysis. However, for the nonconforming elements, due to the reduced continuity of trial and test functions, it becomes more difficult to discuss superconvergence properties and related asymptotic error expansions. Naturally, people want to ask if the accuracy of the nonconforming element approximation can be improved by means of other methods. However, up to present, the work in this field have seldom been found in the literature. For the relatively simple Wilson element, a result of superconvergence in the energy norm has been obtained in [7] for a model situation and, within the same setting, independently, Chen and Li [8] have obtained L^p and $W^{1,p}$ $(1 \leq p \leq \infty)$ error estimates as well as the extrapolation results. For more general equation, Chen and Li[8] have obtained the error expansions and the pointwise superconvergence error estimates for the gradient. For the Carey nonconforming element, the superconvergence estimate of the gradient at the element centroid has been proved in [20]. However, these superconvergence results are only pointwise and particular. In order to get the high accuracy of the nonconforming elements as that for the conforming elements, we carefully analyse the Wilson element in this paper. We find that the Wilson element not only has the pointwise superconvergence, but also has the asymptotic error expansions, the global and local superconvergences, the defect corrections and the extrapolations. The key

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point of analysis is the expansions of some integral identities. And this kind of technical details can be found in [1], [4] and the original paper [10].

It is known that the nonconforming Wilson finite element passes the Irons patch test on general quasi-uniform quadrilateral meshes and the rate of convergence in the energy norm is of first order. It is shown by an example in [5] that this rate of convergence is optimal. Thus, in contrast to conforming quadratic finite element which achieves a second-order rate of convergence in the energy norm, the Wilson finite element loses one order of accuracy because of its nonconforming. In this paper, we present a method that as long as post-processing on the finite element solution, i.e., using a high order interpolation for the finite element solution, we have not only obtained the global superconvergences of a second order or higher order rate of convergence, but also have obtained the local superconvergence and the defect correction schemes.

2. Global Superconvergence

For simplicity, let Ω be the unit square in the xy-plane. We consider the following boundary value problem

$$\begin{cases} -Lu \equiv -\dfrac{\partial}{\partial x}\Big(A_1\dfrac{\partial u}{\partial x}\Big) - \dfrac{\partial}{\partial y}\Big(A_2\dfrac{\partial u}{\partial y}\Big) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on}\ \partial\Omega, \end{cases} \tag{1}$$

where A_1, A_2 and f are sufficiently smooth functions defined on Ω and $A_1, A_2 \geq \alpha = const > 0$. Let $T^h = \{e_{ij}\}_{i,j=1}^{n,m} = \{e\}$ be a rectangular partition of the domain Ω, where n, m are two positive integers, $e_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ are rectangular elements, and

$$0 = x_0 \leq x_1 \leq \cdots, \leq x_n = 1,\ 0 = y_0 \leq y_1 \leq \cdots, \leq y_m = 1$$

are two one-dimensional partitions on the x-axis and y-axis, respectively. Define $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, and the mesh size $h = \max\{h_i, k_j\}_{i,j=1}^{n,m}$. As usual T^h is said to be quasi-uniform if there exists a positive constant c such that

$$ch \leq \min\{h_i, k_j\}_{i,j=1}^{n,m}.$$

Furthermore, T^h is said to be unidirectionally uniform if

$$h_i = h_1,\ i = 1, \cdots, n, \text{ and } \quad k_j = k_1,\ j = 1, \cdots, m.$$

For the mesh T^h, let N_h denote the set of vertices and we define V^h to be the Wilson finite element space which consists of all functions $v \in L^2(\Omega)$ such that v is piecewise quadratic over Ω and continuous on N_h and v vanishes on $N_h \cap \partial\Omega$, i.e., six degrees of freedom on the element e of the Wilson element are uniquely determined by its values at four vertices of element e and two integrals $\displaystyle\int_e \frac{\partial^2 v}{\partial x^2} dxdy$ and $\displaystyle\int_e \frac{\partial^2 v}{\partial y^2} dxdy$. The Wilson finite element solution of the equation (1), $R_h u \in V^h$, is defined through the relation

$$a_h(R_h u, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V^h, \tag{2}$$

where

$$a_h(u,v) = \sum_{e\in T^h} \int_e \Big(A_1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + A_2 \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\Big) dxdy.$$

We introduce the notation $\|.\|_{k,p,h}$,

$$\|.\|_{k,p,h} = \begin{cases} \Big(\sum_{e\in T_h} \|.\|^p_{k,p,e}\Big)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max_{e\in T^h} \|.\|_{k,\infty,e} & p = \infty. \end{cases}$$

Here and below, without confusion, we will use the notation $\|u\|_{k,p}$ to denote $\|u\|_{k,p,h}$.

Furthermore, we introduce the following conforming bilinear finite element space $\overline{V}^h$ defined by

$$\overline{V}^h = \{v \in H^1_0(\Omega) \mid v \quad \text{is piecewise bilinear}\}.$$

Notice that $\overline{V}^h$ is a subspace of V^h. The bilinear finite element solution of the equation (1), $\overline{R}_h u \in \overline{V}^h$ is defined by

$$a(\overline{R}_h u, v) = (f, v), \quad \forall v \in \overline{V}^h, \tag{3}$$

where

$$a(u,v) = \int_\Omega \Big(A_1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + A_2 \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\Big).$$

Obviously,

$$a_h(R_h u - \overline{R}_h u, v) = 0, \quad \forall v \in \overline{V}^h. \tag{4}$$

Let i_h: $C(\Omega) \longrightarrow \overline{V}^h$ be the usual bilinear interpolation operator. From [1], [8], [9] and $a(\overline{R}_h u - i_h u, v) = a(u - i_h u, v)$ for $v \in \overline{V}^h$, we have

Lemma 1. *Assume that A_i $(i = 1,2)$ are smooth enough, T^h is unidirectionally uniform and $u \in W^{5,\infty}(\Omega)$. Then, for any $v \in \overline{V}^h$,*

$$\begin{aligned} a(\overline{R}_h u - i_h u, v) = \sum_{e\in T^h} \Big\{ &- \frac{k_e^2}{3} \int_e A_1 \frac{\partial^3 u}{\partial x \partial y^2}\frac{\partial v}{\partial x} + \frac{h_e^2}{3}\int_e \frac{\partial A_1}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial v}{\partial x} \\ &- \frac{h_e^2}{3}\int_e A_2 \frac{\partial^3 u}{\partial x^2 \partial y}\frac{\partial v}{\partial y} + \frac{k_e^2}{3}\int_e \frac{\partial A_2}{\partial y}\frac{\partial^2 u}{\partial y^2}\frac{\partial v}{\partial y}\Big\} + \mathcal{O}(h^4)|u|_{5,\infty}|v|_{2,1}. \end{aligned}$$

It is easy to prove that (see [8])

Lemma 2. *Under the conditions of the Lemma 1, we have*

$$\partial_s^2 R_h u = -\frac{1}{A_s(p)}(\partial_s A_s(p)\partial_s R_h u(p) + f(p)) + \mathcal{O}(h^2)\|f\|_{2,\infty} + \mathcal{O}(h)\|R_h u\|_{2,2},$$

where $s = 1,2$, $\partial_1^2 = \dfrac{\partial^2}{\partial x^2}$, $\partial_2^2 = \dfrac{\partial^2}{\partial y^2}$, and p is the center of rectangular element e.

From Lemma 1 and Lemma 2, we get

Lemma 3. *Under the conditions of Lemma* 1, *we have, for all* $v \in \overline{V}^h$,

$$\begin{aligned} a(\overline{R}_h u - i_h(R_h u), v) =& \sum_e \int_e \Big\{ A_1 \frac{\partial (R_h u - i_h R_h u)}{\partial x} \frac{\partial v}{\partial x} + A_2 \frac{\partial (R_h u - i_h R_h u)}{\partial y} \frac{\partial v}{\partial y} \Big\} \\ =& \frac{1}{3} \sum_e \int_e \Big(h_e^2 \frac{\partial A_1}{\partial x} \frac{\partial^2 R_h u}{\partial x^2} \frac{\partial v}{\partial x} + k_e^2 \frac{\partial A_2}{\partial y} \frac{\partial^2 R_h u}{\partial y^2} \frac{\partial v}{\partial y} \Big) \\ &+ \mathcal{O}(h^4) |R_h u|_{2,\infty} |v|_{2,1} \\ =& \frac{1}{3} \sum_e \Big(h_e^2 \int_e \frac{\partial A_1}{\partial x} F_1 \frac{\partial v}{\partial x} + k_e^2 \int_e \frac{\partial A_2}{\partial y} F_2 \frac{\partial v}{\partial y} \Big) \\ &+ \mathcal{O}(h^4) |u|_{4,\infty} |v|_{2,1}, \end{aligned}$$

where $F_1 = -\frac{1}{A_1}\Big(\frac{\partial A_1}{\partial x}\frac{\partial u}{\partial x} + f\Big)$ *and* $F_2 = -\frac{1}{A_2}\Big(\frac{\partial A_2}{\partial y}\frac{\partial u}{\partial y} + f\Big)$ *are functions on* Ω *independent of* h.

Lemma 4. *There exists a positive constant* c *such that*

$$||R_h u||_{1,2} \le c||u||_{1,2}, \tag{5}$$

$$||R_h u||_{2,2} \le c||u||_{2,2} \tag{6}$$

and

$$||u - R_h u||_{1,\infty} \le c||u||_{2,2}. \tag{7}$$

Proof. Integrating by parts we come to

$$\begin{aligned} a_h(u - R_h u, R_h u) =& \sum_e \Big\{ \Big(\int_{l_2} - \int_{l_1} \Big) A_1 \frac{\partial u}{\partial x} R_h u + \Big(\int_{l_4} - \int_{l_3} \Big) A_2 \frac{\partial u}{\partial y} R_h u \Big\} \\ =& \sum_e \Big\{ \Big(\int_{l_2} - \int_{l_1} \Big) A_1 \frac{\partial u}{\partial x} (R_h u - i_h R_h u) \\ &+ \Big(\int_{l_4} - \int_{l_3} \Big) A_2 \frac{\partial u}{\partial y} (R_h u - i_h R_h u) \Big\} \le c||u||_{1,2} ||R_h u||_{1,2}, \end{aligned}$$

where we have used the expansion of $R_h u - i_h R_h u$ at nodal point and l_2, l_1 and l_4, l_3 are the two edges of e parallel to x and y axis, respectively. Therefore, we deduce that

$$\begin{aligned} ||R_h u||_{1,2}^2 \le& c a_h(R_h u, R_h u) = c a_h(R_h u - u, R_h u) + c a_h(u, R_h u) \\ \le& c||u||_{1,2} ||R_h u||_{1,2} \end{aligned}$$

and from which inequality (5) follows.

Using the inverse estimate, we get

$$\begin{aligned} ||R_h u - u||_{2,2} \le& ||R_h u - i_h u||_{2,2} + ||i_h u - u||_{2,2} \le ch^{-1} ||R_h u - i_h u||_{1,2} + c||u||_{2,2} \\ \le& ch^{-1}(||R_h u - u||_{1,2} + c||u - i_h u||_{1,2}) + c||u||_{2,2} \le c||u||_{2,2}. \end{aligned}$$

Hence, we deduce that

$$||R_h u||_{2,2} \le ||R_h u - u||_{2,2} + ||u||_{2,2} \le ||u||_{2,2}.$$

Similarly we can prove inequality (7).

Theorem 1. *If* $\overline{R}_h u$ *and* $R_h u$ *satisfy* (3) *and* (2), *respectively, then*

$$\|\overline{R}_h u - i_h(R_h u)\|_{1,2} \leq ch^2 \|u\|_{2,2} \tag{8}$$

and

$$\|\overline{R}_h u - i_h(R_h u)\|_{1,\infty} \leq ch^2 |\ln h| \|u\|_{2,\infty}. \tag{9}$$

Proof. We have by (4) and Lemma 3

$$\begin{aligned} c\|\overline{R}_h u - i_h(R_h u)\|_{1,2}^2 \leq & a(\overline{R}_h u - i_h(R_h u), \overline{R}_h u - i_h(R_h u)) \\ = & a_h(R_h u - i_h(R_h u), \overline{R}_h u - i_h(R_h u)) \\ & + a_h(\overline{R}_h u - R_h u, \overline{R}_h u - i_h(R_h u)) \\ \leq & ch^2 \|\overline{R}_h u - i_h(R_h u)\|_{1,2} \|R_h u\|_{2,2} \\ & + ch^4 \|R_h u\|_{2,2} \|\overline{R}_h u - i_h(R_h u)\|_{2,2} \end{aligned}$$

Thus inequality (8) follows from inequality (6) and the inverse estimate. Similarly, we have (9).

Corollary 1. *If* $u \in H^1$ *and* $D \subset\subset \Omega$, *then there exists a positive constant* c *such that*

$$||\overline{R}_h u - i_h(R_h u)||_{1,2} \leq ch||u||_{1,2}, \tag{10}$$

$$||\overline{R}_h u - i_h(R_h u)||_{0,\infty,D} \leq ch|\ln h|^{\frac{1}{2}} ||u||_{1,2} \tag{11}$$

and

$$||\overline{R}_h u - i_h(R_h u)||_{0,2} \leq ch||u||_{1,2}. \tag{12}$$

Proof. We only prove inequality (11). Inequalities (10) and (12) can be proved similarly. Using $||G_z^h||_{1,2} \leq c|\ln h|^{\frac{1}{2}}$, where $G_z^h \in \overline{V}^h$ denotes the discrete Green function, we get by inequality (5), Lemma 3 and the inverse estimate,

$$\begin{aligned} ||\overline{R}_h u - i_h(R_h u)||_{0,\infty,D} = & a(\overline{R}_h u - i_h R_h u, G_z^h) \\ \leq & ch^2 ||R_h u||_{2,2} ||G_z^h||_{1,2} + c(h^4) ||R_h u||_{2,2} ||G_z^h||_{2,2} \\ \leq & ch|\ln h|^{\frac{1}{2}} ||u||_{1,2}. \end{aligned}$$

We know that (see [1])

Lemma 5. *There exists a positive constant* c *such that*

$$\|\overline{R}_h u - i_h u\|_{1,2} \leq ch^2 \|u\|_{3,2} \tag{13}$$

and

$$\|\overline{R}_h u - i_h u\|_{1,\infty} \leq ch^2 |\ln h| \|u\|_{3,\infty}. \tag{14}$$

We assume that T^h has been obtained from T^{2h} of mesh size $2h$ by uniform subdividing each element into 4 congruent subrectangles of size h on any elements in T^{2h}.

We define the high order interpolation operator I^{2h} on the space of piecewise bilinear function $\overline{V}^h$ and satisfying

$$\begin{aligned} &I^{2h} i_h = I^{2h}, \quad \|I^{2h} v\|_{1,q} \le \|v\|_{1,q}, \quad \forall v \in \overline{V}^h, \\ &\|u - I^{2h} u\|_{1,q} \le ch^2 \|u\|_{3,q}, \quad 2 \le q \le \infty. \end{aligned} \tag{15}$$

Hence, we get the global superconvergence as follows:

Theorem 2. *If u and $R_h u$ satisfy* (1) *and* (2), *respectively, then*

$$\|u - I^{2h}(R_h u)\|_{1,2} \le ch^2 \|u\|_{3,2} \tag{16}$$

and

$$\|u - I^{2h}(R_h u)\|_{1,\infty} \le ch^2 |\ln h| \|u\|_{3,\infty}. \tag{17}$$

Proof. We get by inequalities (8), (13) and (15)

$$\begin{aligned} \|u - I^{2h}(R_h u)\|_{1,2} \le& \|u - I^{2h}(\overline{R}_h u)\|_{1,2} + \|I^{2h}(\overline{R}_h u) - I^{2h}(R_h u)\|_{1,2} \\ \le& c\|I^{2h}(\overline{R}_h u - i_h u)\|_{1,2} + \|I^{2h}(\overline{R}_h u - i_h(R_h u))\|_{1,2} + \|I^{2h} u - u\|_{1,2} \\ \le& c\|\overline{R}_h u - i_h u\|_{1,2} + \|\overline{R}_h u - i_h R_h u\|_{1,2} + ch^2 ||u||_{3,2} \le ch^2 \|u\|_{3,2}. \end{aligned}$$

Similarly, we can prove inequality (17) by inequalities (9), (14) and (15).

3. Local Superconvergence

In this section, we discuss the local superconvergence. We have

Theorem 3. *If $u \in W^{3,\infty}(D) \cap H^2(\Omega)$ and $D \subset\subset \Omega$, then*

$$\|u - I^{2h}(R_h u)\|_{1,\infty,D} \le ch^2 |\ln h| (\|u\|_{3,\infty,D} + ||u||_{2,2,\Omega} + ||f||_{2,\infty})$$

Proof. For all $z \in D$, let $g_z^h = \partial_z G_z^h \in \overline{V}^h$ denote the derivative of the discrete Green function. We take D_1, D_2 and $\omega \in C_0^\infty(\Omega)$ such that $D \subset\subset D_1 \subset\subset D_2 \subset\subset \Omega$, supp $\omega \subset\subset D_2$ and $\omega \equiv 1$ on D_1. Set $u_1 = \omega u, u_2 = u - u_1$. Then we have by Lemma 3 and $||g_z^h||_{1,1} \le c|\ln h|$

$$\begin{aligned} \|\overline{R}_h u - i_h(R_h u)\|_{1,\infty,D} =& a(\overline{R}_h u - i_h(R_h u), g_z^h) \\ =& \sum_e \{ \frac{h_e^2}{3} \int_e \frac{\partial A_1}{\partial x} \frac{\partial^2 R_h u}{\partial x^2} \frac{\partial g_z^h}{\partial x} + \frac{k_e^2}{3} \int_e \frac{\partial A_2}{\partial y} \frac{\partial^2 R_h u}{\partial y^2} \frac{\partial g_z^h}{\partial y} \} \\ &+ \mathcal{O}(h^2) |\ln h| \|R_h u\|_{2,2}. \end{aligned}$$

Hence, we deduce by Lemma 2 that

$$\begin{aligned} \|\overline{R}_h u - i_h(R_h u)\|_{1,\infty,D} \le& \sum_e \Big\{ \frac{h_e^2}{3} \int_e \frac{\partial A_1}{\partial x} B_1 \frac{\partial R_h u}{\partial x} \frac{\partial g_z^h}{\partial x} + \frac{k_e^2}{3} \int_e \frac{\partial A_2}{\partial y} B_2 \frac{\partial R_h u}{\partial y} \frac{\partial g_z^h}{\partial y} \Big\} \\ &+ ch^2 \{\|f\|_{2,\infty} + h\|R_h u\|_{2,2}\} \|g_z^h\|_{1,1} \\ &+ \mathcal{O}(h^2) |\ln h| \|R_h u\|_{2,2}, \end{aligned} \tag{18}$$

where $B_1 = -\dfrac{1}{A_1(p)}\dfrac{\partial A_1(p)}{\partial x}$, $B_2 = -\dfrac{1}{A_2(p)}\dfrac{\partial A_2(p)}{\partial y}$. Using $||g_z^h||_{1,2,\Omega\backslash D} \le c$, we find by inequalities (7), (6) and (18) that

$$\begin{aligned}
\|\overline{R}_h u - i_h(R_h u)\|_{1,\infty,D} \le & ch^2 \int_\Omega \Big\{(A_1)_x B_1 \frac{\partial u}{\partial x}\frac{\partial g_z^h}{\partial x} + (A_2)_y B_2 \frac{\partial u}{\partial y}\frac{\partial g_z^h}{\partial y}\Big\} \\
& + ch^2|\ln h|||u||_{2,2} + ch^2||f||_{2,\infty}||g_z^h||_{1,1} + ch^2||u - R_h u||_{1,\infty}||g_z^h||_{1,1} \\
\le & ch^2\Big\{\int_\Omega (A_1)_x B_1 \frac{\partial u_1}{\partial x}\frac{\partial g_z^h}{\partial x} + \int_\Omega (A_2)_y B_2 \frac{\partial u_1}{\partial y}\frac{\partial g_z^h}{\partial y} \\
& + \int_\Omega (A_1)_x B_1 \frac{\partial u_2}{\partial x}\frac{\partial g_z^h}{\partial x} + \int_\Omega (A_2)_y B_2 \frac{\partial u_2}{\partial y}\frac{\partial g_z^h}{\partial y}\Big\} \\
& + ch^2|\ln h|(\|u\|_{2,2} + ||f||_{2,\infty}) \\
\le & ch^2(\|u_1\|_{1,\infty,D}\|g_z^h\|_{1,1} + \|u_2\|_{1,2,\Omega\backslash D}\|g_z^h\|_{1,2,\Omega\backslash D}) \\
& + ch^2|\ln h|(\|u\|_{2,2,\Omega} + ||f||_{2,\infty}) \\
\le & ch^2|\ln h|(\|u\|_{1,\infty,D} + \|u\|_{2,2,\Omega} + ||f||_{2,\infty}),
\end{aligned} \tag{19}$$

where $(A_1)_x = \dfrac{\partial A_1}{\partial x}$, $(A_2)_y = \dfrac{\partial A_2}{\partial y}$. On the other hand, we know that (see [1])

$$\|u - I^{2h}(\overline{R}_h u)\|_{1,\infty,D} \le ch^2|\ln h|\|u\|_{3,\infty,D} + ch^2||u||_{2,2}. \tag{20}$$

Therefore, it yields from inequalities (15), (19) and (20) that

$$\begin{aligned}
\|u - I^{2h}(R_h u)\|_{1,\infty,D} \le & \|u - I^{2h}(\overline{R}_h u)\|_{1,\infty,D} + \|I^{2h}(\overline{R}_h u - i_h(R_h u))\|_{1,\infty,D} \\
\le & \|u - I^{2h}(\overline{R}_h u)\|_{1,\infty,D} + \|\overline{R}_h u - i_h(R_h u)\|_{1,\infty,D} \\
\le & ch^2|\ln h|(||u||_{3,\infty,D} + ||u||_{2,2} + ||f||_{2,\infty}).
\end{aligned}$$

This completes the proof.

Furthermore, if we note the following relations

$$||u - I^{2h}(\overline{R}_h u)||_{0,\infty,D} \le ch|\ln h|||u||_{1,\infty,D} + ch||u||_{1,2}$$

and

$$||u - I^{2h}(R_h u)||_{0,\infty,D} \le ||\overline{R}_h u - i_h(R_h u)||_{0,\infty,D} + ||u - I^{2h}(\overline{R}_h u)||_{0,\infty,D},$$

using inequality (11), then we have

Corollary 2. *If $D \subset\subset \Omega$, then there exists a positive constant c such that*

$$||u - I^{2h}(R_h u)||_{0,\infty,D} \le ch|\ln h|(||u||_{1,\infty,D} + ||u||_{1,2}).$$

4. Correction Schemes

The defect correction techniques for the conforming finite element were, as we know, initiated by R. Rannacher and developed by H. Blum during the last six years. See a survey in this topic (see [1], [12], [14]). But so far these correction techniques for the nonconforming finite element have not been applied. In this section, we present correction schemes for the nonconforming finite element. It shows that these kinds of techniques are valid for the nonconforming finite element too.

Theorem 4. *If $q = 2$ or ∞ and $u \in W^{4,q}(\Omega) \cap H_0^1(\Omega)$, then there exists a $W_1 \in H_0^1(\Omega)$ such that*

$$\|\overline{R}_h u - i_h(R_h u) - h^2 i_h W_1\|_{1,2} \le ch^3 \tag{21}$$

and

$$\|\overline{R}_h u - i_h(R_h u) - h^2 \overline{R}_h W_1\|_{1,\infty} \le ch^3 |\ln h|. \tag{22}$$

Proof. Consider linear functional on $H_0^1(\Omega)$:

$$L(v) = \frac{1}{3} \sum_e \int_e \Big\{ \lambda_e^2 (A_1)_x F_1 \frac{\partial v}{\partial x} + \mu_e^2 (A_2)_y F_2 \frac{\partial v}{\partial y} \Big\},$$

where $\lambda_e = h_e/h, \mu_e = k_e/h$. Obviously $|L(v)| \le \frac{1}{3}\|u\|_{1,2}\|v\|_{1,2}$.

Hence from the Lax-Milgram theorem, we know that there exists (see [1])

$$W_1 \in W^{2,q}(\Omega) \cap H_0^1(\Omega) \quad (1 < q < \infty)$$

such that

$$a(W_1, v) = L(v) \quad \forall v \in H_0^1(\Omega).$$

Furthermore we have by Lemma 3, for any $v \in \overline{V}^h$,

$$a(\overline{R}_h u - i_h(R_h u), v) = h^2 a(W_1, v) + \mathcal{O}(h^4)\|u\|_{4,2}\|v\|_{2,2}$$

and it suffices to observe that for all $v \in \overline{V}^h$,

$$a(W_1, v) = a(\overline{R}_h W_1, v).$$

Hence, taking $v = \overline{R}_h u - i_h(R_h u) - h^2 \overline{R}_h W_1$, this leads to

$$\|\overline{R}_h u - i_h(R_h u) - h^2 \overline{R}_h W_1\|_{1,2} \le ch^3. \tag{23}$$

Using (see [1])

$$\|\overline{R}_h W_1 - i_h W_1\|_{1,2} \le ch^{2-\epsilon}, \tag{24}$$

and combining (23) and (24), we see that inequality (21) is proved. Furthermore, we have

$$a(\overline{R}_h u - i_h(R_h u) - h^2 \overline{R}_h W_1, v) = \mathcal{O}(h^4)\|u\|_{4,\infty}\|v\|_{2,1}$$

and taking $v = \partial_z G_z^h \in \overline{V}^h$, we get

$$\|\overline{R}_h u - i_h(R_h u) - h^2 \overline{R}_h W_1\|_{1,\infty} \le ch^3 |\ln h| \tag{25}$$

Therefore, inequality (22) is proved.

If $D \subset\subset D_1 \subset\subset \Omega$, the mesh T^h is unidirectionally uniform and u is smooth enough on D, and make use of the following local superconvergence (see [1])

$$\|\overline{R}_h W_1 - i_h W_1\|_{1,2,D} \le ch^2 \left(\|W_1\|_{3,2,D} + \|W_1\|_{2,2,\Omega}\right)$$

and

$$\|\overline{R}_h W_1 - i_h W_1\|_{1,\infty,D} \le ch^2 |\ln h| \left(\|W_1\|_{3,\infty,D} + \|W_1\|_{2,2,\Omega}\right),$$

and note that inequality (23) and (25), then it is easy to deduce the following results.

Corollary 3. *If $D \subset\subset D_1 \subset\subset \Omega$, then*

$$\|\overline{R}_h u - i_h(R_h u) - h^2 i_h W_1\|_{1,2,D} \le ch^3 \tag{26}$$

and

$$\|\overline{R}_h u - i_h(R_h u) - h^2 i_h W_1\|_{1,\infty,D} \le ch^3 |\ln h|. \tag{27}$$

Lemma 6. *There exists a positive constant c such that*

$$||(I - R_h)u||_{0,2} \le ch||u||_{1,2} \tag{28}$$

and

$$||(I - R_h)u||_{0,\infty} \le ch|\ln h|||u||_{1,\infty}. \tag{29}$$

Proof. Let $\phi \in H^2(\Omega) \cap H_0'(\Omega)$ and $f \in L^2(\Omega)$ satisfy

$$-L\phi = f.$$

Then we know that

$$||\phi||_2 \le c||f||_0.$$

Since

$$||u - R_h u||_0 = \sup_{\substack{f\in L^2\\ f\neq 0}} \frac{|(f, u - R_h u)|}{||f||_0} = \sup_{\substack{f\in L^2\\ f\neq 0}} \frac{|(-L\phi, u - R_h u)|}{||f||_0}, \tag{30}$$

we can write by Green's formula

$$\begin{aligned}(-L\phi, u - R_h u) =& a_h(\phi, u - R_h u) - E_h(\phi, u - R_h u)\\ =& a_h(\phi - i_h\phi, u - R_h u) - E_h(\phi, u - R_h u),\end{aligned}$$

where

$$E_h(\phi, u - R_h u) = \sum_e \left\{\Big(\int_{l_2} - \int_{l_1}\Big) A_1 \frac{\partial\phi}{\partial x}(u - R_h u) + \Big(\int_{l_4} - \int_{l_3}\Big) A_2 \frac{\partial\phi}{\partial y}(u - R_h u)\right\}.$$

Using inequality (5), we get

$$\begin{aligned}|a_h(\phi - i_h\phi, u - R_h u)| \le& ||\phi - i_h\phi||_{1,2}||u - R_h u||_{1,2}\\ \le& ch||\phi||_2||u||_1 \le ch||f||_0||u||_1\end{aligned}$$

and similarly to the proof of inequality (5), we have

$$\begin{aligned}|E_h(\phi, u-R_hu)| =&\Big|\sum_e\Big\{\Big(\int_{l_2}-\int_{l_1}\Big)A_1\frac{\partial\phi}{\partial x}(u-R_hu)+\Big(\int_{l_4}-\int_{l_3}\Big)A_2\frac{\partial\phi}{\partial y}(u-R_hu)\Big\}\Big|\\ \leq& ch||R_hu-i_h(R_hu)||_{1,2}||\phi||_{2,2}\leq ch||f||_0||u||_1.\end{aligned}$$

Therefore, we get inequality (28) by (30) and above all relations. Similarly, we can prove inequality (29).

Theorem 5. *If u and R_hu satisfy* (1) *and* (2) *respectively, and if we set*

$$u^\star = I^{2h}R_hu + R_hu - R_hI^{2h}R_hu$$

then

$$||u-u^\star||_{0,2}\leq ch^3,\quad ||u-u^\star||_{0,\infty}\leq ch^3|\ln h|^2$$

Proof. Let I be the identical operator and $u_1=(I-I^{2h}R_h)u$. We get by inequality (29) and (17)

$$\begin{aligned}||u-u^\star||_{0,\infty} =&||(I-R_h)u_1||_{0,\infty}\leq ch|\ln h|||u_1||_{1,\infty}\\ \leq& ch|\ln h|||(I-I^{2h}R_h)u||_{1,\infty}\leq ch^3|\ln h|^2||u||_{3,\infty}\end{aligned}$$

and, using inequality (16) and (28),

$$||u-u^\star||_{0,2}\leq ch||u_1||_{1,2}\leq ch^3||u||_{3,2}.$$

This completes our proof.

We assume that T^h has been obtained from T^{4h} of mesh size $4h$ by uniform subdividing each element into 16 congruent subrectangles of size h on any elements in T^{4h}. We define the high order interpolation operator I^{4h} on the space of piecewise bilinear function $\overline{V}^h$ and satisfying

$$\begin{aligned}&I^{4h}i_h = I^{4h},\quad \|I^{4h}v\|_{1,q}\leq\|v\|_{1,q},\quad\forall v\in\overline{V}^h,\\ &\|I^{4h}u-u\|_{1,q}\leq ch^4\|u\|_{5,q},\quad 2\leq q\leq\infty.\end{aligned}$$

Now we define another correction scheme. Set

$$u^{\star\star} = I^{2h}R_hu + I^{4h}R_hu - I^{2h}R_hI^{4h}R_hu$$

then we get

Theorem 6. *Under the condition of the Lemma 1, if $u\in C^5(D)\cap H_0^1(\Omega)\cap H^4(\Omega)$ and $D\subset\subset\Omega_0\subset\Omega$, then*

$$||u-u^{\star\star}||_{1,\infty,D}\leq ch^3|\ln h| \tag{32}$$

and

$$||u-u^{\star\star}||_{1,2,\Omega}\leq ch^3. \tag{33}$$

Proof. We know that (see [1])

$$\overline{R}_hu - i_hu = h^2 i_h M_1 + q_h.$$

From inequality (21), (26) and (27), we have

$$\overline{R}_h u - i_h(R_h u) = h^2 i_h W_1 + r_h,$$

where M_1 and $W_1 \in H_0^1(\Omega)$ are smooth in D and

$$\begin{aligned}
&\|r_h\|_{1,\infty,D} + \|q_h\|_{1,\infty,D} = \mathcal{O}(h^3|\ln h|),\\
&\|r_h\|_{1,2} + \|q_h\|_{1,2} = \mathcal{O}(h^3).
\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}
u_1 = I^{4h} R_h u - u =&(I^{4h}\overline{R}_h u - u) + I^{4h}(i_h R_h u - \overline{R}_h u)\\
=&h^2 M_1 + (I^{4h} u - u) + I^{4h} q_h + (I^{4h} M_1 - M_1) h^2\\
&- h^2 W_1 - h^2(I^{4h} W_1 - W_1) - I^{4h} r_h = h^2 M_1 - h^2 W_1 + \beta_h,
\end{aligned}$$

where

$$\begin{aligned}
&\beta_h = (I^{4h} u - u) + I^{4h} q_h + h^2(I^{4h} M_1 - M_1) - h^2(I^{4h} W_1 - W_1) - I^{4h} r_h,\\
&\|\beta_h\|_{1,2} \leq ch^3, \quad \|\beta_h\|_{1,\infty,D} \leq ch^3|\ln h|.
\end{aligned}$$

Now, multiplying u_1 by $I - I^{2h} R_h$, we have

$$(I - I^{2h} R_h) u_1 = h^2(I - I^{2h} R_h) M_1 - h^2(I - I^{2h} R_h) W_1 + (I - I^{2h} R_h)\beta_h.$$

Furethermore we have by Theorem 3

$$||W_1 - I^{2h} R_h W_1||_{1,\infty,D} \leq ch^2|\ln h|^2$$

and

$$\|\beta_h - I^{2h} R_h \beta_h\|_{1,\infty,D} \leq ||\beta_h||_{1,\infty,D} \leq ch^3|\ln h|.$$

Therefore, we deduce that

$$\|u^{\star\star} - u\|_{1,\infty,D} = \|(I - I^{2h} R_h) u_1\|_{1,\infty,D} \leq ch^3|\ln h|.$$

Similarly we can prove inequality (33)

Next, we discuss the L^∞ defect correction. Set

$$\begin{aligned}
u^{\star\star\star} =&I^{2h} R_h u + I^{4h} R_h u - I^{2h} R_h I^{4h} R_h u + I^{2h} R_h u\\
&- I^{2h} R_h(I^{2h} R_h u + I^{4h} R_h u - I^{2h} R_h I^{4h} R_h u),
\end{aligned}$$

then we get the following result:

Theorem 7. *Under the conditions of the Theorem* 6, *we have*

$$||u - u^{\star\star\star}||_{0,\infty,D} \leq ch^4|\ln h|^2.$$

Proof. Let $u_2 = (I^{2h} R_h - I)(I^{4h} R_h - I)u$ and $D \subset\subset \Omega$. From Theorem 6 and Corollary 2, we obtain

$$\begin{aligned}
||(I - I^{2h} R_h) u_2||_{0,\infty,D} \leq&ch|\ln h|(||u_2||_{1,\infty,D} + ||u_2||_{1,2,\Omega})\\
\leq&ch|\ln h|(||u - u^{\star\star}||_{1,\infty,D} + ||u - u^{\star\star}||_{1,2,\Omega}) \leq ch^4|\ln h|^2.
\end{aligned}$$

Therefore, Theorem 7 follows.

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