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THE SCHWARZ CHAOTIC RELAXATION METHOD WITH INEXACT SOLVERS ON THE SUBDOMAINS*

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Abstract

In this paper, a S-CR method with inexact solvers on the subdomains is presented, and then its convergence property is proved under very general conditions. This result is important because it guarantees the effectiveness of the Schwarz alternating method when executed on message-passing distributed memory multiprocessor system.

Key words: S-CR method, Chaotic algorithm, Inexact solvers.

1. Introduction

Early in 1869, A.H. Schwarz introduced the technique of domain decomposition and alternative iteration to prove the existence of the solution for some elliptic problem in non-regular domain. In recent years, with the arrival and tremendous development of supercomputer and multiprocessor system, this ancient and profound idea brings about fresh vitality, becomes an important source to the research of large-scale scientific computation.

Besides the ease of parallelization, Schwarz alternating algorithm and many other domain decomposition methods allow one to treat complex geometries or to isolate singular parts of the domain through adaptive refinement. They have attracted much attention all of the world, see e.g. [1], [8] for details. But all of these algorithms are synchronous, which will lead to great overheads in data communication, and severely damage the efficiency of parallelization in practice.

In [5], [6], Kang put forward the S-CR algorithm (Schwarz Chaotic Relaxation algorithm) which first combined the chaotic idea and schwarz relaxation alternating method. This new algorithm was carried out in some message - passing distributed memory multiprocessor system. Numerical experiments have showed its effectiveness[5,6]. In his Ph.D. Thesis, Huang[3,4] gave a rigorous proof for the convergence of the S-CR. This proof depends heavily on the norm estimates of some multiplicative operators.

In this article, the author will go on with the convergence analysis of the S-CR with inexact solvers on the subdomains. It is well known that implementation of the S-CR is mainly at the request of the solving of subproblems assigned on certain separate and interconnected processors. But exact solvers for these subproblems are impossible or

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improper, in practice we have to employ the inexact solvers, e.g. Gauss-Seidel method, SSOR, PCG and other high efficient iterative methods. What influence on the global convergence does this result in? We show under much receivable conditions the S-CR algorithm with inexact solvers is also convergent. This result is important because it guarantees the effectiveness of the S-CR algorithm when executed on the message - passing distributed memory multiprocessor system.

Let $\Omega \subset R^2$ be a bounded polygonal domain, and let

$$\begin{cases} a(u,v)=(f,v), \quad f\in H^{-1}(\Omega), \qquad v\in H_0^1(\Omega), \\ u\in H_0^1(\Omega), \end{cases} \tag{1.1}$$

be the variational form of an elliptic operator defined on it. The bilinear form satisfies: For arbitrary $u, v \in H_0^1(\Omega)$,

$$\begin{cases} a(u,v)=a(v,u), \\ a(u,v)\le C_2\|u\|_1\ \|v\|_1, \\ a(v,v)\ge C_1\|v\|_1^2, \end{cases} \tag{1.2}$$

where $\|.\|_1$ is the conventional Sobolev norm in $H_0^1(\Omega)$, C_1 and C_2 are two positive constants. We will borrow the finite element method to approximate (1.1).

Assume that the triangulation T_h is quasi-uniform[1], and let $V \subset H_0^1(\Omega)$ be the corresponding piecewise linear finite element space defined on it. Then we have the following discretized form of (1.1).

$$\begin{cases} a(u_h,v)=(f,v), \quad v\in V, \\ u_h\in V. \end{cases} \tag{1.3}$$

Thanks to (1.2), in what follows, we will consider V as a Hilbert space with inner product $a(\cdot,\cdot)$, its related induced norm is denoted by $\|.\|$.

Suppose Ω is divided into m subdomains $\Omega_1, \Omega_2, \cdots, \Omega_m$ which satisfy:

1. $\Omega = \sum \Omega_i$;

2. $\partial\Omega_i$ aligns with the triangulation T_h, i.e. the line of $\partial\Omega_i$ either coincides with or does not intersect the triangulation line of ∂T.

Let $V_i = H_0^1(\Omega_i) \cap V$ which can be looked upon as a subspace of V, $M^\perp$ denote the orthogonal complementary subspace of some subspace M, and P_M represent the orthogonal projection operator from V onto M. We assume that

$$V=\sum V_i. \tag{1.4}$$

Let $\omega \in (0,2)$ be a relaxation parameter. The S-CR introduced in [5] and [6] can be abstracted as follows: Let $u^0 \in V$ be an arbitrary guess function, the iterative sequence $\{u^k\}$ for solving (1.3) satisfies that

$$\begin{cases} u_1-u^{k-1}\in V_{\tau(k)}, \\ a(u_1,v)=(f,v), \qquad\qquad v\in V_{\tau(k)}, \\ u^k=(1-\omega)u^{k-1}+\omega u_1, \end{cases} \tag{1.5}$$

where $\tau(k)$ denotes the subscript of the subdomain related to the kth iteration, and for arbitrary natural number i, $1 \leq i \leq m$, it appears infinite often in the index set $\{\tau(k)\}_{k=1}^{\infty}$.

From (1.5), we see that the main work to execute the S-CR is to solve the subproblem defined on the subdomain $\Omega_{\tau(k)}$ at the kth step. But exact solver is not available or improper in general, and only inexact solvers such as Gauss-Seidel method, SSOR, PCG or other high-efficient iterative algorithms can be used. This leads to the following scheme which describes that S-CR implemented in practice.

SCRI (S-CR with inexact solvers). let $u^0 \in V$ be an arbitrary initial guess function, $\omega \in (0,2)$ and $\mu \in (0,1)$ be two parameters. Then the iterative sequence $\{u^k\}$ satisfies that

$$\begin{cases} u_1 - u^{k-1} \in V_{\tau(k)}, & (*) \\ a(u_1, v) = (f, v), & v \in V_{\tau(k)}, \quad (*) \\ u_2 - u^{k-1} \in V_{\tau(k)}, & \\ \|u_2 - u_1\| \leq \mu \|u^{k-1} - u_1\|, & \\ u^k = (1-\omega) u^{k-1} + \omega u_2, & \end{cases} \tag{1.6}$$

where u_2 is the approximate solution of (1.6$*$) via proper iterative method with initial function u^{k-1}, μ means the accuracy restriction.

Let $E^k = u^k - u_h$, $E_1 = u_1 - u_h$ and $E_2 = u_2 - u_h$. It follows from (1.6) that

$$\begin{cases} E_1 - E^{k-1} \in V_{\tau(k)}, & \\ a(E_1, v) = 0, & v \in V_{\tau(k)}, \\ E_2 - E^{k-1} \in V_{\tau(k)}, & \\ \|E_2 - E_1\| \leq \mu \|E^{k-1} - E_1\|, & \\ E^k = (1-\omega) E^{k-1} + \omega E_2. & \end{cases} \tag{1.7}$$

Obviously $E_1 = P'_{\tau(k)} E^{k-1}$ where P'_l denotes the orthogonal projection operator from V onto the subspace $V_l^{\perp}$. Let $\mu_k = \dfrac{\|E_2 - E_1\|}{\|E^{k-1} - E_1\|}$ ($\mu_k = 0$ if $E_1 = E^{k-1}$), then it is clear that $0 \leq \mu_k \leq \mu < 1$.

In order to make out the error propagation of the algorithm SCRI more clearly, we'd like to express (1.7) in operator form. It follows from (1.7), matrix theory and the isomorphism technique that there exists an orthogonal operator Q_k on $V_{\tau(k)}$ satisfying

$$E_2 - E_1 = \mu_k Q_k (E^{k-1} - E_1). \tag{1.8}$$

Here Q_k can be viewed as a linear operator defined on the whole space V by zero extension, i.e. $Q_k v = 0$, for $v \in V_{\tau(k)}^{\perp}$. By the way, from now on we will view an arbitrary orthogonal operator defined on any subspace W of V as the operator on V in the same way. Thus from (1.6), (1.7) and (1.8) we have

$$E_2 = \mu_k Q_k E^{k-1} + (1 - \mu_k Q_k) E_1, \tag{1.9a}$$

$$E^k = [(1-\omega) I + \omega \mu_k Q_k + \omega P'_{\tau(k)}] E^{k-1}. \tag{1.9b}$$

2. Preparation of The Proof

In order to give the rigorous proof of the convergence property, we need the following Lemmas.

Lemma2.1. *Suppose the constants a, μ, ω satisfy that $0 \le a \le \mu < 1$, $0 < \omega < \frac{2}{1+\mu}$, then*

$$\|(1-\omega)I + \omega a Q + \omega P_k^{'}\| \le 1, \tag{2.1}$$

where Q is an arbitrary orthogonal operator on V_k, $1 \le k \le m$.

Proof. For any $v = v_1 + v_2 \in V$, $v_1 \in V_k$, $v_2 \in V_k^{\perp}$,

$$[(1-\omega)I + \omega a Q + \omega P_k']v = (1 - \omega + \omega a Q)v_1 + v_2,$$

and

$$\|[(1-\omega)I + \omega a Q + \omega P_k^{'}]v\|^2 = \|(1-\omega+\omega a Q)v_1\|^2 + \|v_2\|^2.$$

But

$$\begin{aligned}\|(1-\omega+\omega a Q)v_1\|^2 =& (1-\omega)^2\|v_1\|^2 + 2a\omega(1-\omega)(Qv_1, v_1) + a^2\omega^2\|Qv_1\|^2 \\ \le& \begin{cases} (1-\omega+\omega a)^2\|v_1\|^2, & 0 < \omega \le 1 \\ (1-\omega-\omega a)^2\|v_1\|^2, & 1 \le \omega < \dfrac{2}{1+\mu} \end{cases} \le \|v_1\|^2.\end{aligned}$$

Lemma2.1 then follows.

In what follows we will always assume the conditions for ω, μ in Lemma2.1 are satisfied.

Lemma2.2. *There exists a constant $\sigma \in (0,1)$, such that, for arbitrary element M_1 of $\{V_k\}_{k=1}^m$ (here $\{V_k\}_{k=1}^m$ is a finite set with subspaces $V_k, k = 1, 2, \cdots, m$ as its elements), and M_2 which is the sumspace of any subset of $\{V_k\}_{k=1}^m$ (i.e., $M_2 = \sum_{i=1}^{l} V_{t_i}$ for a subset $\{V_{t_i}\}_{i=1}^l \subset \{V_k\}_{k=1}^m$), for any μ_k, $0 \le \mu_k \le \mu$, any orthogonal operator Q_k on subspace M_k, $k = 1, 2$, we have*

$$\Big\| \prod_{k=1}^{2} [(1-\omega)I + \omega\mu_k Q_k + \omega P_{M_k}']v \Big\| \le \sigma\|v\|, \quad v \in M_1 + M_2, \tag{2.2a}$$

and

$$\|[(1-\omega)I + \omega\mu_1 Q_1 + \omega P_{M_1}^{'}]P_{M_2}'v\| \le \sigma\|v\|, \quad v \in M_1 + M_2. \tag{2.2b}$$

Proof. We only prove (2.2a), proof of (2.2b) is similar. Because of the finite choices of M_k, and the continuity of

$$\Big\| \prod_{k=1}^{2} [(1-\omega)I + \omega\mu_k Q_k + \omega P_{M_k}'] \Big\| = \sup_{v \in M_1+M_2, v \ne 0} \Big\| \prod_{k=1}^{2} [(1-\omega)I + \omega\mu_k Q_k + \omega P_{M_k}']v \Big\| / \|v\|$$

with respect to $\mu_k \in [0, \mu]$, and Q_k, $k = 1, 2$, if (2.2a) is not true, then there exist some M_k, μ_k, Q_k, $k = 1, 2$, and v^n, $\|v^n\| = 1$, $v^n \in M_1 + M_2$, such that

$$\Big\| \prod_{k=1}^{2} [(1-\omega)I + \omega\mu_k Q_k + \omega P_{M_k}']v^n \Big\| \to 1. \tag{2.3}$$

Therefore, from Lemma2.1 and (2.3) we have

$$\|[(1-\omega)I+\omega\mu_2 Q_2+\omega P'_{M_2}]v^n\| \to 1.$$

Let $v^n = v_1^n + v_2^n$, $v_1^n \in M_2$, $v_2^n \in M_2^{\perp}$, then

$$\|[(1-\omega)I+\omega\mu_2 Q_2+\omega P'_{M_2}]v^n\|^2 = \|(1-\omega+\omega\mu_2 Q_2)v_1^n\|^2 + \|v_2^n\|^2.$$

So

$$\|v_1^n\|^2 + \|v_2^n\|^2 - \|(1-\omega+\omega\mu_2 Q_2)v_1^n\|^2 - \|v_2^n\|^2 \to 0.$$

But

$$\begin{aligned}
&\|v_1^n\|^2 - \|(1-\omega+\omega\mu_2 Q_2)v_1^n\|^2 \\
&\quad \geq \min\{1-(1-\omega+\omega\mu_2)^2, 1-(1-\omega-\omega\mu_2)^2\}\|v_1^n\|^2 \\
&\quad \geq \min\{\omega(1-\mu)(2-\omega), \omega(2-\omega-\omega\mu)\}\|v_1^n\|^2,
\end{aligned}$$

which leads to $v_1^n \to 0$, i.e. $P_{M_2}v^n \to 0$. Thus from (2.3) we also have

$$\|[(1-\omega)I+\omega\mu_1 Q_1+P'_{M_1})]v^n\| \to 1.$$

With the same argument we have $P_{M_1}v^n \to 0$.

On the other hand, from Lions lemma[7] there exists a positive constant α such that

$$\|v\|^2 \leq \alpha(\|P_{M_1}v\|^2 + \|P_{M_2}v\|^2), \qquad v \in M_1 + M_2.$$

These lead to $v^n \to 0$ which is a contradiction since $\|v^n\| = 1$. Thus Lemma 2.2 is proved.

Lemma 2.3. *Let $\{t_1, t_2\}$ be an arbitrary subset of $\{1, 2, \cdots, m\}$, then for arbitrary K natural numbers $\alpha_i \in \{t_1, t_2\}$, $i = 1, 2, \cdots, K$, and $\{t_1, t_2\} \subset \{\alpha_1, \alpha_2, \cdots, \alpha_K\}$, μ_i, $0 \leq \mu_i \leq \mu$, orthogonal operators Q_{α_i} on V_{α_i}, we have*

$$\Big\|\prod_{k=1}^{K}[(1-\omega)I+\omega\mu_k Q_k+\omega P'_{\alpha_k}]v\Big\| \leq \sigma\|v\|, \quad v \in V_{t_1} + V_{t_2}, \tag{2.4}$$

where $\sigma \in (0,1)$ is defined as in Lemma 2.2.

Lemma 2.3 follows easily from Lemma 2.1 and Lemma 2.2. Now we can obtain the following main result.

Theorem 2.4. *There exists a constant $\sigma \in (0,1)$ such that, for any integer $l(2 \leq l \leq m)$, for arbitrary subset $\{t_1, t_2, \cdots, t_l\} \subset \{1, 2, \cdots, m\}$, arbitrary $\alpha_k \in \{t_1, t_2, \cdots, t_l\}$, $k = 1, 2, \cdots, K$ with $\{t_1, t_2, \cdots, t_l\} \subset \{\alpha_1, \alpha_2, \cdots, \alpha_K\}$, we have*

$$\Big\|\prod_{k=1}^{K}[(1-\omega)I+\omega\mu_k Q_k+\omega P'_{\alpha_k}]v\Big\| \leq \sigma_l\|v\|, \quad v \in \sum_{k=1}^{l} V_{t_k}, \tag{2.5}$$

where $0 \leq \mu_k \leq \mu$, Q_k is arbitrary orthogonal operator on V_{α_k}, and $\sigma_2 = \sigma$, $\sigma_{l+1} = \sigma + (1-\sigma)\sigma_l$.

Proof. By induction. As l =2, the result is followed from Lemma 2.3 directly. Assume the result is true for l $(2 \le l < m)$, we want to prove the correctness for $l+1$.

For arbitrary $\{t_1, t_2, \cdots, t_{l+1}\} \subset \{1, 2, \cdots, m\}$, arbitrary $\alpha_k \in \{t_1, t_2, \cdots, t_{l+1}\}$, $k = 1, 2, \cdots, K$ with $\{t_1, t_2, \cdots, t_{l+1}\} \subset \{\alpha_1, \alpha_2, \cdots, \alpha_K\}$, $v \in \sum_{k=1}^{l+1} V_{t_k}$, consider the estimate of $\prod_{k=1}^{K}[(1-\omega)I + \omega\mu_k Q_k + \omega P'_{\alpha_k}]v$. Without loss of generality, we may assume that $\alpha_1 = t_1$, $t_1 \notin \{\alpha_2, \alpha_2, \cdots, \alpha_K\}$. Otherwise, by the search process in order, there exists some i $(1 \le i \le K)$, such that $\{t_1, t_2, \cdots, t_{l+1}\} \subset \{\alpha_i, \alpha_{i+1}, \cdots, \alpha_K\}$, and $\alpha_i \notin \{\alpha_{i+1}, \alpha_{i+2}, \cdots, \alpha_K\}$. We might as well suppose $\alpha_i = t_1$, then from Lemma 2.1

$$\Big\| \prod_{k=1}^{K}[(1-\omega)I + \omega\mu_k Q_k + \omega P'_{\alpha_k}]v\| \le \Big\| \prod_{k=i}^{K}[(1-\omega)I + \omega\mu_k Q_k + \omega P'_{\alpha_k}]v\|$$

which is converted to the estimate of the assumption case.

Let

$$W = \sum_{k=2}^{l+1} V_{t_k}, \quad E_3 = \prod_{k=2}^{K}[(1-\omega)I + \omega\mu_k Q_k + \omega P'_{\alpha_k}]v.$$

Then from induction assumption, we have

$$\|E_3 - P_{W^\perp}v\| = \Big\| \prod_{k=2}^{K}[(1-\omega)I + \omega\mu_k Q_k + \omega P'_{\alpha_k}](v - P_{W^\perp}v)\Big\| \le \sigma_l\|v - P_{W^\perp}v\|. \tag{2.6}$$

Let $\eta = \dfrac{\|E_3 - P_{W^\perp}v\|}{\|v - P_{W^\perp}v\|}$ $(\eta = 0$ as $v - P_{W^\perp}v = 0)$. Thus

$$0 \le \eta \le \sigma^l. \tag{2.7}$$

We next introduce the following auxiliary function

$$\begin{cases} v^* = P_{W^\perp}v + \dfrac{1}{\eta}(E_3 - P_{W^\perp}v), & (\eta > 0), \\ v^* = v, & (\eta = 0). \end{cases} \tag{2.8}$$

It is easy to see that

$$\begin{cases} \|v^*\|^2 = \dfrac{1}{\eta^2}\|E_3 - P_{W^\perp}v\|^2 + \|P_{W^\perp}v\|^2 = \|v\|^2, \\ E_3 = \eta v^* + (1-\eta)P_{W^\perp}v*, \end{cases} \tag{2.9}$$

since

$$E_3 - P_{M^\perp}v = \prod_{k=2}^{K}[(1-\omega)I + \omega\mu_k Q_k + \omega P'_{\alpha_k}](v - P_{W^\perp}v)$$

and for any $v \in$W,

$$[(1-\omega)I + \omega\mu_k Q_k + \omega P'_{\alpha_k}]v \in W, \quad k = 2, 3, \cdots, K.$$

Therefore,

$$\begin{aligned}\Big\|\prod_{k=1}^{K}[(1-\omega)I+\omega\mu_kQ_k+\omega P'_{\alpha_k}]v\Big\| &= \|[(1-\omega)I+\omega\mu_1Q_1+\omega P'_{\alpha_1}]E_3\|\\ &=\|[(1-\omega)I+\omega\mu_1Q_1+\omega P'_{\alpha_1}][\eta v^*+(1-\eta)P_{W^\perp}v*]\|\\ &\le\eta\|v\|+(1-\eta)\sigma\|v\|\le[\sigma+(1-\sigma)\sigma_l]\|v\|\end{aligned}$$

which proves Theorem 2.4. The last inequalities follow from (2.7), (2.9), Lemma 2.1 and Lemma 2.2, here we also use the fact that $v^*\in V_{\alpha_1}+M$. This is because that $E_3-P_{W^\perp}v\in W$ (see before) and $P_{W^\perp}v\in V_{\alpha_1}+W$ since $v\in V_{\alpha_1}+W$.

Let $l=m$, we have the following lemma.

Lemma 2.5. *For arbitrary natural numbers $\alpha_i\in\{1,2,\cdots,m\}$, $i=1,2,\cdots,K$, and $\{1,2,\cdots,m\}\subset\{\alpha_1,\alpha_2,\cdots,\alpha_K\}$,*

$$\Big\|\prod_{k=1}^{K}[(1-\omega)I+\omega\mu_kQ_k+\omega P'_{\alpha_k}]\Big\|\le\sigma_m<1, \tag{2.10}$$

where $0\le\mu_k\le\mu$, Q_k is an arbitrary orthogonal operator on V_{α_k}, $k=1,2,\cdots,K$, respectively.

3. Proof of The Convergence

Theorem 3.1. *Under the conditions given before, i.e., V is split into m subspaces $\{V_k\}_{k=1}^m$ satisfying*

$$V=\sum_{k=1}^{m}V_k$$

and the relaxation parameter ω and the accuracy parameter μ satisfy

$$0\le\mu<1, 0<\omega<\frac{2}{1+\mu},$$

the SCRI algorithm is convergent.

Proof. There is no harm in assuming that the iterative sequence $\{u^k\}$ can be decomposed into

$$\begin{gathered}u^1\to u^2\to\cdots\to u^{p_1}\\ u^{p_1+1}\to u^{p_1+2}\to\cdots\to u^{p_2}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ u^{p_i+1}\to u^{p_i+2}\to\cdots\to u^{p_{i+1}}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{gathered}$$

$i=1,2,\cdots\cdots$, and $\{1,2,\cdots,m\}\subset\{\tau(p_i+1),\tau(p_i+2),\cdots,\tau(p_{i+1})\}$. Then from (1.8) and Lemma 2.5, for arbitrary k, $p_{l-1}\le k\le p_l$,

$$\|E^k\|\le(\sigma_m)^{l-1}\|E^0\|.$$

Pay attention to the fact that $l \to \infty$ as $k \to \infty$, the theorem is proved.

Remark. It should be pointed out that the techniques and results given in [1], [8] can not lead to our convergence result directly.

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