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A LEAP FROG FINITE DIFFERENCE SCHEME FOR A CLASS OF NONLINEAR SCHRÖDINGER EQUATIONS OF HIGH ORDER*1)

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Abstract

In this paper, the periodic initial value problem for the following class of nonlinear Schrödinger equation of high order

$$i\frac{\partial u}{\partial t}+(-1)^m\frac{\partial^m}{\partial x^m}\Big(a(x)\frac{\partial^m u}{\partial x^m}\Big)+\beta(x)q(|u|^2)u+f(x,t)u=g(x,t)$$

is considered. A leap-frog finite difference scheme is given, and convergence and stability is proved. Finally, it is shown by a numerical example that numerical result is coincident with theoretical result.

Key words: High order nonlinear Schrödinger equation, Leap-Frog difference scheme, Convergence.

1. Introduction

It is well know that the nonlinear equations of Schrödinger type are of great importance to physics and can be used to describe extensive physical phenomena[1].

In this paper,we will consider the periodic initial value problem for the following class of nonlinear Schrödinger equation of high order:

$$\begin{cases} i\dfrac{\partial u}{\partial t}+(-1)^m\dfrac{\partial^m}{\partial x^m}\Big(a(x)\dfrac{\partial^m u}{\partial x^m}\Big)+\beta(x)q(|u|^2)u+f(x,t)u=g(x,t) & (x,t)\in R\times I & (1.1)\\ u|_{t=0}=u_0(x) & x\in R & (1.2)\\ u(x+L,t)=u(x,t) & (x,t)\in R\times I & (1.3)\end{cases}$$

where $i=\sqrt{-1}$, $R=(-\infty,+\infty)$, $I=[0,T]$, $u\equiv u(x,t)$ is an unknow complex valued function of x with period L, and $\overline{u}$ is a conjugate complex function of u; $f(x,t), g(x,t)$, $a(x)$ and $\beta(x)$ are all real-valued function x with period L; $u_0(x)$ is given complex-valued function with period L; $q(\cdot)$ is a continuous real-valued function with real variable, and compound function $z\to q^*(z)=q(|z|^2)$ exist a continuous partial derivative to Rez, Imz. Besides, suppose the following conditions are true:

$$\begin{cases} 0<m'\le a(x)\le M\\ \max\limits_{(x,t)\in R\times I}\{|\beta(x)|,|f(x,t)|\}=M_1\end{cases}\qquad (A)$$

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where m',M and M_1 are all positive constant.

In the paper [2], there have discussed initial Value problem of system such as (1.1)–(1.3), introduced a difference scheme of conservation type, and researched its stability and convergence. Otherwise, it is an implicit method and its difference scheme is a nonlinear system.

In this paper, we introduce a leap-frog finite difference scheme for the periodic initial value problem (1.1)–(1.3) of a class of nonlinear Schrödinger equation of high order, its difference scheme is explicit, easily solved. Its convergence and stability can be proved.

Finally, it is shown by a numerical example that numerical result is coincident with theoretical result.

2. Establishment of the Difference Scheme

First we introduce some notations. Let $Q_T = [0.L] \times I$ be a rectangular region, where $I = [0, T]$. We divide the domain Q_T into small grids by the parallel lines $x = x_j = jh$, $t = t_n = nk$ $(j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N)$, where $Jh = L$, $N = \left[\dfrac{T}{k}\right]$. Let $Q_h = \{(x, t);\ x = jh,\ t = nk,\ j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N)$. And Let ϕ_j^n $(j = 0, 1, \cdots, J;\ n = 0, 1, \cdots, N)$ denote the discrete function on the grid point (x_j, t_n).

Define

$$\Delta_+\phi_j^n = \phi_{j+1}^n - \phi_j^n, \qquad \Delta_-\phi_j^n = \phi_j^n - \phi_{j-1}^n$$
$$D_t\phi_j^n = \frac{1}{2k}(\phi_j^{n+1} - \phi_j^{n-1}), \qquad \delta^{2m}\phi_j^n = \Delta_+^m(a_{j-\frac{m}{2}}\Delta_-^m\phi_j^n)h^{-2m}$$

where $a_{j-\frac{m}{2}} = a((j - \frac{m}{2})h)$, ϕ_j^ndenote the discrete function value on the grid point (jh, nk).

We also introduce the inner product and norms appropriate to function defined on the lattice Q_h, i.e

$$(v, w) = (v, w)_h = h\sum_{j=1}^{J} v(x_j)\overline{w}(x_j) \quad \forall v, w \in c^J$$
$$||v||^2 = ||v||_h^2 = (v, v)_h = (v, v)$$

where C^J is a J-dimensionally complex space.

Corresponding to (1.1)–(1.3),we construct following leap-frog finite difference scheme

$$\begin{cases} iD_t\phi_j^n + (-1)^m\delta^{2m}\phi_j^n + \beta_j q(|\phi_j^n|^2)\phi_j^n + f_j^n\phi_j^n = g_j^n \\ \qquad\qquad (j = 1, 2, \cdots, J; n = 1, 2, \cdots, N = ([T/k]) & (2.1) \\ \phi_j^0 = U_0(jh) \qquad (j = 1, 2, \cdots, J) & (2.2) \\ \phi_{rJ+j}^n = \phi_j^n \begin{cases} j = 1, 2, \cdots, J; r = \pm1, \pm2, \cdots \\ n = 0, 1, \cdots, N \end{cases} & (2.3) \end{cases}$$

In difference scheme (2), if ϕ_j^1 $(j = 1, 2, \cdots, J)$ is given, it can be calculated level by level explicitly. And ϕ_j^1 can calculate by the scheme with same convergence order of the scheme (2), example conservation type difference scheme in [2].

3. Convergence and Stability

For convergence of the difference scheme (2) we have following:

Theorem 1. *Assume u is the solution of the periodic initial value problem* (1.1)–(1.3), $u \in c^3(I; C^{2(m+1)}(R))$, *condition (A) is true, and*

$$||u^0 - \phi^0|| + ||u^1 - \phi^1|| + \max_{0 \le r \le [T/k]} || \widetilde{g}^r ||) = O(h^{\frac{1}{2}})$$

If there a constant σ with $2^{2m} M\lambda \le \sigma < 1$, then there exist positive constants $c_s (s = 1, 2, 3)$ independent of k and h such that

$$\begin{aligned}||u^n - \phi^n|| \le & c_3(k^2 + h^2 + ||u^0 - \phi^0|| + ||u^1 - \phi^1|| \\ & + \max_{0 \le r \le T/k} || \widetilde{g}^r ||) \quad (n = 1, 2, \cdots, N = [T/k])\end{aligned} \tag{3.1}$$

for $k \le c_1, h \le c_2$. Where $\lambda = k/h^{2m}$, M defined by the condition (A), $\widetilde{g}^r_j = g(jh, rk) - g^r_j$.

Proof. Suppose $e^n = u^n - \phi^n$, τ^n is the local truncation error of the difference scheme (2.1), i.e

$$\tau^n_j = iD_t u^n_j + (-1)^m \delta^{2m} u^n_j + \beta_j q(|u^n_j|^2) u^n_j + f^n_j u^n_j - g(jh, nk) \tag{3.2}$$

then expression (3.2)–(2.1) obtain

$$\begin{aligned}\tau^n_j = & iD_t e^n_j + (-1)^m \delta^{2m} e^n_j + \beta_j [q(|u^n_j|^2) u^n_j - q(|\phi^n_j|^2)\phi^n_j] + f^n_j e^n_j - \widetilde{g}^n_j \\ & (j = 1, 2, \cdots, J; n = 1, 2, \cdots, [T/k] - 1)\end{aligned} \tag{3.3}$$

obviously, $\tau^n_j = 0(h^2 + k^2)$, expression(3.3) $\times(\overline{e}^{n+1}_j + \overline{e}^{n-1}_j)$ h, then Summation from 1 to J for j and taken imaginary part:

$$\begin{aligned}(||e^{n+1}||^2 - ||e^{n-1}||^2)/2k + I_m(-1)^m(\delta^{2m} e^n, e^{n+1} + e^{n-1}) \\ = I_m(\tau^n - f^n e^n - \beta[q(|u^n|^2)u^n - q(|\phi^n|^2)\phi^n] + \widetilde{g}^n, e^{n+1} + e^{n-1})\end{aligned} \tag{3.4}$$

where $\beta, f^n, q(|u^n|^2), q(|\phi^n|^2)$ are all net function. Because

$$I_m(\delta^{2m} e^n, e^{n+1} + e^{n-1}) = I_m(\delta^{2m} e^n, e^{n+1}) - I_m(\delta^{2m} e^{n-1}, e^n)$$

Let

$$E^n = ||e^{n-1}||^2 + ||e^n||^2 + 2k(-1)^m I_m(\delta^{2m} e^{n-1}, e^n) \tag{3.5}$$

Expression (3.4) $\times 2k$ and substitute into it by expression (3.5), we obtain

$$E^{n+1} - E^n = 2kI_m(\tau^n - f^n e^n - \beta[q(|u^n|^2)u^n - q(|\phi^n|^2)\phi^n] + \widetilde{g}^n, e^{n+1} + e^{n-1}) \tag{3.6}$$

summation from 1 to N on n for the expression (3.6), then we obtain

$$\begin{aligned} E^{N+1} \leq & |E^1| + 2k \sum_{n=1}^{N} |\tau^n - f^n e^n - \beta[q(|u^n|^2)u^n \\ & - q(|\phi^n|^2)\phi^n] + \tilde{g}^n, e^{n+1} + e^{n-1})| \end{aligned} \tag{3.7}$$

Suppose $q^*, \partial q^*/\partial$ $(\operatorname{Im} Z), \partial q^*/\partial$ $(\operatorname{Re} Z)$ are bounded provisionally, where $q^*(Z) = q(|Z|^2)$, and let $\max_{z \in c}\{|q^*(Z)|, |q*'(Z)|\} = M_2$, therefore,

$$\begin{aligned} |\beta[q(|u^n|^2)u^n - q(|\phi^n|^2)|\phi^n], e^{n+1} + e^{n-1})| \leq & |\beta\, q^*(\phi^n)e^n, e^{n+1} + e^{n-1})| \\ & + |(\beta[q^*(u^n) - q^*(\phi^n)]u^n, e^{n+1} + e^{n-1})| \\ \leq & M_4(||e^n||^2 + ||e^{n+1} + e^{n-1}||^2)/2 \end{aligned} \tag{3.8}$$

where $M_3 = \max_{(x,t) \in R \times I} |u(x,t)|$, $M_4 = M_1 M_2 (1 + M_3)$ are all positive constant.

Let $\lambda = k/h^{2m}$, prove easily, when $2^{2m} M \lambda < 1$, we have

$$\begin{aligned} 0 < & (1 - 2^{2m} M \lambda)(||e^n||^2 + ||e^{n-1}||^2) \leq E^n \\ \leq & (1 + 2^{2m} M \lambda)(||e^n||^2 + ||e^{n-1}||^2) \end{aligned} \tag{3.9}$$

by expressions (3.7), (3.8) and (3.9), we obtain

$$\begin{aligned} (1 - 2^{2m} M \lambda)(||e^{N+1}||^2 + ||e^N||^2) \leq & E^1 \\ & + 2k \sum_{n=1}^{N} [(||\tau^n|| + M_1 ||e^n|| + || \tilde{g}^n ||)||e^{n+1} + e^{n-1}|| \\ & + \frac{1}{4} M_4 (||e^n||^2 + ||e^{n+1} + e^{n-1}||^2)] \\ \leq & E^1 + k \sum_{n=1}^{N} (||\tau^n||^2 + || \tilde{g}^n ||^2) + k \sum_{n=1}^{N} \Big(M_1 + \frac{M_4}{2}\Big) ||e^n||^2 \\ & + k \sum_{n=1}^{N} (2M_1 + 4 + M_4)(||e^{n+1}||^2 + ||e^{n-1}||^2) \\ \leq & M_5[||e^0||^2 + ||e^1||^2 + \max_{0 \leq n \leq [T/k]} (||\tau^n||^2 + || \tilde{g}^n ||^2] \\ & + k \sum_{n=1}^{N} \frac{M_6}{1 - \sigma} (1 - \sigma)(||e^n||^2 + ||e^{n+1}||^2) \end{aligned}$$

where $M_5 = \max(1 + \sigma, T)$, $M_6 = \max\{2M_1 + M_4, 4M_1 + 8 + 2M_4\}$ are all positive constant.

We can obtain by Gronwall inequality

$$(1 - \sigma)(||e^{N+1}||^2 + ||e^N||^2) \leq e^{M_6 T/(1-\sigma)} M_5 [||e^0||^2 + ||e^1||^2 + \max_{0 \leq n \leq [T/k]} (||\tau^n||^2 + || \tilde{g}^2 ||^2)]$$

So

$$||e^{N+1}||^2 \leq M_7[||e^0||^2 + ||e^1||^2 + \max_{0\leq n\leq[T/k]}(||\tau^n||^2 + ||\tilde{g}^n||^2)]$$

where $M_7 = (M_5 e^{M_6 T/(1-\sigma)})(1-\sigma)$ is a positive constant, therefore, expression (3.1) is true.

Finally, we point out the supposed of boundary of $q^*, \partial q^*/\partial(ImZ),\ \partial q^*/\partial(ReZ)$ can be offset[4]. The proof is over.

Corollary 1. *Under the suppose of Theorem* 1, $c_s^{'}$ $(s=1,2,3)$ *is existent and they are a positive constant, when* $h \leq c_1^{'}$, $k \leq c_2^{'}$, *we have*

$$||\phi^n||_\infty \leq c_3^{'} \quad n = 1, 2, \cdots, [T/k].$$

With the proof of theorem 1, *we can get the following stability theorem:*

Theorem 2. *If condition* (A) *is true, and* $2^{2m}M\lambda \leq \sigma$ *is true for any positive constant* $\sigma < 1$, *and* $||\phi^1 - \tilde{\phi}^1|| \leq c_4||u_0 - \tilde{u}_0||$ *is true, then difference scheme* (2.1)–(2.3) *is stable on square norm for the initial value and right term, i.e* C_5 *is existent and it is independent of* h *and* k,

$$||\phi^n - \tilde{\phi}^n|| \leq c_5||u_0 - \tilde{u}_0|| + \max_{0\leq r\leq[T/k]}||g_r - \approx{g}^r|| \tag{3.10}$$

where $\tilde{\phi}$ *is the solution of corresponding difference problem* (2.1)–(2.3), *under the condition that initial value and right side term of problem* (1) *have disturbance* $u_0 - \tilde{u}_0$ *and* $g - \approx{g}$ *correspondly.*

Note. Difference scheme (2.1)–(2.3) is conditional convergence, we must select h and k approx satisfy $2^{2m}M\lambda \leq \sigma \leq 1$ to guarantee difference scheme (2) convergence and stability.

4. Numerical Example

Consider the following problem

$$\begin{cases} iu_t + u_{xxxx} + 6|u|^2u - 150(sin^2x)u = 0 & (4.1)\\ u(x,0) = \frac{5}{2}\sqrt{2}(1+i)\sin x & (4.2)\\ u(x+2\pi,t) = u(x,t) & (4.3)\end{cases}$$

It has a classical solution $u = u(x,t) = 5\exp(i(t+\frac{\pi}{4}))\sin x$. Let $h = \pi/10$, $k = 1/2 \times 10^{-3}$ then $\lambda = k/h^4 = (5/\pi^4) < (1/16)$, u^1 is calculated by the accurate value, when it is calculated until $N = 2000$ (i.e, $T = 1$) by the scheme is this paper, $||\,|u|^2 - |u^N|^2||_\infty \leq 10^{-4}$. The accuracy is the same with that of conservation scheme in [2],see table 1) when choose $h = \pi/10, k = 10^{-3}$, and $(1/16) < \lambda < (1/8)$ then overflow at $N = 25$. It is shown that numerical result is coincident with theoretical result.

Tabel 1. Result at $t = 1$, when $h = \pi/10$, $k = 1/2 \times 10^{-3}$, $N = 2000$

	classical solu $\|u\|^2$	Num. solu $\|u_h^N\|^2$	Error $\|u\|^2 - \|u_h^N\|^2$
$\pi/10$	2.38728757	2.38728432	0.00000324
$2\pi/10$	8.63728755	8.63727872	0.00000883
$3\pi/10$	16.36271240	16.36270065	0.00001176
$4\pi/10$	22.61271241	22.61270032	0.00001209
$5\pi/10$	25.0000000	24.99998816	0.00001184
$6\pi/10$	22.61271246	22.61270035	0.00001211
$7\pi/10$	16.36271249	16.36270041	0.00001208
$8\pi/10$	8.63728764	8.63727810	0.00000953
$9\pi/10$	2.38728762	2.38728378	0.00000384
π	0	0	0
$11\pi/10$	2.38728751	2.38728487	0.00000264
$12\pi/10$	8.63728747	8.63727932	0.00000815
$13\pi/10$	16.36271232	16.36270106	0.00001126
$14\pi/10$	22.61271236	22.61270062	0.00001173
$15\pi/10$	25.00000000	24.99998837	0.00001163
$16\pi/10$	22.61271251	22.61270072	0.00001179
$17\pi/10$	16.36271257	16.36270095	0.00001163
$18\pi/10$	8.63728772	8.63727887	0.00000885
$19\pi/10$	2.38728767	2.38728440	0.00000327
2π	0	0	0

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