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NONLINEAR GALERKIN METHOD AND CRANK-NICOLSON METHOD FOR VISCOUS INCOMPRESSIBLE FLOW*

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Abstract

In this article we discuss a new full discrete scheme for the numerical solution of the Navier-Stokes equations modeling viscous incompressible flow. This scheme consists of nonlinear Galerkin method using mixed finite elements and Crank-Nicolson method. Next, we provide the second-order convergence accuracy of numerical solution corresponding to this scheme. Compared with the usual Galerkin scheme, this scheme can save a large amount of computational time under the same convergence accuracy.

Key words: Nonlinear Galerkin method, Crank-Nicolson method, Viscous incompressible flow.

1. Introduction

Nonlinear Galerkin method is numerical method for dissipative evolution partial differential equations where the spatial discretization relies on a nonlinear manifold instead of a linear space as in the classical Galerkin method. More precisely, one considers a finite dimensional space $V_h - h$ being some parameter related to the spatial discretization$-$which is splitted as $V_h = V_H + W_h$, where $H \gg h$ and W_h is a convenient supplementary of V_H in V_h. One looks for an approximate solution u^h lying in a maniflod $\Sigma = \text{graph}\phi$ of $V_h; u^h$ takes the form $u^h = v^H + \phi(v^H)$ where v^H lies in V_H and ϕ is a mapping from V_H into W_h. The method reduces to an evolution equation for v^H, obtained by projecting the equations under consideration on the manifold $\Sigma = \text{graph}\phi$. The related works see [1, 2, 3]. In a classical Galerkin method, typically, we have $\phi = 0$.

The papers[2,3] have extended the nonlinear Galekrin method to the Navier-Stokes equations in the framework of mixed finite elements. However, the paper[2] does not deal with the case of time discretization and the paper[3] only obtains the first-order convergence accaracy for time discretization. Our purpose here is to modify the approximate scheme of [2] and consider the discretization with respect to time of the modified scheme by the Crank-Nicolson method[4]. Also, we aim to derive the full second-order convergence accuracy of numerical solution corresponding to this full discrete scheme. Finally, we compare the full discerete scheme with the usual Galerkin scheme, which shows that the new full discrete scheme is more simple than the usual Galerkin scheme.

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2. The Navier-Stokes Equations

Let Ω be a bounded domain in R^2 assumed to have a Lipschitz-continuous boundary Γ. We consider the time-dependent Navier-Stokes equations describing the flow of a viscous incompressible fluid confined in Ω:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in} \quad \Omega\times R^+ \\ &\operatorname{div} u = 0 \qquad \text{in} \ \ \Omega\times R^+ \\ &u = 0 \qquad \text{on} \ \ \Gamma\times R^+ \\ &u(0) = u_0 \qquad \text{in} \ \ \Omega \end{aligned} \tag{2.1}$$

where $u = (u_1, u_2)$ is the velocity, p is the pressure, f represents the density of body force, $\nu > 0$ is the viscosity and u_0 is the initial velocity with $\operatorname{div} u_0 = 0$.

In order to introduce a variational formulation, we set

$$Y = L^2(\Omega)^2, M = L_0^2(\Omega) = \Big\{q \in L^2(\Omega); \int_\Omega q dx = 0\Big\}$$

We denote by $(\cdot,\cdot), |\cdot|$ the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^2$ and identify $L^2(\Omega)$ with its dual space. We set

$$Au = -\nu\Delta u, \quad B(u,v) = (u\cdot\nabla)v + \frac{1}{2}(\operatorname{div} u)v$$

It is well known that A is a linear unbounded self-adjoint operator in Y with domain $D(A) = (H^2(\Omega)\cap H_0^1(\Omega))^2$ dense in Y; and A is positive closed and the inverse A^{-1} of A is compact, self-adjoint in Y. We then can define the powers A^s of A for any $s \in R$; the space $D(A^s)$ is a Hilbert space when endowed with the scalar product $(A^s\cdot, A^s\cdot)$ and norm $|A^s\cdot|$. We set

$$X = D(A^{\frac{1}{2}}) = H_0^1(\Omega)^2, \|\cdot\| = |A^{\frac{1}{2}}\cdot|, ((\cdot,\cdot)) = (A^{\frac{1}{2}}\cdot, A^{\frac{1}{2}}\cdot)$$

Next, we define the bilinear forms

$$\begin{aligned} &a(u,v) = \nu\langle Au, v\rangle \quad \forall u,v\in X \\ &D(v,q) = (q, \operatorname{div} v) \quad \forall v\in X, q\in M \end{aligned}$$

and the trilinear form

$$b(u,v,w) = \langle B(u,v), w\rangle \quad \forall u,v,w \in W$$

So, we obtain the variational formulation of problem (2.1):

For any $t > 0$, find a pair $(u(t), p(t)) \in X\times M$ such that

$$\begin{aligned} &(u_t, v) + a(u,v) + b(u,u,v) - D(v,p) = (f,v) \quad \forall v\in X \\ &D(u,q) = 0 \quad \forall q\in M \end{aligned} \tag{2.2}$$

$$u(0) = u_0$$

With the above notations and statements, the following estimates hold

$$|a(u,v)| \le \nu \| u \| \| v \|, a(u,u) = \nu \| u \|^2 \quad \forall u, v \in X \tag{2.3}$$
$$b(u,v,w) = -b(u,w,v) \tag{2.4}$$
$$|b(u,v,w)| \le c_0 \| u \| \| v \| \| w \|$$
$$|b(u,v,w)| \le \frac{c_0}{2}(|u| \| u \| |w| \| w \|)^{\frac{1}{2}} \| v \| + \frac{c_0}{2} \| u \| (|v| \| v \| |w| \| w \|)^{\frac{1}{2}}$$
$$|b(u,v,w)| \le \frac{c_0}{2}(|u| \| u \| |w| \| w \|)^{\frac{1}{2}} \| v \| + \frac{c_0}{2} \| w \| (|u| \| u \| |v| \| v \|)^{\frac{1}{2}}$$
$$\forall u, v, w \in X$$
$$|b(u,v,w)| \le \frac{c_0}{2}(|u||Au|)^{\frac{1}{2}} \| v \| |w| + \frac{c_0}{2}(\| u \| |Au|)^{\frac{1}{2}}(|v| \| v \|)^{\frac{1}{2}}|w|$$
$$|b(u,v,w) \le \frac{c_0}{2}(|u| \| u \| \| v \| |Av|)^{\frac{1}{2}}|w| + \frac{c_0}{2} \| u \| (|v||Av|)^{\frac{1}{2}}|w|$$
$$|b(u,v,w)| \le \frac{c_0}{2}|u|(\| v \| |Av||w| \| w \|)^{\frac{1}{2}} + \frac{c_0}{2}|u|(|v||Av|)^{\frac{1}{2}} \| w \|$$
$$\forall u, v \in D(A), w \in X$$
$$\| B(u,v) \| \le c_0|Au||Av| \quad \forall u, v \in D(A) \tag{2.5}$$
$$|u| \le c_0 \| u \| \quad \forall u \in X \tag{2.6}$$
$$|q| \le \beta_0^{-1} \sup_{v \in X} \frac{D(v,q)}{\| v \|} \quad \forall q \in M \tag{2.7}$$
$$|D(v,q)| \le c_0 \| v \| |q| \quad \forall v \in X, q \in M$$

According to the estimates (2.3)—(2.7), we can prove the following existence and regularity results:

Theorem 2.1. *If $u_0 \in D(A)$ with $\operatorname{div} u_0 = 0$ and $f \in L^\infty(R^+; X)$ then problem (2.2) admits a unique solution (u,p) such that*

$$|u_t(t)|^2 + |Au(t)|^2 \le M_1^2, \int_0^t \| Au \|^2 \, ds \le M_1^2\Big(1 + \int_0^t \| f \|^2 \, ds\Big) \tag{2.8}$$
$$\int_0^t (\| u_s \|^2 + \| p \|_2^2) ds \le M_1^2\Big(1 + \int_0^t \| f \|^2 \, ds\Big)$$

Moreover, if $u_0 \in D(A^2)$ with $\operatorname{div} u_0 = 0$, $f \in L^\infty(R^+; D(A))$, $f_t \in L^\infty(R^+; X)$ and $f_{tt} \in L^\infty(R^+; X')$ then

$$\int_0^t |Au_s|^2 ds \le M_1^2\Big(1 + \int_0^t |Af|^2 ds\Big) \tag{2.9}$$
$$\int_0^t \| u_{ss} \|^2 \, ds \le M_1^2\Big(1 + \int_0^t \| f_s \|^2 \, ds\Big)$$
$$\int_0^t (|p_{ss}|^2 + \| u_{sss} \|_{-1}^2) ds \le M_1^2\Big(1 + \int_0^t (\| f_s \|^2 + \| f_{ss} \|_{-1}^2) ds$$
$$|u_{tt}(t)|^2 \le M_1^2$$

where $M_1 > 0$ *is a constant,* $\|\cdot\|_2$ *denotes the norm on* $H^2(\Omega)$ *and* $\|\cdot\|_{-1}$ *denotes the norm on* X' *defined by*

$$\| f \|_{-1} = \sup_{v\in X} \frac{(f,v)}{\| v \|}$$

This proof can refer to [1, 3, 5, 6].

3. Galerkin Scheme

From now on, h will be a real positive parameter tending to 0. We introduce two finite-dimensional subspaces X_h and M_h of X and M respectively and we define the L^2-orthogonal projection operators $P_h : Y \to X_h$ and $\rho_h : M \to M_h$ as follows

$$(P_h v, v_h) = (v, v_h) \quad \forall v_h \in X_h, \quad \forall v \in Y$$
$$(\rho_h q, q_h) = (q, q_h) \quad \forall q_h \in M_h, \quad \forall q \in M$$

We assume that the couple (X_h, M_h) satisfies the following approximation properties:

For each $v \in D(A^{3/2})$ and $q \in H^2(\Omega) \cap M$, there exist approximations $I_h v \in X_h$ and $J_h q \in M_h$ such that

$$\| v - I_h v \| \le c h^{1+i} |A^{1+\frac{i}{2}} v| \tag{3.1}$$
$$|q - J_h q| \le c h^{1+i} \| q \|_{1+i} \quad i = 0, 1$$

together with the inverse inequality

$$c_1 h \| v \| \le |v| \quad \forall v \in X_h \tag{3.2}$$

and the so-called inf-sup condition

$$|q_h| \le \beta^{-1} \sup_{v\in X_h} \frac{D(v, q_h)}{\| v \|} \quad \forall q_h \in M_h \tag{3.3}$$

The following properties which are classical consequences of (3.1)–(3.3)[2,5,7] will be very useful

$$\| P_h v \| \le c \| v \| \quad \forall v \in X \tag{3.4}$$
$$|v - P_h v| \le c h^2 |Av| \quad \forall v \in D(A) \tag{3.5}$$
$$\| v - P_h v \| \le c h^2 \| Av \| \quad \forall v \in D(A^{3/2})$$
$$|q - \rho_h q| \le c h^2 \| q \|_2 \quad \forall q \in H^2(\Omega) \cap M \tag{3.6}$$

Referring to [5], we give a example of subspaces X_h and M_h such that the assumptions (3.1)–(3.3) are satisfied. Let Ω be a polyhedral domain and let $\{\tau_h\}, h > 0$, be a uniformly regular family of triangulation of Ω made of the closed triangle elements K with the diameters boundeed by h, vertices a_i, mid-points a_{ij} of the sides $[a_i, a_j]$ and barycenter a_{123}. Then the basis functions of this element K are

$$\varphi_i = \lambda_i(2\lambda_i - 1) + 3\lambda_1\lambda_2\lambda_3, \quad i = 1, 2, 3$$

$$\varphi_{ij} = 4\lambda_i\lambda_j - 12\lambda_1\lambda_2\lambda_3, \quad 1 \le i < j \le 3$$
$$\varphi_{123} = 27\lambda_1\lambda_2\lambda_3$$
$$\psi_1 = 1, \psi_2 = x_1 - x_1^0, \psi_3 = x_2 - x_2^0$$

where λ_1, λ_2 and λ_3 are the barycentric coordinates corresponding to the vertices a_1, a_2 and $a_3, a_{123} = (x_1^0, x_2^0), x = (x_1, x_2)$. We write

$$P_K = \text{span}\,\{\varphi_1, \varphi_2, \varphi_3, \varphi_{12}, \varphi_{13}, \varphi_{23}, \varphi_{123}\}$$

Then X_h and M_h are defined by

$$S_h = \{w_h \in C(\bar{\Omega}); w_h|_K \in P_K, \forall K \in \tau_h\}, X_h = S_h^2 \cap X$$
$$O_h = \{q_h \in L^2(\Omega); q_h|_K \in \text{span}\{\psi_1, \psi_2, \psi_3\} \forall K \in \tau_h\}, M_h = O_h \cap M$$

The Galerkin method of (2.2) based on (X_h, M_h) reads:
For any $t > 0$, find $(u_h(t), p_h(t)) \in X_h \times M_h$ such that

$$\begin{aligned} &(u_{h,t}, v) + a(u_h, v) + b(u_h, u_h, v) - D(v, p_h) = (f, v) \quad \forall v \in X_h \\ &D(u_h, q) = 0 \quad \forall q \in M_h \\ &u_h(0) = P_h u_0 \end{aligned} \tag{3.7}$$

The following error estimates are the usual results

Theorem 3.1. *Under the assumptions of Theorem 2.1 and* (3.1)–(3.3), *the following error estimates hold:*

$$|u(t) - u_h(t)|^2 + \int_0^t \| u - u_h \|^2 \, ds \le c(t) h^4 \tag{3.8}$$
$$\Big| \int_0^t (p - p_h) ds \Big|^2 \le c(t) h^4$$

where the constant $c(t)$ is continuous with respect to t.

This proof can refere to [8].

Next, we consider the discretization with respect to time of the semidiscrete Galerkin approximate problem (3.7) by the Crank-Nicolson method. Let Δt denote the timestep. Then the Galerkin Scheme consisting of the Galerkin method and Crank-Nicolson method is defined as follows

Galerkin Scheme (G Scheme)

$$u^0 = P_h u_0 \tag{3.9}$$
$$\Big(\frac{u^n - u^{n-1}}{\Delta t}, v\Big) + a(\hat{u}^n, v) + b(\hat{u}^n, \hat{u}^n, v) - D(v, \hat{p}_h^n) = (\hat{f}(t_n), v) \quad \forall v \in X_h \tag{3.10}$$
$$D(\hat{u}^n, q) = 0 \quad \forall q \in M_h \tag{3.11}$$

where (u^n, p_h^n) is expected to be the approximation $(u_h(t_n), p_h(t_n))$ and

$$\hat{u}^n = \frac{1}{2}(u^n + u^{n-1}), \quad \hat{f}(t_n) = \frac{1}{2}(f(t_n) + f(t_{n-1})).$$

$$\hat{u}^0 = u^0 \quad , \hat{f}(0) = f(0)$$

Now, we construct the approximat solution $(u_\Delta(t), p_\Delta(t))$ of $(u(t), p(t))$ as follows:

$$\Big(\frac{du_\Delta}{dt}(t), v\Big) + a(u_\Delta(t), v) + b(u_\Delta, u_\Delta, v) - D(v, p_\Delta) = (f(t), v) \quad \forall v \in X_h \tag{3.12}$$

$$D(u_\Delta, q) = 0 \quad \forall q \in M_h \tag{3.13}$$

$$u_\Delta(t_{n-1}) = u^{n-1} \tag{3.14}$$

for any $t \in [t_{n-1}, t_n)$. So, we can obtain the error estimates of (u_Δ, p_Δ) produced by G Scheme.

Theorem 3.2. *Under the assumptions of Theorem* 3.1, *the numerical solution* $(u_\Delta(t), p_\Delta(t))$ *satisfies*

$$|u(t) - u_\Delta(t)|^2 + \int_0^t \| u - u_\Delta \|^2 \, ds \le c(t_m)(h^4 + \Delta t^4) \tag{3.5}$$

$$\Big|\int_0^t (p - p_\Delta) ds\Big|^2 \le c(t_m)(h^4 + \Delta t^4) \tag{3.16}$$

for any $t \in [0, t_m)$, *when* Δt *satisfies*

$$256 c_0^6 M_2^4 \Delta t < 1, \frac{\Delta t^2}{c_1 h} \textit{is bounded.}$$

$M_2 > 0$ *is constant given in Section* 4.

This proof is very similar to ones of the error estimates of numerical solution produced by NG Scheme, it can be omitted.

4. Nonlinear Galerkin Scheme

In this section, we are given two parameters h and H, tending to 0, with $H > h > 0$. We consider three spaces X_h, X_H and M_h with $X_H \subset X_h$ and we write

$$X_h = X_H + X_h^H, \quad \text{with } X_h^H = (I - P_H) X_h$$

Note that X_H and X_h^H are orthogonal with respect to the scalar product $(\cdot,\cdot)$. In the applications, X_h and M_h correspond to spaces associated to a fine grid, while X_H corresponds to space associated to a coarse grid. The following properties of X_H and X_h^H will be often used.

Lemma 4.1. *Assume that* (3.1)–(3.3) *hold. Then*

i) *We have*

$$|w| \le c_2 H \| w \| \quad \forall w \in X_h^H \tag{4.1}$$

ii) *There exists* $0 < \gamma < 1$ *such that*

$$|((v, w))| \le (1 - \gamma) \| v \| \| w \| \tag{4.2}$$

$$(v, w) = 0 \quad \forall v \in X_H, w \in X_h^H$$

which implies readily that

$$\gamma(\| v \|^2 + \| w \|^2) \leq \| v + w \|^2 \tag{4.3}$$
$$|v|^2 + |w|^2 = |v + w|^2 \quad \forall v \in X_H, w \in X_h^H$$

The nonlinear Galerkin method associated to (X_h, X_H, X_h^H, M_h) *consists in looking for an approximate solution* (u^h, p^h) *of the form*

$$u^h = y + z, \textit{with} \quad y \in X_H, z \in X_h^H, p^h \in M_h$$

such that, for any $t > 0, u^h = y + z$ *and* p^h *satisfy*

$$\begin{aligned}&(y_t, v) + (z_t, w) + a(y+z, v+w) + b(y, y, v+w) + b(y, z, v) + b(z, y, v)\\&- D(v+w, p^h) = (f, v+w) \quad \forall v \in X_M, w \in X_h^H\\&D(y+z, q) = 0 \quad \forall q \in M_h\\&y(0) = P_H u_0, z(0) = (P_h - P_H)u_0\end{aligned} \tag{4.4}$$

Here, (4.4) *is the modification of nonlinear Galerkin approximation in* [2]*, where the term* (z_t, w) *was neglected.*

Now, we consider the time discretization of nonlinear Galerkin approximation (4.4) by the Crank-Nicolson method.

NONLINEAR GALERKIN SCHEME (NG Scheme)

$$y^0 = y(0), z^0 = z(0), p^0 = p^h(0) \tag{4.5}$$

$$\begin{aligned}&\frac{1}{\Delta t}(y^n - y^{n-1}, v) + \frac{1}{\Delta}(z^n - z^{n-1}, w) + a(\hat{y}^n + \hat{z}^n, v+w)\\&\quad + b(\hat{y}^n, \hat{z}^n, v+w) + b(\hat{y}^n, \hat{z}^n, v) + b(\hat{z}^n, \hat{y}^n, v)\\&\quad - D(v+w), \hat{p}^n) = (\hat{f}(t_n), v+w) \quad \forall v \in X_H, w \in X_h^H\end{aligned} \tag{4.6}$$

$$D(\hat{y}^n + \hat{z}^n, q) = 0 \quad \forall q \in M_n \tag{4.7}$$

where $p^h(0)$ is determined by (4.4).

By NG Scheme, we can construct the numerical approximation $(u_\Delta(t), p_\Delta(t))$ of the solution $(u(t), p(t))$ of problem (2.1). Here, $u_\Delta(t)$ and $p_\Delta(t)$ are defined by

$$u_\Delta(t) = y_\Delta(t) + z_\Delta(t) \tag{4.8}$$

$$\begin{aligned}&\Big(\frac{d}{dt}y_\Delta, v\Big) + \Big(\frac{d}{dt}z_\Delta, w\Big) + a(y_\Delta + z_\Delta, v+w) + b(y_\Delta, y_\Delta, v+w)\\&\quad + b(y_\Delta, y_\Delta, v) + b(z_\Delta, y_\Delta, v) - D(v+w), p_\Delta) = (f, v+w)\\&\quad\quad \forall v \in X_H, w \in X_h^H\end{aligned} \tag{4.9}$$

$$D(u_\Delta, q) = 0 \quad \forall q \in M_h \tag{4.10}$$

$$y_\Delta(t_{n-1}) = y^{n-1}, z_\Delta(t_{n-1}) = z^{n-1} \tag{4.11}$$

for any $t \in [t_{n-1}, t_n)$.

For the numerical solutions (u_h, p_h) and (u^h, p^h), we can obtain the similar regularity results to ones of (u, p).

Theorem 4.2. *If $u_0 \in D(A)$ with $\operatorname{div} u_0 = 0, f \in L^\infty(R^+;X)$, then*

$$|u_{h,t}(t)|^2 + |Au_h(t)|^2 \le M_2^2 \tag{4.12}$$
$$\int_0^t (|Au_h|^2 + \| u_{h,s} \|^2)ds \le M_2^2\Big(1 + \int_0^t \| f \|^2 \, ds\Big)$$
$$|u_t^h(t)|^2 + |Au^h(t)|^2 \le M_2^2$$
$$\int_0^t (|Au^h|^2 + \| u_s^h \|^2)ds \le M_2^2\Big(1 + \int_0^t \| f \|^2 \, ds\Big)$$

Moreover, if $u_0 \in D(A^2)$ with $\operatorname{div} u_0 = 0, f \in L^\infty(R^+;D(A))$, $f_t \in L^\infty(R^+;X)$ and $f_{tt} \in L^\infty(R^+;X')$ then

$$\| u_{h,t}(t) \|^2 + \| u_t^h(t) \|^2 \le M_2^2 \tag{4.13}$$
$$|u_{h,tt}(t)|^2 + |u_{tt}^h(t)|^2 \le M_2^2$$
$$\int_0^t (\| u_{h,ss} \|^2 + \| u_{ss}^h \|^2 + \| p_{h,ss}|^2 + |p_{ss}^h|^2)ds$$
$$\le M_2^2\Big(1 + \int_0^t (\| f_s \|^2 + \| f_{ss} \|_{-1}^2)ds)\Big) \tag{4.14}$$
$$\int_0^t (\| u_{h,sss} \|_{-1}^2 + \| y_{sss} \|_{-1}^2 + \| z_{sss} \|_{-1}^2)ds$$
$$\le M_2^2\Big(1 + \int_0^t (\| f_s \|^2 + \| f_{ss} \|_{-1}^2)ds\Big)$$

where $u_{h,t} = \dfrac{\partial u_h}{\partial t}$, $u_{h,tt} = \dfrac{\partial}{\partial t} u_{h,t}$ and so on, $M_2 > 0$ is constant.

5. Error Estimates: Semidiscrete Case

In this section, we aim to derive error estimates for the nonlinear Galerkin approximate problem (4.4) in terms of the two parameters H and h.

Let us write

$$u_h = v_H + w_h, v_H = P_H u_h, w_h = (I - P_H)u_h$$
$$e = v_H - y, \varepsilon = w_h - z, \eta = p_h - p^h$$

Then we derive from (3.7) and (4.4) that

$$\Big(\frac{d}{dt}(e+\varepsilon), v + w\Big) + a(e+\varepsilon, v+w) + b(e+\varepsilon, u_h, v+w)$$
$$+ b(u^h, e+\varepsilon, v+w) + b(y,z,w) + b(z,y,w)$$
$$+ b(z,z,v+w) - D(v+w,\eta) = 0 \quad \forall v \in X_H, w \in X_h^H \tag{5.1}$$
$$D(e+\varepsilon, q) = 0 \quad \forall q \in M_h \tag{5.2}$$
$$e(0) = \varepsilon(0) = 0 \tag{5.3}$$

This gives, by taking $v = e, w = \varepsilon$ in (5.1), $q = \eta$ in (5.2) and using (2.3)–(2.4),

$$\frac{1}{2}\frac{d}{dt}|e+\varepsilon|^2 + \nu \| e+\varepsilon \|^2 + b(e+\varepsilon, u_h, e+\varepsilon)$$

$$+ b(y,z,\varepsilon) + b(z,y,\varepsilon) + b(z,z,e+\varepsilon) = 0 \tag{5.4}$$

We aim to derive bounds for the trilinear terms in (5.4). Using (2.4), (2.6) and (4.1), we have

$$\begin{aligned}|b(e+\varepsilon, u_h, e+\varepsilon)| \leq& \frac{c_0}{2}|e+\varepsilon| \parallel e+\varepsilon \parallel\parallel u_h \parallel + \frac{c_0}{2}|e+\varepsilon|^{\frac{1}{2}} \parallel e+\varepsilon \parallel^{3/2} |u_h|^{\frac{1}{2}} \parallel u_h \parallel^{\frac{1}{2}} \\ \leq& \frac{3\nu}{16} \parallel e+\varepsilon \parallel^2 + \frac{c_0^2}{\nu} \parallel u_h \parallel^2 (1+4c_0^2\nu^{-2}|u_h|^2)|e+\varepsilon|^2 \end{aligned} \tag{5.5}$$

$$\begin{aligned}|b(z,z,e+\varepsilon)| \leq& c_0 \parallel z \parallel\parallel z \parallel\parallel e+\varepsilon \parallel \leq c_0 c_2^3 H^3 |Az| \parallel Az \parallel\parallel e+\varepsilon \parallel \\ \leq& \frac{\nu}{16} \parallel e+\varepsilon \parallel^2 + \frac{4}{\nu} c_0^2 c_2^6 H^6 |Az|^2 \parallel Az \parallel^2 \end{aligned} \tag{5.6}$$

$$|b(y,z,\varepsilon) + b(z,y,\varepsilon)| \leq 2c_0^2 \parallel z \parallel |Ay||\varepsilon| \leq \frac{\nu}{16} \parallel \varepsilon \parallel^2 + \frac{16}{\nu} c_0^2 c_2^6 H^6 |Ay|^2 \parallel Az \parallel^2 \tag{5.7}$$

Thanks to (4.2)–(4.3), we have

$$\begin{aligned}&|Ay|^2 + |Az|^2 = |Au^h|^2 \\ &\gamma \parallel Az \parallel^2 \leq \parallel Au^h \parallel^2, \gamma \parallel \varepsilon \parallel^2 \leq \parallel e+\varepsilon \parallel^2\end{aligned} \tag{5.8}$$

So, (5.4) and (5.5)–(5.8) give

$$\frac{d}{dt}|e+\varepsilon|^2 + \nu \parallel e+\varepsilon \parallel^2 \leq \frac{2}{\nu}(16+4\gamma)\gamma^{-2} c_0^2 c_2^6 H^6 |Au^h| \parallel Au^h \parallel + g(t)|e+\varepsilon|^2 \tag{5.9}$$

where $g(t) = \frac{2}{\nu} c_0^2 (1 + 4c_0^2 \nu^{-2} |u_h(t)|^2) \parallel u_h(t) \parallel^2$.

Moreover, thanks to Theorem 4.2, we have

$$|u^h(t)|^2 \leq M_2^2, \int_0^t \parallel Au^h \parallel^2 ds \leq M_2^2 \Big(1 + \int_0^t \parallel f \parallel^2 ds\Big) \tag{5.10}$$

So, by integrating (5.9) and using (5.3), we obtain

$$\begin{aligned}&|e(t)+\varepsilon(t)|^2 + \nu \int_0^t \parallel e+\varepsilon \parallel^2 ds \\ &\leq \frac{2}{\nu}(16+4\gamma)\gamma^{-2} c_0^2 c_2^6 M_2^4 \exp\Big(\int_0^t g(s)ds\Big)\Big(1+\int_0^t \parallel f \parallel^2 ds\Big) H^6\end{aligned} \tag{5.11}$$

Hence, by Theorem 3.1 and the triangle inequality, we obtain readily the following error estimates.

Theorem 5.1. *If the assumptions of Theorem* 3.1 *hold, then* (u^h, p^h) *satisfies the following error estimates:*

$$|u(t) - u^h(t)|^2 + \int_0^t \parallel u - u^h \parallel^2 ds \leq c(t)(H^6 + h^4) \tag{5.12}$$

$$\Big|\int_0^t (p - p^h) ds\Big|^2 \leq c(t)(H^6 + h^4) \tag{5.13}$$

Proof (5.11) is readily obtained by (5.12) and Theorem 3.1.

We aim to prove (5.13). Due to (5.1)–(5.3), we obtain

$$\begin{aligned} D\Big(v+w,\int_0^t \eta ds\Big) =&(e(t)+\varepsilon(t),v+w)+a\Big(\int_0^t (e+\varepsilon)ds, v+w\Big) \\ &+\int_0^t b(e+\varepsilon,u_h,v+w)ds+\int_0^t b(u^h,e+\varepsilon,v+w)ds \\ &+\int_0^t [b(y,z,w)+b(z,y,w)+b(z,z,v+w)]ds \end{aligned} \tag{5.14}$$

But, thanks to (2.3)–(2.7) and (4.1)–(4.3), we have

$$\begin{aligned} \Big|a\Big(\int_0^t (e+\varepsilon)ds, v+w\Big)\Big| \le&\nu\Big\|\int_0^t (e+\varepsilon)ds\Big\| \parallel v+w\parallel \\ \le&\nu t^{\frac{1}{2}}\Big(\int_0^t \parallel e+\varepsilon\parallel^2 ds\Big)^{1/2} \parallel v+w\parallel \end{aligned} \tag{5.15}$$

$$\begin{aligned} \Big|\int_0^t b(e+\varepsilon,u_h,v+w)ds+\int_0^t b(u^h,e+\varepsilon,v+w)ds\Big| \\ \le& c_0\int_0^t (\parallel u_h\parallel+\parallel u^h\parallel)\parallel e+\varepsilon\parallel ds \parallel v+w\parallel \\ \le& 2c_0^2 M_1 t^{\frac{1}{2}}\Big(\int_0^t \parallel e+\varepsilon\parallel^2 ds\Big)^{1/2} \parallel v+w\parallel \end{aligned}$$

$$\Big|\int_0^t [b(y,z,w)+b(z,y,w)+b(z,z,v+w))ds\Big| \tag{5.16}$$

$$\begin{aligned} &\le \int_0^t (2c_0^2|Ay|\parallel z\parallel |w|+c_0\parallel z\parallel^2\parallel v+w\parallel)ds \\ &\le \Big(2c_0^2c_2^3H^3\gamma^{-\frac{1}{2}}\int_0^t |Ay|\parallel Az\parallel ds+c_0c_3^2H^3\int_0^t |Az|\parallel Az\parallel ds\Big)\parallel v+w\parallel \\ &\le (2c_0^2c_2^3\gamma^{-1}M_2+c_0c_2^3\gamma^{-\frac{1}{2}}M_2)t^{\frac{1}{2}}H^3\Big(\int_0^t \parallel Au^h\parallel^2 ds\Big)^{1/2}\parallel v+w\parallel \end{aligned} \tag{5.17}$$

So, according to (3.3) and Theorem 4.2, we imply from (5.12) that

$$\Big|\int_0^t \eta ds\Big|^2 \le c(t)H^6 \tag{5.18}$$

Hence, by the triangle inequality and Theorem 3.1, we obtain

$$\Big|\int_0^t (p-p^h)ds\Big|^2 \le c(t)(h^4+H^6) \tag{5.19}$$

namely (5.13) holds.

The proof ends.

6. Error Estimates: Full Discrete Case

In this section, we aim to derive the error estimates for the numerical solution $(u_\triangle(t), p_\triangle(t))$ obtained by NG Scheme.

Theorem 6.1. *If $u_0 \in D(A^2)$, with $\operatorname{div} u_0 = 0$, $f \in L^\infty(R^+; D(A))$, $f_t \in L^\infty(R^+; X)$, $f_{tt} \in L^\infty(R^+; X')$ and (3.1)–(3.3) hold, then the sequence $(y^n + z^n, p^n)$ obtained by NG Scheme satisfies the following error estimates*

$$|u^h(t_m) - y^m - z^m|^2 + \sum_{n=1}^{m} \| \hat{u}^h(t_n) - y^n - z^n \|^2 \Delta t \le c(t_m)\Delta t^4 \tag{6.1}$$

$$\Big| \sum_{n=1}^{m} (p^h(t_n) - p^n)\Delta t|^2 \le c(t_m)\Delta t^4 \tag{6.2}$$

when Δt satisfies

$$256 c_0^6 (\gamma\nu)^{-3}(1 + (\gamma\nu)^{-2}) M_2^4 \Delta t < 1, \frac{\Delta t^2}{c_1 h} \quad \textit{is bounded}$$

Proof. Integrating (4.4) for $t \in [t_{n-1}, t_n)$, we obtain

$$\begin{aligned}
&\frac{1}{\Delta t}(y(t_n) - y(t_{n-1}), v) + \frac{1}{\Delta t}(z(t_n) - z(t_{n-1}), w) \\
&\qquad + \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} a(y(t) + z(t), v + w)dt - \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} D(v + w, p^h(t))dt \\
&\qquad + \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} (b(y(t), y(t), v + w) + b(y(t), z(t), v) + b(z(t), y(t), v)dt \\
&\quad = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} (f(t), v + w)dt \quad \forall v \in X_H, w \in X_H^H
\end{aligned} \tag{6.3}$$

$$D(\hat{y}(t_n) + \hat{z}(t_n), q) = 0 \quad \forall q \in M_h \tag{6.4}$$

$$y(0) = y^0, z(0) = z^0 \tag{6.5}$$

We set

$$e^n = y(t_n) - y^n, \varepsilon^n = z(t_n) - z^n, \eta^n = p^h(t_n) - p^n$$

Then (6.3)–(6.5) and (4.5)–(4.7) give

$$\begin{aligned}
&\frac{1}{\Delta t}(e^n + \varepsilon^n - e^{n-1} - \varepsilon^{n-1}, v + w) + a(\hat{e}^n + \hat{\varepsilon}^n, v + w) \\
&\qquad + b(\hat{y}(t_n), \hat{y}(t_n), v + w) - b(\hat{y}^n, \hat{y}^n, v + w) + b(\hat{y}(t_n), \hat{z}(t_n), v) - b(\hat{y}^n, \hat{z}^n, v) \\
&\qquad + b(\hat{z}(t_n), \hat{y}, v) - b(\hat{z}^n, \hat{y}^n, v) - D(v + w, \hat{\eta}^n) \\
&\quad = (e_n, v) + (\varepsilon_n, w) \quad \forall v \in X_H, w \in X_h^H
\end{aligned} \tag{6.6}$$

$$D(\hat{e}_n + \hat{\varepsilon}_n, q) = 0 \quad \forall q \in M_h \tag{6.7}$$

$$e^0 = 0, \varepsilon^0 = 0 \tag{6.8}$$

where

$$\begin{aligned}
(e_n, v) + (\varepsilon_n, w) &= a(\hat{y}(t_n) + \hat{z}(t_n), v + w) - \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} a(y(t) + z(t), v + w)dt \\
&\quad + b(\hat{y}(t_n), \hat{y}(t_n), v + w) - \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} b(y(t), y(t), v + w)dt + b(\hat{y}(t_n, \hat{z}(t_n), v)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} b(y(t),z(t),v)dt + b(\hat{z}(t_n),\hat{y}(t_n),v) - \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} b(z(t),y(t),v)dt \\
&+ (\hat{f}(t_n),v+w) - \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n}(f(t),v+w)dt - D(v+w,\hat{p}^h(t_n)) \\
&+ \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} D(v+w,p^h(t))dt
\end{aligned}
\tag{6.9}
$$

By using the formula:for any smooth function $g(t)$,

$$
\hat{g}(t_n) - \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} g(t)dt = \frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}\beta_n(t)g_{tt}(t)dt
$$

where $\beta_n(t) = (t_n - t)(t - t_{n-1}) \le \Delta t^2$, we can write (6.9) as follows:

$$
\begin{aligned}
(e_n,v) =&\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}\beta_n(t)a(y_{tt}+z_{tt},v)dt + \frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}\beta_n(t)((f_{tt},v) - D(v,p_{tt}^h))dt \\
&+ \frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}\beta_n(t)(b_{tt}(y,z,v)+b_{tt}(z,y,v))dt + \frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}\beta_n(t)b_{tt}(y,y,v)dt \\
&- \frac{1}{4}b(y(t_n)-y(t_{n-1}),y(t_n)-y(t_{n-1}),v) \\
&- \frac{1}{4}b(y(t_n)-y(t_{n-1}),z(t_n)-z(t_{n-1}),v) \\
&- \frac{1}{4}b(z(t_n)-z(t_{n-1}),y(t_n)-y(t_{n-1}),v)
\end{aligned}
\tag{6.10}
$$

$$
\begin{aligned}
(\varepsilon_n,w) =&\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}\beta_n(t)a(y_{tt}+z_{tt},w)dt + \frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}\beta_n(t)((f_{tt},w) - D(w,p_{tt}^h))dt \\
&+ \frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}\beta_n(t)b_{tt}(y,y,w)dt - \frac{1}{4}b(y(t_n)-y(t_{n-1}),y(t_n)-y(t_{n-1}),w)
\end{aligned}
\tag{6.11}
$$

Here, the following formula holds

$$
b_{tt}(y(t),z(t),v) = b(y_{tt}(t),z(t),v) + b(y(t),z_{tt}(t),v) + 2b(y_t(t),z_t(t),v)
$$

Now, we aim to estimate $\| e_n \|_{-1}$ and $\| \varepsilon_n \|_{-1}$. Thanks to (2.3)–(2.7), (4.3) and Theorem 4.2, we have

$$
\begin{aligned}
\Big|\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}\beta_n(t)a(y_{tt}+z_{tt},v)dt\Big| &\le \frac{1}{2}\Delta t\int_{t_{n-1}}^{t_n}|a(y_{ttt}+z_{tt},v)|dt \\
&\le\frac{1}{2}\Delta t\int_{t_{n-1}}^{t_n}\| y_{tt}+z_{tt}\| dt \| v \| \le \frac{1}{2}\Delta t^{3/2}\int_{t_{n-1}}^{t_n}\| u_{tt}^h \|^2 dt)^{1/2}\| v \|
\end{aligned}
\tag{6.12}
$$

$$
\begin{aligned}
&\Big|\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}\beta_n(t)((f_{tt},v) - D(v,p_{tt}^h))dt\Big| \\
&\qquad \le\Delta t^{3/2}\Big(\int_{t_{n-1}}^{t_n}(\| f_{tt}\|_{-1}^2 + c_0^2|p_{tt}^h|^2)dt\Big)^{1/2}\| v \|
\end{aligned}
\tag{6.13}
$$

$$\Big|\frac{1}{2\Delta t}\int_{t_{n-1}}^{t_n}\beta_n(t)(b_{tt}(y,y,v)+b_{tt}(y,z,v)+b_{tt}(z,y,v))dt\Big|$$
$$\le c_0\Delta t\int_{t_{n-1}}^{t_n}(\| y_{tt}\|(\| y\|+\| z\|)+\| z_{tt}\|\| y\|)dt\| v\|$$
$$+c_0\Delta t\int_{t_{n-1}}^{t_n}(\| y_t\|^2+2\| y_t\|\| z_t\|)dt\| v\|$$
$$\le 4c_0\Delta t\gamma^{-1}\sup_{t\in R^+}(\| u^h(t)\|+\| u_t^h(t)\|)\int_{t_{n-1}}^{t_n}(\| u_{tt}^h\|+\| u_t^h\|)dt\| v\|$$
$$\le 12c_0\gamma^{-1}\Delta t^{3/2}M_2\Big(\int_{t_{n-1}}^{t_n}(\| u_{tt}^h\|^2+\| u_t^h\|^2)dt\Big)^{1/2}\| v\| \tag{6.14}$$

$$\Big|\frac{1}{4}b(y(t_n)-y(t_{n-1}),y(t_n)-y(t_{n-1}),v)\Big|\le\frac{c_0}{4}\| y(t_n)-y(t_{n-1})\|^2\| v\|$$
$$=\frac{c_0}{4}\Big\|\frac{d}{dt}y(\theta)\Big\|\Big\|\int_{t_{n-1}}^{t_n}y_t dt\Big\|\Delta t\| v\|\quad \theta\in(t_{n-1},t_n)$$
$$\le\frac{c_0}{4}\gamma^{-1}\Delta t\sup_{t\in R^+}\| u^h(t)\|\int_{t_{n-1}}^{t_n}\| u_t^h\| dt\| v\|_{\Delta t}$$
$$\le\frac{c_0}{4}\gamma^{-1}M_1\Delta t^{3/2}\Big(\int_{t_{n-1}}^{t_n}\| u_t^h\|^2 dt\Big)^{1/2}\| v\| \tag{6.15}$$

$$\Big|\frac{1}{4}b(y(t_n)-y(t_{n-1}),z(t_n)-z(t_{n-1}),v)\Big|$$
$$\le\frac{c_0}{4}\gamma^{-1}M_2\Delta t^{3/2}\Big(\int_{t_{n-1}}^{t_n}\| u_t^h\|^2 dt\Big)^{1/2}\| v\| \tag{6.16}$$

$$\Big|\frac{1}{4}b(z(t_n)-z(t_{n-1}),y(t_n)-y(t_{n-1}),v)\Big|$$
$$\le\frac{c_0}{4}\gamma^{-1}M_2\Delta t^{3/2}\Big(\int_{t_{n-1}}^{t_n}\| u_t^h\|^2 dt\Big)^{1/2}\| v\| \tag{6.17}$$

So, (6.10) and (6.12)–(6.17) give

$$\| e_n\|_{-1}\le\sup_{v\in X_H}\frac{(e_n,v)}{\| v\|}\le(2+14c_0\gamma^{-1}M_1)\Delta t^{3/2}\Big(\int_{t_{n-1}}^{t_n}(\| u_t^h\|^2$$
$$+\| u_{tt}^h\|^2+\| f_{tt}\|_{-1}^2+c_0^2\| p_{tt}^h\|^2)dt\Big)^{\frac{1}{2}} \tag{6.18}$$

Similarly, we can imply

$$\|\varepsilon_n\|\le(2+14c_0\gamma^{-1}M_1)\Delta t^{3/2}\Big(\int_{t_{n-1}}^{t_n}(\| u_t^h\|^2+\| u_{tt}^h\|^2+\| f_{tt}\|_{-1}^2+c_0^2|p_{tt}^h|^2)dt\Big)^{\frac{1}{2}} \tag{6.19}$$

Hence, we have

$$\sum_{n=1}^m(\| e_n\|_{-1}^2+\|\varepsilon_n\|_{-1}^2)\Delta t\le c_3\Delta t^4\int_0^{t_n}(\| u_t^h\|^2$$
$$+\| u_{tt}^h\|^2+\| f_{tt}\|_{-1}^2+|p_{tt}^h|^2)dt \tag{6.20}$$

In order to prove (6.1), we take $v = \hat{e}^h, w = \hat{\varepsilon}^n$ in (6.6) and $q = \hat{\eta}^n$ in (6.7) and add (6.6) and (6.7). Then thanks to (2.3)–(2.4), we obtain

$$\begin{aligned}\frac{1}{2\Delta t}(|e^n+\varepsilon^n|^2-|e^{n-1}+\varepsilon^{n-1}|^2)+\nu \parallel \hat{e}^n+\hat{\varepsilon}^n \parallel^2 &+b(\hat{e}^n,\hat{y}(t_n)+\hat{z}(t_n),\hat{e}^n)\\ +b(\hat{e}^n,\hat{y}(t_n),\hat{\varepsilon}^n)+b(\hat{\varepsilon}^n,\hat{y}(t_n),\hat{e}^n)&=(e_n,\hat{e}^n)+(\varepsilon_n,\hat{\varepsilon}^n)\end{aligned} \tag{6.21}$$

Due to (2.4) and (2.6), we have

$$\begin{aligned}|b(\hat{e}^n,\hat{u}^h(t_n),\hat{e}^n)| \le & c_0^{3/2}|\hat{e}^n|^{\frac{1}{2}} \parallel \hat{e}^n \parallel^{3/2}\parallel \hat{u}^h(t_n) \parallel\\ \le & \frac{2\gamma\nu}{16} \parallel \hat{e}^n \parallel^2 +16c_0^6(\gamma\nu)^{-3} \parallel \hat{u}^h(t_n) \parallel^4 |\hat{e}^n|^2\end{aligned} \tag{6.22}$$

$$\begin{aligned}|b(\hat{e}^n,\hat{y}(t_n),\hat{\varepsilon}^n)+b(\hat{\varepsilon}^n,\hat{y}(t_n),\hat{e}^n)| \le & 2c_0^{3/2}|\hat{\varepsilon}^n|^{\frac{1}{2}} \parallel \hat{\varepsilon}^n \parallel^{\frac{1}{2}}\parallel \hat{e}^n \parallel\parallel \hat{y}(t_n) \parallel\\ \le & \frac{2\gamma\nu}{16}(\parallel \hat{e}^n \parallel^2+\parallel \hat{\varepsilon}^n \parallel^2)+32c_0^6(\gamma\nu)^{-3} \parallel \hat{u}^h(t_n) \parallel^4 |\hat{\varepsilon}^n|^2\end{aligned} \tag{6.23}$$

$$|(e_n,\hat{e}^n)+(\varepsilon_n,\hat{\varepsilon}^n)| \le \frac{\gamma\nu}{16}(\parallel \hat{e}^n \parallel^2+\parallel \hat{\varepsilon}^n \parallel^2)+\frac{4}{\gamma\nu}(\parallel e_n \parallel_{-1}^2+\parallel \varepsilon_n \parallel_{-1}^2) \tag{6.24}$$

According to (4.3) and Theorem 4.2, we have

$$\gamma \parallel \hat{e}^n \parallel^2 +\gamma \parallel \hat{\varepsilon}^n \parallel^2 \le \parallel \hat{e}^n+\hat{\varepsilon}^n \parallel^2 \tag{6.25}$$

$$|\hat{e}^n|^2+|\hat{\varepsilon}^2| = |\hat{e}^n+\hat{\varepsilon}^n|^2 \le \frac{1}{2}(|e^n+\varepsilon^n|^2+|e^{n-1}+\varepsilon^{n-1}|^2)$$

$$\parallel \hat{u}^h(t_n) \parallel \le M_1 \tag{6.26}$$

So, (6.21) and (6.22)–(6.26) give

$$\begin{aligned}&|e^n+\varepsilon^n|^2-|e^{n-1}+\varepsilon^{n-1}|^2+\nu \parallel \hat{e}^n+\hat{\varepsilon}^n \parallel^2 \Delta t\\ &\le 64c_0^6(\gamma\nu)^{-3}(1+(\gamma\nu)^{-2})M_1^2\Delta t(|e^n+\varepsilon^n|^2+|e^{n-1}+\varepsilon^{n-1}|^2) \parallel \hat{u}^h(t_n) \parallel^2\\ &+\frac{8}{\gamma\nu}(\parallel e_n \parallel_{-1}^2+\parallel \varepsilon_n \parallel_{-1}^2)\Delta t\end{aligned} \tag{6.27}$$

Summing (6.27) for $n = 1, \cdots, m$ and noticing $e^0 + \varepsilon^0 = 0$, we obtain

$$\begin{aligned}&|e^m+\varepsilon^m|^2+\nu\sum_{n=0}^{m} \parallel \hat{e}^n+\hat{\varepsilon}^n \parallel^2 \Delta t\\ &\le \frac{8}{\gamma\nu}\sum_{n=0}^{m}(\parallel e_n \parallel_{-1}^2+\parallel \varepsilon_n \parallel_{-1}^2)\Delta t+\sum_{n=0}^{m} d_n|e^n+\varepsilon^n|^2\Delta t\end{aligned} \tag{6.28}$$

where $e_0 = 0, \varepsilon_0 = 0, d_0 = 0$ and

$$d_n = 64c_0^6(\gamma\nu)^{-3}(1+(\gamma\nu)^{-2})M_1^2(\parallel \hat{u}^h(t_n) \parallel^2+\parallel \hat{u}^h(t_{n-1}) \parallel^2)$$

To prove (6.1), let us recall the Discrete Gronwall Lemma[4]:

Let $\Delta t, \beta$ and $a_n, b_n, c_n, d_n, n \geq 0$, be nonnegative number such that

$$a_m + \Delta t \sum_{n=0}^{m} b_n \leq \Delta t \sum_{n=0}^{m} d_n a_n + \Delta t \sum_{n=0}^{m} c_n + \beta \quad \forall m \geq 0$$

Suppose that $\Delta t d_n < 1, n \geq 0$, and set $\sigma_n = (1 - \Delta t d_n)^{-1}$. Then

$$a_m + \Delta t \sum_{n=0}^{m} b_n \leq \exp\Big(\Delta t \sum_{n=0}^{m} \sigma_n d_n\Big)\Big\{\Delta t \sum_{n=0}^{m} c_n + \beta\Big\}$$

Due to

$$\| \hat{u}^h(t_n) \|^2 \leq \frac{1}{2}(\| u^h(t_n) \|^2 + \| u^h(t_{n-1}) \|^2) \leq M_2^2$$
$$d_n \Delta t \leq 128 c_0^6 (\gamma\nu)^{-3}(1 + (\gamma\nu)^{-2}) M_2^4 \Delta t < \frac{1}{2}$$

then

$$\sigma_n = (1 - d_n \Delta t)^{-1} \leq 2$$

Thus, applying the Discrete Gronwall Lemma to (6.28) with

$$a_n = |e^n + \varepsilon^n|^2, b_n = \nu \| \hat{e}^n + \hat{\varepsilon}^n \|^2$$
$$c_n = \frac{8}{\gamma}(\| e_n \|_{-1}^2 + \| \varepsilon_n \|_{-1}^2), \sigma_n \leq 2$$

We obtain (6.1).

Now, we aim to estimate (6.2). Thanks to (6.6), we obtain

$$\begin{aligned} D\Big(v + w, \sum_{n=1}^{m} \hat{\eta}^n \Delta t\Big) \leq & |(e^m + \varepsilon^m, v + w)| + \nu \sum_{n=1}^{m} \| \hat{e}^n + \hat{\varepsilon}^n \| \Delta t \| v + w \| \\ & + \Delta t \gamma^{-\frac{1}{2}} \sum_{n=1}^{m} (\| e_n \|_{-1}^2 + \| \varepsilon_n \|_{-1}^2)^{\frac{1}{2}} \| v + w \| \\ & + \sum_{n=1}^{m} [b(\hat{y}^n, \hat{e}^n, v + w) + b(\hat{y}^n, \hat{\varepsilon}^n, v) + b(\hat{z}^n, \hat{e}^n, v)]\Delta t \\ & + \sum_{n=1}^{m} [b(\hat{e}^n, \hat{y}(t_n), v + w) + b(\hat{e}^n, \hat{z}(t_n), v) + b(\hat{\varepsilon}^n, \hat{y}(t_n), v)]\Delta t \end{aligned} \tag{6.29}$$

Thanks to (6.1), (3.2) and $\dfrac{\Delta t^2}{c_1 h}$ being bounded, we have that

$$\begin{aligned} \| \hat{y}^n \| \leq & \| \hat{y}(t_n) \| + \| \hat{e}^n \| \leq \| \hat{y}(t_n) \| + (c_1 h)^{-1} |\hat{e}^n| \\ \leq & \hat{y}(t_n) \| + \frac{1}{2}(c_1 h)^{-1}(|e^n| + |e^{n-1}|) \\ \leq & \| \hat{y}(t_n) \| + (c_1 h^{-1}) c(t_m)^{\frac{1}{2}} \Delta t^2 \quad \forall n \leq m \end{aligned}$$

Similar, we have

$$\| \hat{z}^n \| \le \| \hat{z}(t_n) \| + (c_1 h^{-1}) c(t_m)^{\frac{1}{2}} \Delta \quad \forall n \le m$$

Hence, we obtain

$$\| \hat{y}^n \| \quad \text{and} \quad \| \hat{z}^n \| \quad \text{are bounded for any } n \le m \tag{6.30}$$

So, we obtain from (2.4), (6.29)–(6.30) that

$$\begin{aligned}
&\sum_{n=1}^{m} [b(\hat{y}^n, \hat{e}^n, v+w) + b(\hat{y}^n, \hat{\varepsilon}^n, v) + b(\hat{z}^n, \hat{e}^n, v)]\Delta t \\
&\qquad \le 4c_0(1+\gamma^{-\frac{1}{2}}) \| v+w \| \sup_{n\le m} (\| \hat{y}^n \| + \| \hat{z}^n \|) \sum_{n=1}^{m} (\| \hat{e}^n \| + \| \varepsilon^n \|)\Delta t \\
&\qquad \le 4c_0(1+\gamma^{-\frac{1}{2}}) \| v+w \| \sup_{n\le m} (\| \hat{y}^n \| + \| \hat{z}^n \|) t_m^{\frac{1}{2}} \\
&\qquad\quad \cdot \Big(\sum_{n=1}^{m} (\| \hat{e}^n \|^2 + \| \hat{\varepsilon}^n \|^2)\Delta t \Big)^{1/2}
\end{aligned} \tag{6.31}$$

$$\begin{aligned}
&\sum_{n=1}^{m} [b(\hat{e}^n, \hat{y}(t_n), v+w) + b(\hat{\varepsilon}^n, \hat{y}(t_n), v) + b(\hat{e}^n, \hat{z}(t_n), v)]\Delta t \\
&\qquad \le 4c_0(1+\gamma^{-\frac{1}{2}}) M_1 \| v+w \| t_m^{\frac{1}{2}} \Big(\sum_{n=1}^{m} (\| \hat{e}^n \|^2 + \| \hat{\varepsilon}^n \|^2)\Delta t \Big)^{\frac{1}{2}}
\end{aligned} \tag{6.32}$$

Hence, by (2.6), (3.3) and (6.29)–(6.32), we obtain

$$\begin{aligned}
\beta\Big| \sum_{n=1}^{m} \hat{\eta}^n \Delta t| \le & c_0 |e^m + \varepsilon^m| + c_5 t_m^{\frac{1}{2}} \Big\{ \Big(\sum_{n=1}^{m} \| \hat{e}^n + \hat{\varepsilon}^n \| \Delta t \Big)^{1/2} \\
& + \Big(\sum_{n=1}^{m} (\| e_n \|_{-1}^2 + \| \varepsilon_n \|_{-1}^2 \Delta t \Big)^{\frac{1}{2}} \Big\}
\end{aligned} \tag{6.33}$$

So, by (6.1), (6.20) and Theorem 4.2, we obtain (6.2).

The proof ends.

Theorem 6.2. *Under the assumptions of Theorem* 5.1 *and Theorem* 6.1, *the numerical solution* $(u_\Delta(t), p_\Delta(t))$ *produced by NG scheme satisfying the following error estimates:*

$$|u(t) - u_\Delta(t)|^2 + \int_0^t \| u - u_\Delta \|^2 \, ds \le c(t_m)(h^4 + H^6 + \Delta t^4) \tag{6.34}$$

$$\Big| \int_0^t (p - p_\Delta) ds \Big|^2 \le c(t_m)(h^4 + H^6 + \Delta t^4) \tag{6.35}$$

for any $t \in [0, t_m)$.

Proof. We set

$$e_\Delta(t) = y(t) - y_\Delta(t), \varepsilon_\Delta(t) = z(t) - z_\Delta(t)$$

$$\eta_\Delta(t) = p^h(t) - P_\Delta(t)$$

Then (4.4) and (4.8)–(4.10) yield

$$\begin{aligned}\Big(\frac{d}{dt}e_\Delta, v\Big) + \Big(\frac{d}{dt}\varepsilon_\Delta, w\Big) &+ a(e_\Delta+\varepsilon_\Delta, v+w) + b(e_\Delta, y, v+w) + b(e_\Delta, z, v)\\ &+ b(\varepsilon_\Delta, y, v) + b(y_\Delta, \varepsilon_\Delta, v) + b(y_\Delta, e_\Delta, v+w) + b(z_\Delta, e_\Delta, v)\\ &- D(v+w, \eta_\Delta) = 0 \quad \forall v\in X_H, w\in X_h^H \qquad (6.36)\\ D(e_\Delta+\varepsilon_\Delta, q) &= 0 \quad \forall q\in M_h \qquad (6.37)\\ e_\Delta(t_{n-1}) = e^{n-1}&, \varepsilon_\Delta(t) = \varepsilon^{n-1} \qquad (6.38)\end{aligned}$$

for any $t\in[t_{n-1}, t_n)$. Taking $v = e_\Delta, w = \varepsilon_\Delta$ in (6.36), $q = \eta_\Delta$ in (6.37) and adding the corresponding relations, we derive from (2.3)–(2.4) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|e_\Delta+\varepsilon_\Delta|^2 + \nu\parallel e_\Delta+\varepsilon_\Delta\parallel^2 + b(e_\Delta, y+z, e_\Delta)\\ &+ b(e_\Delta, y, \varepsilon_\Delta) + b(\varepsilon_\Delta, y, e_\Delta) = 0 \qquad (6.39)\end{aligned}$$

using (2.4), we can imply

$$\frac{d}{dt}|e_\Delta+\varepsilon_\Delta|^2 + \parallel e_\Delta+\varepsilon_\Delta\parallel^2 \le g(t)|e_\Delta+\varepsilon_\Delta|^2 \qquad (6.40)$$

where

$$g(t) = c_6\parallel u^h(t)\parallel^2$$

By integrating (6.40) and

$$e_\Delta(t_{n-1}) + \varepsilon_\Delta(t_{n-1}) = e^{n-1} + \varepsilon^{n-1}$$

we have that for any $t\in[t_{n-1}, t_n)$,

$$|e_\Delta(t)+\varepsilon_\Delta(t)|^2 + \nu\int_{t_{n-1}}^t \parallel e_\Delta+\varepsilon_\Delta\parallel^2 ds \le |e^{n-1}+\varepsilon^{n-1}|^2 \exp\Big(\int_{t_{n-1}}^t g(s)ds\Big) \qquad (6.41)$$

Applying Theorem 6.1 and noticing

$$\int_0^t \parallel e_\Delta+\varepsilon_\Delta\parallel^2 ds = \sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n-1}} \parallel e_\Delta+\varepsilon_\Delta\parallel^2 ds + \int_{t_{m-1}}^t \parallel e_\Delta+\varepsilon_\Delta\parallel^2 ds \qquad (6.42)$$

we imply

$$|e_\Delta(t)+\varepsilon_\Delta(t)|^2 + \nu\int_0^t \parallel e_\Delta+\varepsilon_\Delta\parallel^2 ds \le c(t_m)\Delta t^4 \qquad (6.43)$$

Using again (6.36), we have

$$\begin{aligned}D\Big(v+w, \int_0^t \eta ds\Big) &= (e_\Delta(t)+\varepsilon_\Delta(t), v+w)\\ &+ \int_0^t a(e_\Delta+\varepsilon_\Delta, v+w)ds + \int_0^t (b(e_\Delta, y, v+w) + b(y_\Delta, e_\Delta, v+w))ds\end{aligned}$$

$$+\int_0^t [b(e_\Delta, z, v) + b(y_\Delta, \varepsilon_\Delta, v)]ds + \int_0^t [b(\varepsilon_\Delta, y, v) + b(z_\Delta, e_\Delta, v)]ds \tag{6.44}$$

Due to the assumptions of Theorem 6.1, we can imply

$$\| y^n \|^2 \quad \text{and} \quad \| z^n \|^2 \quad \text{are bounded for any } n \leq m$$

Hence, (4.8)–(4.10) can imply

$$\| y_\Delta(t) \|^2 \leq M_3^2, \| z_\Delta(t) \|^2 \leq M_3^2 \quad \forall t[0, t_m) \tag{6.45}$$

So, applying (3.3) to (6.44), we can derive from (6.43) that

$$\Big| \int_0^t \eta_\Delta ds \Big|^2 \leq c(t_m)\Delta t^4 \quad \forall t \in [0, t_m) \tag{6.46}$$

Finally, by the triangle inequality and Theorem 5.1, (6.43) and (6.46) imply (6.34)–(6.35).

The proof ends.

7. Numerical Test

We describe here the results of numerical test performed with the full discrete nonlinear Galerkin method (4.5)–(4.7). Comparison is also made with the usual Galerkin method (3.9)–(3.11).

Here we set that $\Omega = [0, \pi] \times [0, \pi], \nu = 0.005$ and $T = \pi/2$. Then the complexity of the flow is described by the Reynolds number

$$Re = (\text{ vol } \Omega)^{1/2} |f|^{1/2}/\nu = 200\pi |f|^{1/2}$$

Moreover, the solution (u, p) of (2.1) is $p = 0, u = (u_1, u_2)$:

$$u_1 = G(y)\cos t, u_2 = G(x)\sin t$$
$$G(x) = 0.1 \times (x^4 - 4\pi x^3 + 3\pi^2 x^2)$$

where f and u_0 are determined from (2.1). By computing, we obtain

$$Re = 3004$$

Hence the exact solution of (2.1) is known and it is easy to check the accuracy of numerical test.

One starts with a fine mesh: mesh 1 (as Fig.1.a). Then (X_h, M_h) is constructed on mesh 1 by the nine-node element of velocity and the four-node element of pressure. Next, X_H is constructed on mesh 2 (as Fig.1.b) by the nine-node element. So, X_h^H is constructed as in Section 4.

We set $|u(t) - u_h(t)|/|u(t)|$, $|p(t) - p_h(t)|$ denote the relative error of velocity and the obsolute error of pressure, where t is taken in $[0, T]$.

a) Comparison of error and CPU time

We take $\Delta t = \pi/160, h = \frac{\pi}{8}, H = \frac{\pi}{4}$. The comparison of errors of two methods is showed by Fig.2 and Fig.3. For G method, the absolute error of pressure decreases, but the relative error of velocity increases. And CPU time of G method is 821 seconds, CPU time of NLG method is 401 seconds.

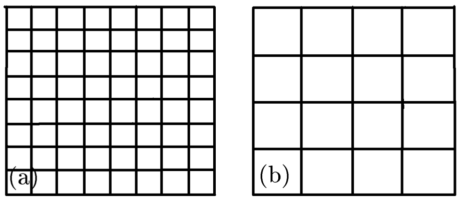

Fig.1. Generated meshs and elements: a: mesh 1, b: mesh 2.

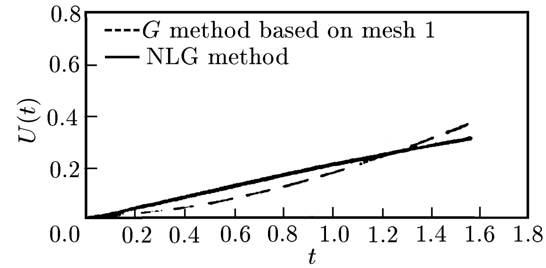

Fig.2. Error curves of velocity $U(t)$

Thus shows that NLG method is superior to G method. Hence we choose the NLG method to solve (2.1).

b) Comparison of numerical velocity and exact velocity

We consider the norm $|u(t)|$ of exact solution and the norm $|u_h(t)|$ of numerical solution of (2.1) for $0 \le t \le T$. Then FIG 4 shows that the maximal relative error of numerical velocity is

$$\max_{t\in[0,T]} ||u(t)| - |u_h(t)||/|u(t)| = 0.067$$

c) Comparison of numerical orbit and exact orbit

By the following average of exact velocity and numerical velocity

$$\bar{u}_1(t) = \Big(\frac{1}{\text{vol }(\Omega)} \int_\Omega u_1^2(x,t)dx\Big)^{1/2}, \bar{u}_2(t) = \Big(\frac{1}{\text{vol }(\Omega)} \int_\Omega u_2^2(x,t)dx\Big)^{1/2}$$
$$\bar{u}_{1h}(t) = \Big(\frac{1}{\text{vol }(\Omega)} \int_\Omega u_{1h}^2(x,t)dx\Big)^{1/2}, \bar{u}_{2h}(t) = \Big(\frac{1}{\text{vol }(\Omega)} \int_\Omega u_{2h}^2(x,t)dx\Big)^{1/2}$$

we obtain the curves produced by $(\bar{u}_1(t), \bar{u}_2(t))$ and $(\bar{u}_{1h}(t), \bar{u}_{2h}(t))$, $0 \le t \le T/2$. By symmetry, we obtain the exact orbit (as Fig.5) and the numerical orbit (as Fig.6) of flow.

Above a), b) and c) show that NLG method is a successful method to solve the viscous incompressible flow under the large Reynolds number.

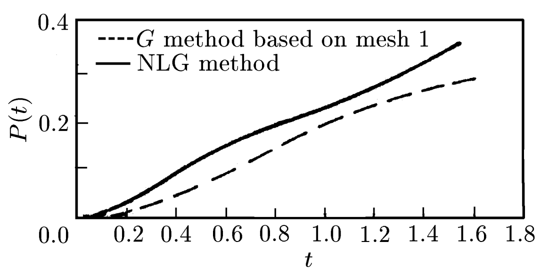

Fig.3. Error curves of pressure $P(t)$

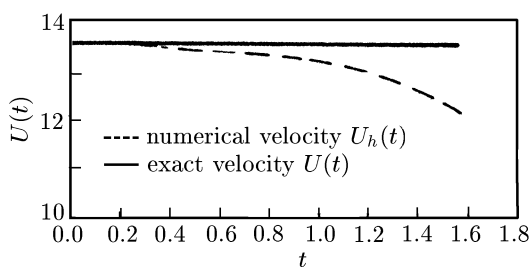

Fig.4. L^2-norm curves of velocity $U(t)$

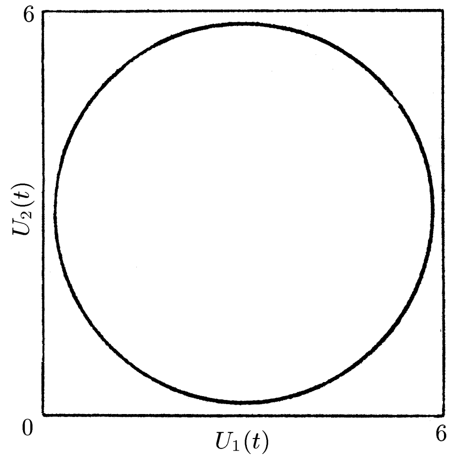

Fig.5. Orbit of exact velocity $U(t)$

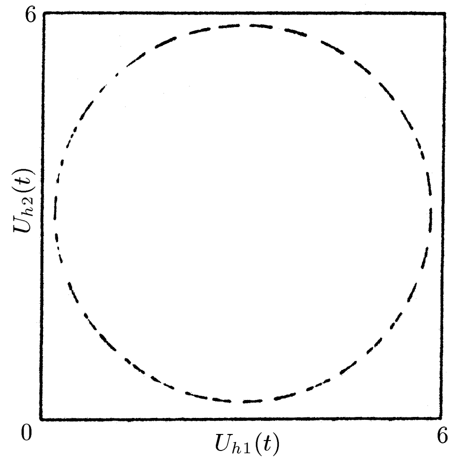

Fig.6. Orbit of numerical velocity $U_h(t)$

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