Journal of Computational Mathematics, Vol.17, No.2, 1999, 159–170.

NUMERICAL COMPUTATION OF BOUNDED SOLUTIONS FOR A SEMILINEAR ELLIPTIC EQUATION ON AN INFINITE STRIP*

Fu-ming Ma
(*Department of Mathematics, Jilin University, Changchun 130023, China*)

Abstract

In this paper, we consider the computation of bounded solutions of a semilinear elliptic equation on an infinite strip. The dynamical system approach and reduction on center manifold are used to overcome the difficulties in numerical procedure.

Key words: Numerical method, Nonlinear PDE, Center manifold.

1. Introduction

Consider elliptic problem on an infinite strip of R^2

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+f(\lambda,y,u)+g(\lambda,\epsilon,x,y)=0 \tag{1.1}$$

where $(x,y)\in(-\infty,\infty)\times(0,1)$ and f,g are smooth functions of thier arguments, λ,ϵ two real parameters. We are interested in the bounded solutions of (1.1) with the following conditions

$$u(x,0)=u(x,1)=0,\quad x\in R \tag{1.2}$$

and

$$\lim_{x\to-\infty}u(x,y)=0,\quad y\in(0,1). \tag{1.3}$$

Some problems arising in applied mathematics are given by the formulation (1.1) with conditions (1.2) and (1.3), for example, the description of the steady flow of an inviscid nondiffusive fluid through a channel of varying depth(see A. Meilke[7]). Here we concern the numerical computation of the bounded solutions of (1.1) with (1.2) and (1.3), i.e., that solution satisfying

$$\sup_{x,y}|u(x,y)|<+\infty. \tag{1.4}$$

To do this, we shall meet some difficulties from two aspects, unboundedness of domain and nonlinearity of function f. In order to overcome the difficulty from unboundedness of domain, the boundary conditions at an artificial boundary are often used and then the boundary-value problems on the finite domain are solved (see T.Hagstrom and H.B.Keller [2] and its references). However, the multi-solution of our problem which

* Received January 11, 1995.

is from the nonliearity makes it difficult to compute numerically even though artificial boundary conditions are used.

We propose here another approach to compute the solutions of problem (1.1). The first step of our approach is to transform (1.1) with boundary condition (1.2) into infinite dimensional formally dynamical system which is follows the idea of K.Kirschgässner[3], A.Mielke[7] and Ma[5]. Then, the bounded solutions of (1.1) will be found as the special orbits–homoclinic or heteroclinic or half-periodic orbits of the formally dynamical system. The second step of our approach is to study numerically the planar dynamical system reduced from infinite dimentional system by use of center manifold theory. The porpuse of this step is to provid good prediction of special orbits of system obtained by the first step. Finally, we calculate numerically the special solutions of the formally dynamical system. To this end, of course, it is neccessary to approximate the infinite dimensional system by a finite dimensional one and to give a artificial boundary condition. We use the semi-discrezation on y and the projection boundary conditions. Meanwhile, we also use a predict-correct procedure with an initial prediction which is constructed by use of the results in step two.

The outline of this paper is as follows: in Sec. 2, we describe the procedures to transform (1.1) and (1.2) into the infinite dimensional system and to reduce it into a planar dynamical system by use of the center manifold theory. In Sec. 3, we give a numerical study of reduced system. In Sec.4, the predict-correct procedure to solve problem (1.1)–(1.4) is described. In last section, a numerical implementation of our approach is given by a example.

2. Formally Dynamical System Formulation of Problem

Following Kirschgässner[3] and Mielke[7], we now transform our problem (1.1)–(1.2) into an infinite dimensional system. Assume that $f(\lambda, y, 0) \equiv 0$ and define the linear operator $L(\lambda)$ in $L^2(0,1)$ as

$$
\begin{aligned}
L(\lambda)\phi := -\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial f}{\partial u}(\lambda, y, 0)\phi, \\
\forall \phi \in D(L(\lambda)) = H_0^1(0,1) \bigcap H^2(0,1).
\end{aligned}
$$

Then (1.1) with (1.2) can be understood as a nonlinear differential equation

$$
\frac{d^2 u}{dx^2} - L(\lambda)u + \bar{f}(\lambda, u) + \bar{g}(\lambda, \epsilon, x) = 0, \tag{2.1}
$$

where $u : (-\infty, \infty) \to L^2(0,1)$, $\bar{f} : \Lambda \times \mathrm{L}^2(0,1) \to L^2(0,1)$, $\bar{g} : \Lambda \times (-\epsilon_0, \epsilon_0) \times (-\infty, \infty) \to L^2(0,1)$ are defined by

$$
\begin{aligned}
u(x)(y) &= u(x,y), \\
\bar{f}(\lambda, u)(y) &= f(\lambda, y, u) - \frac{\partial f}{\partial u}(\lambda, y, 0)u, \\
\bar{g}(\lambda, \epsilon, x)(y) &= g(\lambda, \epsilon, x, y)
\end{aligned}
$$

respectively. Setting $v = \frac{du}{dx}$, we can transform the (1.1)–(1.2) into a formally dynamical system

$$\begin{aligned} \frac{du}{dx} =& v, \\ \frac{dv}{dx} =& L(\lambda)u - \bar{f}(\lambda, u) - \bar{g}(\lambda, \epsilon, x), \end{aligned} \tag{2.3}$$

on the infinite dimensional space $L^2(0,1) \times L^2(0,1)$. we can find the solutions of problem (1.1)–(1.4) by understanding the behavier of solutions for (2.3) on $(-\infty, \infty)$. It is well known that $L(\lambda)$ is selfadjoint and has an infinite seqence of eigenvalues $\mu_i(\lambda)$, $i = 1, 2, \cdots$. We assume that $\mu_1(\lambda) < \mu_2(\lambda) < \cdots$ and there exists $\lambda_0 \in \Lambda$ such that $\mu_1(\lambda_0) = 0$, $\mu_1'(\lambda_0) < 0$. We are interested in the solutions of (1.1)–(1.4) for λ near λ_0. Denote by ω the eigenfunction of $L(\lambda_0)$ corresponding to $\mu_1(\lambda_0) = 0$ and normalized in $L_2(0,1)$. Define the projections

$$\begin{aligned} P\phi :=& \int_0^1 \omega(y)\phi(y)dy, \\ Q\phi :=& \phi - (P\phi)\omega \end{aligned} \tag{2.4}$$

and set $\gamma = P\phi = (\gamma_1, \gamma_2)$, $\theta = Q\phi = (\theta_1, \theta_2)$. Then (2.3) yields

$$\frac{d}{dx}\begin{pmatrix}\gamma_1 \\ \gamma_2\end{pmatrix} = \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\gamma_1 \\ \gamma_2\end{pmatrix} - \begin{pmatrix}0 \\ PF(\lambda, \epsilon, x, \gamma_1\omega + \theta)\end{pmatrix} \tag{2.5}$$

$$\frac{d}{dx}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = \begin{pmatrix}0 & 1 \\ L_0 & 0\end{pmatrix}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} - \begin{pmatrix}0 \\ QF(\lambda, \epsilon, x, \gamma_1\omega + \theta)\end{pmatrix} \tag{2.6}$$

where L_0 is the restriction of $L(\lambda_0)$ on $QL^2(0,1)$ and F is defined by

$$F(\lambda, \epsilon, x, \phi)(y) = f(\lambda, y, \phi) - \frac{\partial}{\partial u} f(\lambda_0, y, 0)\phi + g(\lambda, \epsilon, x, y). \tag{2.7}$$

The varibles (γ_1, γ_2) and (θ_1, θ_2) are supposed to be elements of $X_1 = R^2$, and $X_2 = QH_0^1(0,1) \times QL_2(0,1)$.

It has shown by Mielke[7] that all small bounded solutions of system (2.6)–(2.7) lie on a two-dimensional manifold modelled over the (γ_1, γ_2)-space. In more accurate words, there exist a positive number κ and a reduction function

$$h \in C_b^4((\lambda_0 - \kappa, \lambda_0 + \kappa) \times (\kappa, \kappa) \times R \times (-\kappa, \kappa)^2, QH_0^1(0,1))$$

such that all solution of (2.5)–(2.6) which are sufficiently small in $X_1 \times X_2$ for all x satisfy $\theta_1 = h(\lambda, \epsilon, x, \gamma_1, \gamma_2)$, where $C_b^4(A, B)$ is the space of bounded 4-times continouly differentable functions. Thus the investigation of (2.5)–(2.6) for all small bounded solutions is completely reduced to the ordinary differential equation

$$\gamma'' + PF(\lambda, \epsilon, x, \gamma\omega + h(\lambda, \epsilon, x, \gamma, \gamma')) = 0, \quad |\gamma|, |\gamma'| < \kappa. \tag{2.8}$$

On the other hand, every solution γ of (2.7) yields a solution (γ, θ) of (2.5)–(2.6) via $\theta(x) = h(\lambda, \epsilon, x, \gamma(x), \gamma'(x))$.

As an application of the above result, Mielke[7] analized the Problem (1.1)–(1.4) when g satisfies

$$|\partial g(\lambda, \epsilon, x, y)| \leq Ce^{-|x|}. \tag{2.9}$$

It is proved that there exist three kinds of solutions for (1.1)–(1.4), satisfying
(i) $\lim\limits_{x\to+\infty} u(x,y) = 0$,
(ii) $\lim\limits_{x\to+\infty} u(x,y) = u_0(y) \not\equiv 0$,
(iii) there exist $x_0 > 0$, $T > 0$ such that $u(x+T, y) = u(x,y)$ for $x > x_0$
respectively for different values of λ and ϵ near λ_0 and 0. They are corresponding to special orbits of (2.8), (i) homoclinic, (ii) heteroclinic and (iii) half-periodic orbits respectively.

3. Numerical Analysis of Reduced System

Our goal in this paper is to calculate the solutions of (1.1)–(1.4) numerically. As we describe in Sec. 1, it is more difficult if we do it directly. In this section, we give a numerical stretagy for study of reduced system (2.8) on plan. The results provided by this section will be initial date to get the numerical solutions of (1.1)–(1.4) in next section.

Assume that, after dropping the term $O(u^3)$, (2.8) can be reformulated as

$$\frac{d^2u}{dx^2} - au + bu^2 + \epsilon f(x) = 0. \tag{3.1}$$

What we want is the solutions of (3.1) satisfying

$$u(-\infty) = 0, \quad \sup_{x\in R} |u(x)| < +\infty. \tag{3.2}$$

We assume, in addition, that $a > 0$ and set

$$\sigma = a^{\frac{1}{2}}, \quad t = \sigma x, \quad v(t) = \frac{b}{\sigma^2} u(x). \tag{3.3}$$

Now we rescale (3.1) by (3.3) and get that

$$\frac{d^2v}{dt^2} - v + v^2 + \epsilon g(t) = 0. \tag{3.4}$$

We will find the solutions of (3.4) with

$$v(-\infty) = 0, \quad \sup_{t\in R} |v(t)| < +\infty. \tag{3.5}$$

Furthermore, we assume that

$$\text{supp } g(t) \subset [-T, T] \tag{3.6}$$

for some $0 < T < +\infty$. In case of $g(t) \equiv 0$, we can give a complete description of solutions for (3.4) with (3.5). In Fig.1, we draw down the (v, v')-phase picture for (3.4) when $g(t) \equiv 0$, where curves C_i, $i = 1, 2, 3, 4$ are

$$C_1 : v' = \sqrt{v^2 - \frac{2}{3}v^3}, \quad 0 \leq v \leq \frac{3}{2};$$

$$C_2 : v' = -\sqrt{v^2 - \frac{2}{3}v^3}, \quad 0 \le v \le \frac{3}{2};$$
$$C_3 : v' = \sqrt{v^2 - \frac{2}{3}v^3}, \quad v < 0;$$
$$C_4 : v' = -\sqrt{v^2 - \frac{2}{3}v^3}, \quad v < 0.$$

When $g(t) \not\equiv 0$, the solution of (3.4) with (3.5) must satisfy

$$\frac{d^2v}{dt^2} - v + v^2 + \epsilon g(t) = 0, \quad t \in (-T, T), \tag{3.7}$$
$$(v(-T), v'(-T)) \in D_-, \quad (v(T), v'(T)) \in D_+, \tag{3.8}$$

where $D_- = C_1 \bigcup C_2 \bigcup C_4$, $D_+ = C_3 \bigcup \overline{D}$ and D is domain surrounded by C_1 and C_2.

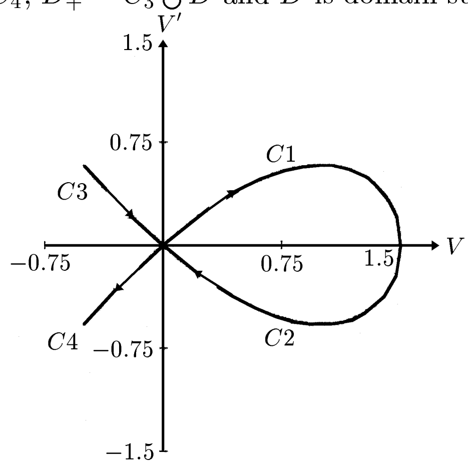

Fig.1

Let $v(t; v_-, v'_-)$ be the solution of initial values problem

$$\frac{d^2v}{dt^2} - v + v^2 + \epsilon g(t) = 0, \quad t > -T, \tag{3.9}$$
$$v(-T) = v_-, \quad \frac{dv}{dt}(-T) = v'_- \tag{3.10}$$

Define $\xi(v_-, v'_-) = (v(T; v_-, v'_-), \frac{d}{dt}v(T; v_-, v'_-))^T$, then (3.9) and (3.10) can be rewritten as the shooting formulation:find $(v_-, v'_-) \in D_-$, such that

$$\xi(v_-, v'_-) \in D_+. \tag{3.11}$$

In particular, because $\xi(v_-, v'_-)$ depends on (v_-, v'_-) continuously, the points (v_-, v'_-) on D_- which make $\xi(v_-, v'_-) \in C_1 \bigcup C_2 \bigcup C_3$ become important and the orbits corresponding to these points are homoclinic orbits of reduced system.

In order to provid good initial date for numerical solution of problem (1.1)–(1.4), we are to determin numerically set S of solutions for (3.11) in D_-. This can be realized by shooting method. In fact, we only need to find the boundary points of S then to determin S by the continuty of $\xi(v_-, v'_-)$ on (v_-, v'_-). In the example in Sec.5, we use dichotomous search to find the boundary points of S. Of course, other way also can be used to realized the numerical analysis described in this section.

4. Predict-Correct Procedure

To get the numerical solution of (1.1)–(1.4), we first discrete the equation (1.1) only in direction y. It is proved in Ma[6] that an approximate solution can be obtained in neighborhood of every regular solution of (1.1) satisfying $\lim_{x\to\pm\infty} u(x,y) = 0$ under appropriate conditions. To realize this discretization we can chose finite dfference method, spectral method or finite element method.

After this discretization, we obtain a finite dimentional system

$$\frac{d^2u_h}{dx^2} + A_h u_h + F_h(\lambda, u_h) + g_h(\lambda, \epsilon, x) = 0, \tag{4.1}$$

where $u_h \in X_h$, X_h a finite dimentional space, A_h a matrix, F_h, g_h vector-value functions. Now the problem becomes to find the solution of (4.1) satisfying

$$u_h(-\infty) = 0, \quad \|u_h(x)\| < +\infty. \tag{4.2}$$

In fact, we are particularly interested in the solutions of (4.1) satisfying

$$u_h(-\infty) = u_h(+\infty) = 0, \tag{4.3}$$

which are correponding to the homoclinic orbits of system

$$\begin{aligned} \frac{du_h}{dx} &= w_h, \\ \frac{dw_h}{dx} &= -A_h u_h - F_h(\lambda, u_h) - g_h(\lambda, \epsilon, x) \end{aligned} \tag{4.4}$$

in phase space $X_h \times X_h$. Thus, what we face is numerical computation of the homoclinic orbits of dynamical system. Here, we will suffer difficulties from two aspects. One is from the infinity of x and another one is from the ununiqueness of the solutions.

At first, we are referred Beyn[1] to restricte (4.4) on $x \in [-T, T]$ and give a artificial boundary-value condition on $x = \pm T$. (also see H.B.Keller et.[4]) To this end, we suppose that $g_h(\lambda, \epsilon, x)$ tends rapidly to zero as $|x| \to \infty$ so that we can regard to $g_h(\lambda, \epsilon, x) = 0$ for $|x|$ sufficeintly large.

Let

$$\begin{aligned} L_h(\lambda) &= A_h + \frac{\partial}{\partial u_h} F_h(\lambda, 0), \\ \tilde{F}_h(\lambda, u_h) &= F_h(\lambda, u_h) - \frac{\partial}{\partial u_h} F_h(\lambda, 0) u_h \end{aligned}$$

and μ_i^h, ϕ_i^h, $i = 1, 2, \cdots, N, N = \dim X_h$, be eigenvalues and eigenvectors. Thus we know that the matrix

$$\begin{pmatrix} 0 & 1 \\ -L_h(\lambda) & 0 \end{pmatrix}$$

has the eigenvalues $\sigma_{\pm i}^h = \pm\sqrt{\mu_i^h}$ and eigenvectors $(\phi_i^h, \pm\sqrt{\mu_i^h}\phi_i^h)^T$, $i = 1, 2, \cdots, N$ where we assume that $\mu_i^h > 0$ for $i = 1, 2, \cdots, N$. Acorrding to Beyn[1] and Keller et.[4], we introduce the projection boundary-value conditions for (4.4) on $x = \pm T$, i.e.,

$$(u_h(-T), w_h(-T))^T \in \text{span}\{(\phi_i^h, \sqrt{\mu_i^h}\phi_i^h)^T,\ i = 1, 2, \cdots, N\} \tag{4.5}$$

and

$$(u_h(T), w_h(T))^T \in \text{span}\{(\phi_i^h, -\sqrt{\mu_i^h}\phi_i^h)^T,\ i = 1, 2, \cdots, N\}. \tag{4.6}$$

In matrix formulation, (4.5) and (4.6) can be written as

$$w_h(-T) = (-L_h)^{\frac{1}{2}} u_h(-T), \tag{4.7}$$

$$w_h(T) = -(-L_h)^{\frac{1}{2}} u_h(T) \tag{4.8}$$

or

$$\frac{\partial}{\partial x} u_h(-T) = (-L_h)^{\frac{1}{2}} u_h(-T), \tag{4.9}$$

$$\frac{\partial}{\partial x} u_h(T) = -(-L_h)^{\frac{1}{2}} u_h(T) \tag{4.10}$$

Therefore, we need to solve (4.1) on $(-T, T)$ with artificial boundary value conditions (4.9) and (4.10).

In order to overcome the difficulty from the ununiquesnees of solution, we use the numerical results of the section 3 to constructe a prediction for solution of (4.1) with (4.9) and (4.10). Suppose that $\bar{v}(x)$ is a numerical solution of (3.1) with $\lim_{|x|\to\infty} \bar{v}(x) = 0$. By center manifold theory, we regard

$$\bar{u}(x, y) = \bar{v}(x)\phi(y) + h(\lambda, \epsilon, x, \bar{v}(x), \bar{v}'(x))$$

as an approximation solution of (1.1) with $\lim\limits_{|x|\to\infty} \bar{u}(x, y) = 0$. Then, we can take $u_h^0 = P_h\bar{u}$ as a prediction of solution for (4.1) with (4.9) and (4.10). In particular, for simplicity, it is enough to take $\bar{u}(x, y) = \bar{v}(x)\phi(y)$ directly.

5. Numerical Implementation

In this section, we illustrate our approach by a numerical example. Consider following problem

$$\begin{gathered}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (2\lambda\beta - \beta^2)u - (\lambda + 4\beta)\beta^2 e^{\beta y} u^2 + \lambda\epsilon g_1(x, y) + \epsilon g_2(x, y) = 0, \\
(x, y) \in \Omega \\
u(x, 0) = u(x, 1) = 0, \\
\lim_{x\to-\infty} u(x, y) = 0,
\end{gathered} \tag{5.1}$$

where λ, ϵ, $\beta \in R$, $\Omega = (-\infty, \infty) \times (0,1)$, and

$$g_1(x,y) = \frac{8\beta e^{-\beta y} \sin(1-y)}{e^x + e^{-x} + 2\cos(1-y)},$$
$$g_2(x,y) = 2\beta e^{-\beta y} \frac{8 + (e^x + e^{-x})\cos(1-y)}{[e^x + e^{-x} + 2\cos(1-y)]^2}.$$

The above equation is from a classical problem to describe the steady flow of an inviscid nondiffusive fluid through a channel of varing depth. Starting with the stationary Euler equations, introducing a stream function and using a conformal mapping for the boundary of channel, we obtain the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a(\lambda, y)u + b(\lambda, y)u^2 + \epsilon g(\lambda, x, y) + O(|u|^3 + |\epsilon u| + \epsilon^2) = 0, \tag{5.3}$$

where

$$a(\lambda, y) = 2\lambda\beta - \beta^2, \quad b(\lambda, y) = -(\lambda - 4\beta)\beta^2 e^{\beta y}$$

and

$$g(\lambda, x, y) = \lambda g_1(x,y) + g_2(x,y), \quad \text{for } |\epsilon| \ll 1.$$

Because we are only concerned with "small" solution of (5.3), we drop down terms $O(|u|^3 + |\epsilon u| + \epsilon^2)$ and get (5.1) (see Mielkea [7]).

To note that, setting

$$f(\lambda, y, u) = (2\lambda\beta - \beta^2)u - (\lambda + 4\beta)\beta^2 e^{\beta y} u^2$$

and

$$g(\lambda, \epsilon, x, y) = \lambda\epsilon g_1(x,y) + \epsilon g_2(x,y),$$

we identify the equation (5.1) with (1.1). We can apply the stretage described in previous sections. Specially, we have linear operator

$$L(\lambda)\phi := -\frac{\partial^2}{\partial y^2}\phi - (2\lambda\beta - \beta^2)\phi$$

and its eigenvalues

$$\mu_k(\lambda) = k^2\pi^2 - (2\lambda\beta - \beta^2), \quad k = 1, 2, \cdots.$$

When $\lambda = \lambda_0 = (\pi^2 + \beta^2)/2\beta$, we have that eigenvalue $\mu_1(\lambda_0) = 0$ and eigenfunction $\omega(y) = \sqrt{2}\sin y$. Using the procedure of reduction in the section 2 and 3, we obtain the reduced system

$$\frac{d^2 u}{dt^2} + (2\lambda\beta - \beta^2 - \pi^2)u - (\lambda + 4\beta)\beta^2 c_0 u^2 + \lambda\epsilon f_1(x) + \epsilon f_2(x) = 0, \tag{5.3}$$

where

$$c_0 = \int_0^1 e^{\beta y}\omega(y)dy,$$

$$f_1(x) = \int_0^1 g_1(x,y)\omega(y)dy$$

and

$$f_2(x) = \int_0^1 g_2(x,y)\omega(y)dy.$$

Applying the method in the section 3, we analyse the system (5.3) numerically. After rescalling, we rewrite (5.3) as (3.4). In Fig.2, we show two approximate solutions of (3.4) for $\lambda = \lambda_0 - 0.125$, $\epsilon = 0.001$ which satisfy

$$\lim_{|x|\to\infty} u(x) = 0.$$

Here, we use dichotomous search in shooting procedure. Perhaps the accuracy of theses approximate solutions is not high, but it is enough for our next step.

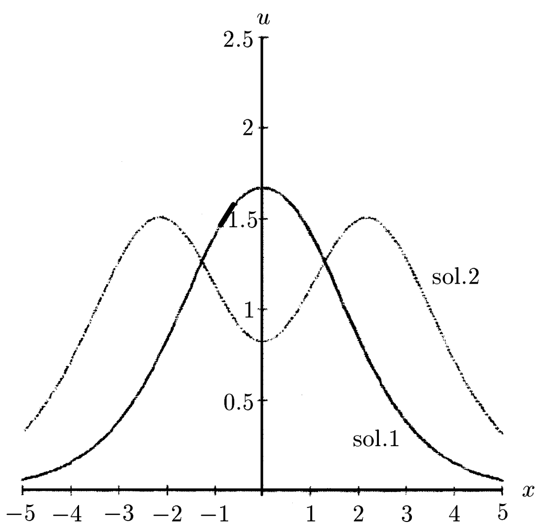

Fig.2

Now let us return to the equation (5.1). Assume that $\lambda < \lambda_0$ and $\sigma = \sqrt{2(\lambda_0 - \lambda)}$. We rescale (5.1) by use of $u = \sigma^2\phi$ and get

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + (2\lambda\beta - \beta^2)\phi - \sigma^2(\lambda + 4\beta)\beta^2\phi^2 + \frac{\lambda\epsilon}{\sigma^2}g_1 + \frac{\epsilon}{\sigma^2}g_2 = 0. \tag{5.4}$$

Using central difference on y, we get the semi-discretzation of (5.4)

$$\begin{aligned}\frac{d^2\phi_i}{dx^2} + &\frac{1}{(\triangle y)^2}(\phi_{i+1} - 2\phi_i + \phi_{i-1}) + (2\lambda\beta - \beta^2)\phi_i - \sigma^2(\lambda + 4\beta)\beta^2 e^{\beta y_i}\phi_i^2 \\ &+ \frac{\lambda\epsilon}{\sigma^2}g_1(x, y_i) + \frac{\epsilon}{\sigma^2}g_2(x, y_i) = 0, \quad i = 1, 2, \cdots, N, \end{aligned} \tag{5.5}$$

where $\triangle y$ is difference step-size and $y_i = i\triangle y$. (5.5) is a finite dimensional system on

x. Setting $u_h = (\phi_1, \phi_2, \cdots, \phi_N)^T$, we identifies (5.5) with (4.1), where

$$A_h = \frac{1}{(\triangle y)^2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

and

$$F_h(\lambda, u_h) = (2\lambda\beta - \beta^2) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} - \sigma^2(\lambda + 4\beta)\beta^2 \begin{pmatrix} e^{\beta y_1}\phi_1^2 \\ e^{\beta y_2}\phi_2^2 \\ \vdots \\ e^{\beta y_N}\phi_N^2 \end{pmatrix},$$

$$g_h(\lambda, \epsilon, x) = \frac{\epsilon}{\sigma^2} \begin{pmatrix} \lambda g_1(x, y_1) + g_2(x, y_1) \\ \lambda g_1(x, y_2) + g_2(x, y_2) \\ \vdots \\ \lambda g_1(x, y_N) + g_2(x, y_N) \end{pmatrix}.$$

Therefore, we have that

$$L_h(\lambda) = A_h + (2\lambda\beta - \beta^2)I$$

and $-L_h(\lambda)$ has eigenelements

$$\mu_i = \frac{4}{(\triangle y)^2}\sin^2\frac{i\pi\triangle y}{2} + (2\lambda\beta - \beta^2)$$

and

$$\phi_i^h = (\sin i\pi\triangle y, \sin 2i\pi\triangle y, \cdots, \sin Ni\pi\triangle y)^T.$$

Let

$$Q = (\phi_1^h, \phi_2^h, \cdots, \phi_N^h).$$

Then we have that

$$(-L_h)^{\frac{1}{2}} = Q\Lambda^{\frac{1}{2}}Q^{-1},$$

where

$$\Lambda^{\frac{1}{2}} = \text{diag}\{\mu_1^{\frac{1}{2}}, \mu_2^{\frac{1}{2}}, \cdots, \mu_N^{\frac{1}{2}}\}.$$

Now we choose a $T > 0$ large enough and consider to solve (4.1) on $(-T, T)$ with artificial boundary value conditions (4.9) and (4.10). Using second order central difference quotient to approximate $\frac{d^2\phi_i}{dx^2}$, we get the following discrete system

$$\begin{aligned} &u_h^{j+1} - 2u_h^j + u_h^{j-1} + (\triangle x)^2 L_h u_h^j + (\triangle x)^2 F_h(\lambda, u_h^j) \\ &\qquad + (\triangle x)^2 g_h^j = 0, \quad j = 1, 2, \cdots, M-1, \\ &u_h^1 - u_h^0 = \triangle x(-L_h)^{\frac{1}{2}}u_h^0, \quad u_h^M - u_h^{M-1} = -\triangle x(-L_h)^{\frac{1}{2}}u_h^M, \end{aligned} \tag{5.6}$$

where $\triangle x = \frac{2T}{M}$ and $g_h^j = g_h(\lambda, \epsilon x_j)$, $x_j = -T + j\triangle x$.

Finally, we use the Newton method with a relaxation factor to solve system (5.6). We describe this procedure brifely as follows: let (5.6) can be reformulated

$$F(u) = 0. \tag{5.7}$$

The Newton method with a relaxation factor ω is the iteration

$$\begin{array}{l} DF(u^{(k)})\triangle u^{(k+1)} = -F(u^{(k)}), \\ u^{(k+1)} = u^{(k)} + \omega \triangle u^{(k+1)}, \end{array} \quad k = 0, 1, \cdots. \tag{5.8}$$

The relaxation factor ω is determined by

$$\|F(u^{(k+1)})\| < \|F(u^{(k)})\|.$$

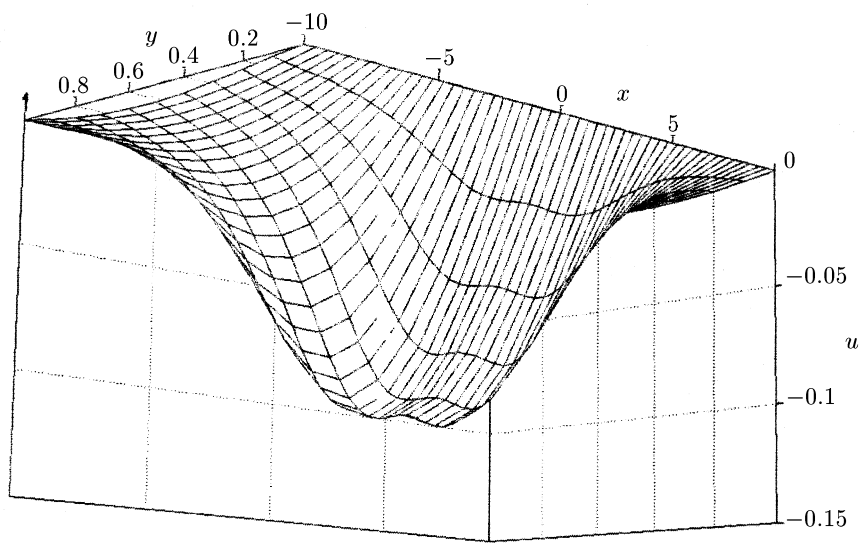

Fig.3

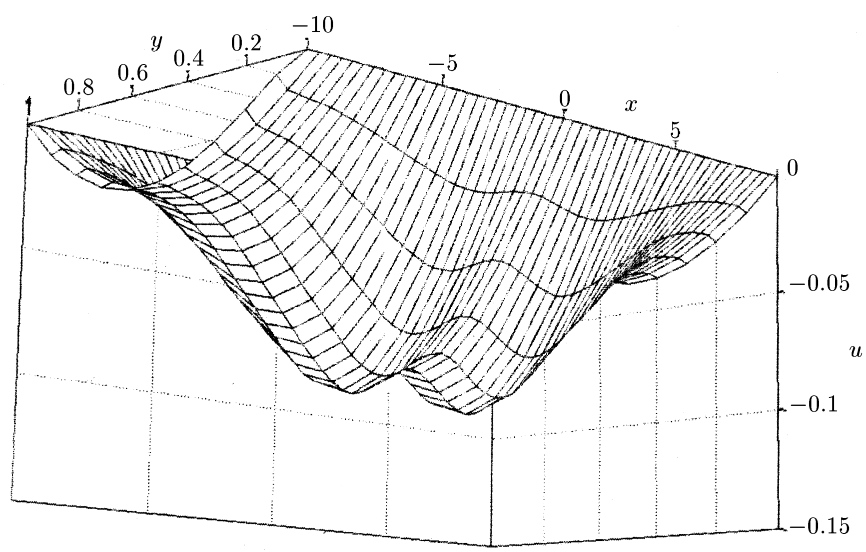

Fig.4

The initial date of iteration is constructed by procedure described in the section 4 using the numerical solutions of (5.3). The motive using Newton method with a relaxation factor is to overcome the difficulty from ununiquesnees of solutions for (5.6). In the each step of iteration (5.8), we need to solve a $(M-1)\times(N-1)$ linear algebric system. We use block elimination in computation to reduce storage. Finally, we show two numerical solutions for $\lambda=\lambda_0-0.125$, $\epsilon=0.001$ in Fig.3 and Fig.4. The solution in Fig.4 is not easy to obtain by usual methods.

Acknowledgement This work was partially supported by the National Natural Science Foundation of China and National Education Commission of China.

References

[1] W-J. Beyn, The numerical computation of connecting orbits in dynamical systems, *IMA J. Numer. Anal.,* **9** (1990), 379–405.

[2] T. Hagstrom, H.B. Keller, Exact boundary conditions at an artificial boundary for partial differential equations in cylinders, *SIAM J. Math. Anal.,* **17**:2 (1986), 322–341.

[3] K. Kirschgässner, Wave-solutions of reversible systems and applications, *J. Diff. Equa.,* **45** (1988), 113–127.

[4] H.B. Keller, M. Lentini, Boundary value problems over semi-infinite intervals and their numerical solution, *SIAM J. Numer. Anal.,* **17** (1980), 577–604.

[5] Ma F., Numerical calculation of center manifolds for a class of infinite dimensional systems with applications, *SIAM J. Numer. Anal.,* **32**:3 (1995), 952–968.

[6] Ma F., Numerical approximation of bounded solutions for semilinear elliptic equations in an unbounded cylindrical domain, *Numerical Method for PDE,* **9** (1993).

[7] A. Mielke, Steady flows of inviscid fluids under localized perturbations, *J. Diff. Equas.,* **65**:1 (1986).