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ON THE LINEAR CONVERGENCE OF PC-METHOD FOR A CLASS OF LINEAR VARIATIONAL INEQUALITIES*

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Abstract

This paper studies the linear convergence properties of a class of the projection and contraction methods for the affine variational inequalities, and proposes a necessary and sufficient condition under which PC-Method has a globally linear convergence rate.

Key words: Affine variational inequality, Projection and contraction method, Linear convergence.

1. Introduction

Let M be an $n \times n$ matrix and let q be a vector in R^n, the n-dimensional Enclidean space. Let Ω be an nonempty closed convex set. The linear variational inequality problem (denoted by (LVI)) is to find $x^* \in \Omega$ such that

$$(x - x^*)^T (Mx^* + q) \geq 0, \quad \forall x \in \Omega. \tag{1.1}$$

The problem (1.1) is well known in optimization and contains as special cases linear (and quadratic) programming, bimatrix game, etc. (see Cottle and Dantzig [1]). When Ω is a polyhedral set, for convenience expressed as

$$X = \{x \in R^n | Ax \geq b\}, A \in R^{m \times n}, b \in R^m, \tag{1.2}$$

it is called the affine variational inequality problem (AVI). When $\Omega = R^n_+$, the nonnegative orthant in R^n, it is again called the linear complementarity problem (LCP). For these subjects, many computational methods and theoretical results have been developed (See Harker and Pang [2], Cottle, Pang and Stone [3], Isac [4] etc.). An important class of methods is the projection-type method, originally proposed by Goldstein [5], Levitin and Polyak [6] for solving convex programming. More recently, He [7–12] has proposed a special class of the projection methods for problem (1.1). The iterative form is as follows. Given $x^k \in R^n$ (or Ω), find the search direction $d(x^k)$ such that it satisfies

$$x^{k+1} = x^k - \alpha_k \cdot d(x^k), \quad \text{or} \quad x^{k+1} = P_\Omega[x^k - \alpha_k d(x^k)], \tag{1.3a}$$

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$$\|x^{k+1} - x^*\|_G^2 \leq \|x^k - x^*\|_G^2 - \rho_k \cdot \|e(x^k)\|^2, \tag{1.3b}$$

where $\alpha_k > 0$ is the search step length, ρ_k is a positive number, $G \in R^{n \times n}$ is symmetric and positive definite, $\|x\|_G = (x^T G x)^{\frac{1}{2}}, P_\Omega[\cdot]$ denotes the projection from R^n onto Ω, i.e.,

$$P_\Omega[x] = \arg\min\{\|x - y\| \quad |\forall y \in \Omega\}, \tag{1.4}$$

$e(x) = x - P_\Omega[x - (Mx + q)]$, and $x^* \in \Omega^*$, which denotes the set of solutions of problem (1.1). From (1.3) we readily see that the sequence $\{\|x^k - x^*\|_G^2\}$ has a contractive property. Therefore, He defines this class of methods as the projection and contraction method (PC-Method). The main advantages of the method are its simplicity, robustness and ability to handle the large-scale problems.

In [12], He has summerized the basic idea of finding the search direction $d(x)$ of PC-Method, i.e., for any $x^* \in \Omega^*$, it holds that

$$(x - x^*)^T d(x) \geq r \cdot \|e(x)\|^2, \quad r > 0, \tag{1.5}$$

and proven that the PC methods of He [7–11] are all globally convergent for varieties of monotone problems. However, He only prove that PC-Method is globally linearly convergent for the monotone linear complementarity problem.

The purpose of this paper is to develop the linear convergence theory of PC-Method. The main results obtained in this paper are as follows.

(a) For the monotone problem (AVI), a class of PC methods is linearly convergent. Furthermore, $x^k \to x^*$ Q-linearly, $\|e(x^k)\| \to 0$ R-linearly.

(b) For strongly monotone problem (AVI), the necessary and sufficient condition under which a class of PC methods has linearly convergent rate is the search direction $d(x)$ to be strongly descent (see Theorem 4.2).

This paper is organized as follows. In section 2, we give the definitions of the strictly descent direction and strongly descent direction, and discuss their convergence properties, which extend the previous convergence theory. In Section 3, we investigate the linear convergence of PC-Method when it is applied to solve the monotone problem (AVI). Finally, Section 4 considers the special case of (AVI) where M is positive definite.

We adopt the following notations throughout. For any $x \in R^n$ and $y \in R^n$, we denote by $x^T y$ the Euclidean inner product of x with y. For any $x \in R^n$, we define $\|x\| = (x^T x)^{\frac{1}{2}}$. For any $C_1, C_2 \subseteq R^n$, we denote by $\text{dist}(C_1, C_2)$ the usual Euclidean distance between two sets C_1 and C_2, that is,

$$\text{dist}(C_1, C_2) = \inf\{\|x - y\| \; |x \in C_1, y \in C_2\}.$$

For any symmetric matrix $A \in R^{n \times n}$, we denote by $\lambda_{\min}(A)$ (and $\lambda_{\max}(A)$) the minimum (and maximum) eigenvalue of A. Other notations have the usual meaning. Throughout this paper we assume that (H_1) $\Omega^* \neq \phi$, and (H_2) M is positive semidefinite (but not necessarily symmetric).

2. A Class of Descent Directions and Convergence for (LVI)

For any $x^* \in \Omega^*$, we define the function $g(x) = \frac{1}{2}\|x - x^*\|^2$. Obviously, $-d(x)$ satisfying (1.5) is a descent direction of $g(x)$ at point x. We clearly give the following definition.

Definition 2.1. *A direction $-d(x)$ is said to be strictly descent for $g(x)$ at x, if $d(x)$ is continuous on Ω (or R^n), and there exists $r(x) > 0$ such that*

$$(x - x^*)^T d(x) \geq r(x) \cdot \|e(x)\|^2, \quad \forall x \in \Omega \ (or\ R^n). \tag{2.1}$$

Moreover, if $r(x) \equiv r > 0$, i.e., the inequality

$$(x - x^*)^T d(x) \geq r \cdot \|e(x)\|^2, \quad \forall x \in \Omega \ (orR^n) \tag{2.2}$$

holds, then we say that $-d(x)$ is strongly descent.

In this section we study the convergence of PC-Method, where $-d(x)$ is strictly descent, applied to solve (LVI). The main result is the following theorem.

Theorem 2.2. *Let $-d(x)$ be strictly descent direction, the sequence $\{x^k\}$ be generated by iteration form*

$$x^{k+1} = x^k - \alpha_k d(x^k), \quad \forall x^0 \in R^n, \tag{2.3a}$$

$$\alpha_k = r_k \cdot \frac{\|e(x^k)\|^2}{\|d(x^k)\|^2}, \quad r_k = r(x^k). \tag{2.3b}$$

If the sequence $\{r_k\}$ satisfies condition

$$\sum_{k=0}^{\infty} r_k^2 = +\infty, \tag{2.4}$$

then $\{x^k\}$ converges to a solution point x^∞ of (1.1).

Proof. For any $x^* \in \Omega^*$, by (2.1) and (2.3), we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|x^k - x^* - \alpha_k d^k\|^2 \\ &\leq \|x^k - x^*\|^2 - 2\alpha_k r_k \cdot \|e(x^k)\|^2 + \alpha_k^2 \cdot \|d^k\|^2 \\ &= \|x^k - x^*\|^2 - r_k \cdot \alpha_k \cdot \|e(x^k)\|^2. \end{aligned} \tag{2.5}$$

Hence the sequence $\{x^k\}$ is bounded, and $\{\|x^k - x^*\|^2\} \downarrow$. Further, we readily obtain

1) $\exists C > 0, \ni \ \|d(x^k)\|^2 \leq C,\ \forall k \in N \triangleq \{0, 1, 2, \cdots\}$;

2) $r_k \cdot \alpha_k \cdot \|e(x^k)\|^2 \geq r_k^2 \frac{\|e(x^k)\|^4}{C}$.

From (2.5) and the above inequalities, we have

$$\sum_{k=0}^{\infty} \frac{1}{C} r_k^2 \cdot \|e(x^k)\|^4 < +\infty. \tag{2.6}$$

Assume that $\exists \varepsilon_0 > 0$ such that

$$\|e(x^k)\| \geq \varepsilon_0, \quad \forall k \in N, \tag{2.7}$$

then from (2.6), we implies that $\sum_{k=0}^{\infty} r_k^2 < +\infty$, a contradiction to (2.4). So (2.7) does not hold, i.e., there is a subset $N_1 \subseteq N$ such that $\lim_{k\in N_1, k\to\infty} \|e(x^k)\| = 0$. By the boundedness of $\{x^k\}$, we assume, without loss of the generality, that

$$\lim_{k\in N_1, k\to\infty} x^k = x^\infty, \|e(x^\infty)\| = \lim_{k\in N_1, k\to\infty} \|e(x^k)\| = 0. \tag{2.8}$$

This shows that $x^\infty \in \Omega^*$. Again using (2.5), for any $k, k' \in N_1$ and $k' > k$,

$$0 \leftarrow \|x^{k'} - x^\infty\|^2 \le \|x^{k+1} - x^\infty\|^2 < \|x^k - x^\infty\|^2 \to 0.$$

It follows that $\lim_{k\to\infty} x^k = x^\infty \in \Omega^*$. The proof is completed. □

This result shows that $-d(x)$ is not necessarily strongly descent when we only demand the PC method to be convergent. That is, it suffices that $-d(x)$ is a strictly descent direction with conditions $r_k \to 0$ and $\sum_{k=0}^{\infty} r_k^2 = +\infty$. It is interesting to note that this consistents with a class of descent directions

$$g_k^T d_k \ge \varepsilon_k \cdot \|g_k\| \cdot \|d_k\|, \sum_{k=0}^{\infty} \varepsilon_k^2 = +\infty \tag{2.9}$$

for the unconstrained optimization (See Yuan [13, P75]).

If the iterative form (2.3) is replaced by the following form

$$x^{k+1} = P_\Omega[\overline{x}^{k+1}], \tag{2.10a}$$
$$\overline{x}^{k+1} = x^k - \alpha_k d^k, \quad \forall x^0 \in \Omega, \tag{2.10b}$$
$$\alpha_k = r_k \cdot \|e(x^k)\|^2 / \|d^k\|^2, \tag{2.10c}$$

we can obtain the result similar to Theorem 2.2.

Theorem 2.3. *Under the conditions of Theorem* 2.2, *if* $\{x^k\}$ *is generated by* (2.10), *then it converges to a solution point* x^∞ *of* (1.1).

Proof. Using inequality

$$\|x^{k+1} - x^*\|^2 = \|P_\Omega[\overline{x}^{k+1}] - P_\Omega[x^*]\|^2 \le \|\overline{x}^{k+1} - x^*\|^2$$

and the proof of Theorem 2.2, it is easy to get the proof of this theorem. □

We now consider the linesearch in (2.3) or (2.10). Let

$$\varphi_k(\alpha) \triangleq 2\alpha r_k \|e(x^k)\|^2 - \alpha^2 \cdot \|d^k\|^2, \quad \alpha \ge 0,$$
$$\alpha_k^{opt} \triangleq r_k \cdot \|e(x^k)\|^2 / \|d^k\|^2.$$

Then we easily prove the properties

(i) $\varphi_k(\alpha) \ge 0$, for $0 \le \alpha \le 2\alpha_k^{opt}$;

(ii) For $0 < \delta \le 1$, when α satisfies $\delta \cdot \alpha_k^{opt} \le \alpha \le (2-\delta) \cdot \alpha_k^{opt}$, we have

$$\varphi_k(\alpha) \ge \delta(2-\delta) \cdot \varphi_k(\alpha_k^{opt}). \tag{2.11}$$

This says that the sequence $\{\|x^k - x^*\|^2\}$ must be strictly decreacing for $\alpha_k \in [\delta, 2-\delta] \cdot \alpha_k^{opt}$. For convenience we introduce the following concept.

Definition 2.4. *α_k is called exact linesearch if $\alpha_k = \alpha_k^{opt}$, and inexact linesearch if α_k satisfies* (2.11).

From the above definition, Theorems 2.2 and 2.3, we immediately deduce a general conclusion.

Theorem 2.5. *For problem* (1.1)*, if $-d(x)$ is strictly descent, and $\{x^k\}$ is generated by iterative form*

$$\begin{cases} x^{k+1} = x^k - \alpha_k d^k, \quad \text{or} \quad x^{k+1} = P_\Omega[x^k - \alpha_k d^k], \\ \alpha_k \quad \text{by exact/inexact linesearch,} \end{cases}$$

then $\lim_{k\to\infty} x^k = x^\infty \in \Omega^*$ *when* $\sum_{k=0}^{\infty} r_k^2 = +\infty$.

3. Linear Convergence of PC-Method for (AVI)

In this section, we consider the problem (AVI). Let X^* denote the set of solutions of (AVI). We first quote an error bound result of Luo and Tseng [14, Theorem 2.3], which will play an important role in linear convergence analysis of this section. By studying carefully the proof of this theorem, we readily find that the condition "$x \in X$" can be relaxed to "$x \in R^n$", i.e., feasibility for x is not demanded. So we have

Lemma 3.1 (the extended error bound result). *There is a constant $\varepsilon > 0$ such that for any $x \in R^n$ and $\|e(x)\| \le \varepsilon$,*

$$dist(x, X^*) \le \tau \|e(x)\|, \tag{3.1}$$

where $\tau > 0$ is some constant.

Based on the above result, we can now establish the main result of this section.

Theorem 3.2. *For problem (AVI), if the sequence $\{x^k\}$ generated by PC-Method satisfies condition*

$$\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 - \rho\|e(x^k)\|^2, \quad \rho > 0, \forall x^* \in X^*, \tag{3.2}$$

then $\{x^k\}$ converges to a solution point x^∞ at least Q-linearly.

Proof. By the proof technique of Theorem 2.2, we can easily prove that the sequence $\{x^k\}$ converges to a point $x^\infty \in X^*$. So it suffices that the convergence rate of $\{x^k\}$ is Q-linear. Let

$$x^*(k) = \arg\min\{\|x^k - x\| \ \ |x \in X^*\}, \quad \forall k \in N.$$

By (3.2) we have for any $k \in N$,

$$\begin{aligned} \|x^{k+1} - x^*(k)\|^2 &\le \|x^k - x^*(k)\|^2 - \rho\|e(x^k)\|^2 \\ &= \mathrm{dist}(x^k, X^*)^2 - \rho\|e(x^k)\|^2, \end{aligned}$$

which implies that

$$\mathrm{dist}(x^{k+1}, X^*)^2 \le \mathrm{dist}(x^k, X^*)^2 - \rho\|e(x^k)\|^2. \tag{3.3}$$

Using (3.3) and Lemma 3.1 (since $\|e(x^k)\| \to 0), \exists K_\varepsilon > 0$ such that for any $k \geq K_\varepsilon$,

$$\begin{aligned} \operatorname{dist}(x^{k+1}, X^*)^2 &\leq \operatorname{dist}(x^k, X^*)^2 - \frac{\rho}{\tau^2}\operatorname{dist}(x^k, X^*)^2 \\ &= \delta_1 \cdot \operatorname{dist}(x^k, X^*)^2, \end{aligned} \tag{3.4}$$

where $\delta_1 = 1 - \dfrac{\rho}{\tau^2}$ and $0 < \delta_1 < 1$. Let again

$$\delta_2 = \max\left\{1 - \frac{\rho \cdot \|e(x^0)\|^2}{\operatorname{dist}(x^0, X^*)^2}, \cdots, 1 - \frac{\rho\|e(x^{K_\varepsilon})\|^2}{\operatorname{dist}(x^{K_\varepsilon}, X^*)^2}\right\}.$$

By (3.2), we have $0 < \delta_2 < 1$ and

$$\operatorname{dist}(x^{k+1}, X^*)^2 \leq \delta_2 \cdot \operatorname{dist}(x^k, X^*)^2, \quad \forall k < K_\varepsilon. \tag{3.5}$$

Set $\delta = \max\{\delta_1, \delta_2\}$, then $0 < \delta < 1$. Combining (3.4), (3.5) with $x^k \to x^\infty$, we can obtain the desired result. □

The above theorem shows the convergence behavior of the sequence $\{x^k\}$. In practice, we also concerns with convergence properties of $\{e(x^k)\}$. The following theorem tell us an important fact. In order to attain the goal, we first prove a significant lemma.

Lemma 3.3. *For problem (AVI) and any $x^* \in X^*$, we have*

$$\|e(x)\| \leq (\|I + M\|) \cdot \|x - x^*\|, \quad \forall x \in R^n. \tag{3.6}$$

Proof. By the definition of $e(x)$, $x - e(x) = P_X[x - Mx - q]$. So we have for any $x^* \in X^*$,

$$(x - e(x) - x^*)^T(Mx^* + q) \geq 0, \quad \forall x \in R^n. \tag{3.7}$$

By the essential property of the projection operator, we again obtain

$$(x^* - (x - e(x)))^T((Mx + q) - e(x)) \geq 0, \quad \forall x \in R^n. \tag{3.8}$$

Adding (3.7) and (3.8), and rearranging terms, we deduce

$$\begin{aligned} (x - x^*)^T M(x - x^*) &\leq e(x)^T[(Mx + q) - (Mx^* + q) + x - x^*] - \|e(x)\|^2 \\ &= e(x)^T(I + M)(x - x^*) - \|e(x)\|^2. \end{aligned} \tag{3.9}$$

By $M \geq 0$ and the Cauchy-Schwartz inequality, it follows that

$$\|e(x)\|^2 \leq e(x)^T(I + M)(x - x^*) \leq \|e(x)\| \cdot \|(I + M)\| \cdot \|x - x^*\|.$$

Therefore the lemma is true. □

Theorem 3.4. *For problem (AVI), the sequence $\{e(x^k)\}$ generated by PC-Method with* (3.2) *is R-linearly convergent.*

Proof. By Theorem 3.2, $\{x^k\}$ converges Q-linearly to some point $x^* \in X^*$, and $\exists 0 < \delta < 1$ such that

$$\|x^{k+1} - x^*\|^2 \leq \delta \cdot \|x^k - x^*\|^2, \quad \forall k \in N.$$

That is,

$$\|x^k - x^*\|^2 \le \delta^k \cdot \|x^0 - x^*\|^2, \quad \forall k \in N.$$

From the above inequality and Lemma 3.3, we have

$$\begin{aligned}\|e(x^k)\|^2 &\le (\|I+M\|)^2 \cdot \|x^k - x^*\|^2 \le \|I+M\|^2 \cdot \delta^k \cdot \|x^0 - x^*\|^2 \\ &= C \cdot \delta^k, \quad \forall k \in N,\end{aligned}$$

where $C \triangleq \|I+M\|^2 \cdot \|x^0 - x^*\|^2$. This completes the proof. □

We close this section by mentioning Theorems 3.2 and 3.4's applications to some PC methods.

In 1987, Khobotov [15] proposed a modified extragradient method for monotone variational inequality problem:

$$\overline{x}^k = P_\Omega[x^k - \alpha_k F(x^k)], \tag{3.10a}$$
$$x^{k+1} = P_\Omega[x^k - \alpha_k F(\overline{x}^k)], \tag{3.10b}$$

where α_k is determined by some linesearch rule. Sun [16, 17, 18] used the different linesearch rules. When the problem degenerates into (AVI), it can be obtained the inequality similar to (3.2). So the following result be gained.

Corollary 3.5. *For the problem (AVI), the modified extragradient methods of* [15, 16, 17, 18] *are globally linearly convergent (Here we mean $\{x^k\}$ at Q-linearly, and $\{e(x^k)\}$ at R-linearly).*

In [9], He introduced a PC algorithm for the monotone problem (LVI) :

$$x^{k+1} = x^k - \alpha_k d^k, \quad \forall x^0 \in R^n, \tag{3.11a}$$
$$d^k = (I + M^T)e(x^k), \ \alpha_k = \|e(x^k)\|^2 / \|d^k\|^2, \tag{3.11b}$$

It has been proven that the sequence $\{x^k\}$ generated by (3.11) satisfies the condition (3.2). Hence we have

Corollary 3.6. *For the problem (AVI), the PC algorithm* (3.11) *of He* [9] *is globally linearly convergent.*

Similarly, we can also handle the algorithms in [10, 11]. In Theorem 3.2, however, it is not given the estimate of Q-factor. The next section will discuss this problem.

4. Further Discussion for $M > 0$

In this section, we consider the problem (AVI), and assume that M is positive definite but not necessarily symmetric (denoted by $M > 0$). So the problem (AVI) has the unique solution which we shall denote by x^*, and $\text{dist}(x, X^*) = \|x - x^*\|, \forall x \in R^n$. To start, we derive the global upper and lower error bounds of the solution x^* to such a problem, the idea of proof of which comes from Pang [19].

Lemma 4.1. *for the problem (AVI), if $M > 0$ and $X^* \ne \phi$, the we have*

$$(a) \qquad \frac{1}{\|I+M\|}\|e(x)\| \le \|x - x^*\| \le \frac{\|I+M\|}{\alpha}\|e(x)\|, \quad \forall x \in R^n. \tag{4.1}$$

$$(b) \qquad \frac{\alpha}{\|I+M\|^2}\frac{\|e(x)\|}{\|P_X[-q]\|} \leq \frac{\|x-x^*\|}{\|x^*\|} \leq \frac{\|I+M\|^2}{\alpha}\frac{\|e(x)\|}{\|P_X[-q]\|}, \quad \forall x \in R^n. \tag{4.2}$$

where $\alpha = \lambda_{\min}(\hat{M}) > 0, \hat{M} = \dfrac{M+M^T}{2}$.

Proof. From (3.9) and $M > 0$, we have

$$\alpha\|x-x^*\|^2 \leq e(x)^T(I+M)(x-x^*) - \|e(x)\|^2 \leq \|e(x)\| \cdot \|I+M\| \cdot \|x-x^*\|,$$

which readily lead to the right-hand side inequality of the part (a). This together with (3.6) implies (4.1). Now, setting $x = 0$ in (4.1), we obtain

$$\frac{1}{\|I+M\|}\|P_X[-q]\| \leq \|x^*\| \leq \frac{\|I+M\|}{\alpha}\|P_X[-q]\|. \tag{4.3}$$

Combining (4.1) with (4.3), we obtain (4.2) readily. □

By using Lemma 4.1, we can now study the linear convergence property of the sequence $\{x^k\}$ generated by PC-Method without 'feasibility'.

Theorem 4.2. *For the problem (AVI), if $M > 0, X^* \neq \phi$, and $-d(x)$ is a strictly descent direction satisfying*

$$m_1 \cdot \|e(x)\| \leq \|d(x)\| \leq m_2 \cdot \|e(x)\|, \quad m_1, m_2 > 0, \forall x \in R^n, \tag{4.4}$$

then the sequence $\{x^k\}$ generated by (2.3) converges to the unique solution x^* at Q-linearly if and only if $-d(x)$ is strongly descent.

Proof. By Theorem 2.2, $\{x^k\}$ must converge to x^*, and (2.5) holds, i.e.,

$$\|x^{k+1}-x^*\|^2 \leq \|x^k-x^*\|^2 - r_k \cdot \alpha_k \cdot \|e(x^k)\|^2. \tag{4.5}$$

"$\Longleftarrow$". If $-d(x)$ is strongly descent, then we have from (4.5) and (4.4),

$$\|x^{k+1}-x^*\|^2 \leq \|x^k-x^*\|^2 - \rho\|e(x^k)\|^2, \tag{4.6}$$

where $\rho = r^2/m_2^2 > 0$. This together with (4.1) yields

$$\|x^{k+1}-x^*\|^2 \leq \left(1 - \rho\frac{\alpha^2}{\|I+M\|^2}\right)\|x^k-x^*\|^2. \tag{4.7}$$

This shows that $\{x^k\}$ converges to x^* at Q-linearly.

"$\Longrightarrow$". By contrary, we assume that there is a subsequence $r_{k_i} \to 0$ satisfying

$$(x^{k_i}-x^*)^T d(x^{k_i}) = r_{k_i} \cdot \|e(x^{k_i})\|^2. \tag{4.8}$$

Similarly we can calculate

$$\|x^{k_i+1}-x^*\|^2 = \|x^{k_i}-x^*\|^2 - r_{k_i}^2\frac{\|e(x^{k_i})\|^4}{\|d^{k_i}\|^2}. \tag{4.9}$$

From (4.9) and inequalities

$$\|e(x^{k_i})\| \le \|I+M\|\cdot\|x-x^*\|, \quad \text{(by Lemma 4.1)}$$
$$\|e(x^{k_i})\| \le \frac{1}{m_1}\|d(x^{k_i})\|, \quad \text{(by (4.4))}$$

we readily imply that

$$\frac{\|x^{k_i+1}-x^*\|^2}{\|x^{k_i}-x^*\|^2} = 1 - \frac{r_{k_i}^2\cdot\|e(x^{k_i})\|^4}{\|d^{k_i}\|^2\cdot\|x^{k_i}-x^*\|^2}$$
$$\ge 1 - \frac{r_{k_i}^2\cdot\|I+M\|^2}{m_1^2} \to 1 (i\to\infty)$$

This is to say that $\{x^{k_i}\}$ does not linearly converge to x^*, the proof is completed. □

It is easy to see that the PC algorithm (3.11) of He [9] satisfies the assumptions of Theorem 4.2. Hence a direct result is

Corollary 4.3. *For strongly monotone problem* (AVI), *the PC algorithm of He [9] is globally linearly convergent, and Q-factor* $q \le \left(1-\rho\frac{\alpha^2}{\|I+M\|^2}\right)^{\frac{1}{2}}$.

We here point out that the PC algorithm in [11] has also similar result. The inequality (4.7) clearly indicates that the convergence rate of $\{x^k\}$ depends seriously on the least eigenvalue α of the symmetric part $\hat{M}$ of M (when M is symmetric, α is the least eigenvalue). The more α is small, the more $\delta \triangleq \left(1-\rho\frac{\alpha^2}{\|I+M\|^2}\right)^{\frac{1}{2}}$ approximates to 1, and the convergence rate is slow.

References

[1] Cottle R.W., Dantzig G.B., Complementarity pivot theory of mathematical programming, *Linear Algebra and its Applications,* **1** (1968), 103–125.

[2] Harker P.T., Pang J.S., Finite-Dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications, *Math. Programming,* **48** (1990), 161–220.

[3] Cottle R.W., Pang J.S., Stone R.E., The linear complementarity problem, Academic Press, New York, 1992.

[4] Isac G., Complementarity Problems, Lecture Notes in Mathemtiacs 1528, Springer Verlag, Berlin, 1992.

[5] Goldstein A.A., Convex programming in Hilbert space, *Bull. Am. Math. Soc.,* **70** (1964), 709–710.

[6] Levitin F.S., Polyak B.T., Constrained minimization methods, *USSR Comput. Math. and Math. Phys.,* **6** (1965), 1–50.

[7] He B.S., A projection and contraction method for a class of linear complementarity problems and its application in convex quadratic programming, *Appl. Math. and Optim.,* **25** (1992), 247–262.

[8] He B.S., Stoer J., Solution of projection problems over polytopes, *Numer. Math.,* **61** (1992), 73–90.

[9] He B.S., A new method for a class of linear variational inequalities, *Math. Programming,* **66** (1994), 137–144.
[10] He B.S., Solving a class of linear projection equations, *Numer. Math.,* **68** (1994), 71–80.
[11] He B.S., A class of projection and contraction methods for monotone variational inequalities, *Appl. Math. and Optim.,* **35** (1997), 69–76.
[12] He B.S., On some projection and contraction methods for solving monotone variational inequalities, *Numer. Math. Sinica,* **18** (1996), 54–60.
[13] Yuan Y., Numerical methods for nonlinear programming, Shanghai Scientific and Technical Publishers, 1993.
[14] Luo Z.Q., Tseng P., Error bound and convergence analysis of matrix splitting algorithms for the affine variational inequality problem, *SIAM J. Optim.,* **2** (1992), 43–54.
[15] Khobotov E.N., Modification of the extragradient method for solving variational inequalities and certain optimization problem, *USSR Comput. Math. Phys.,* **27 :** 5 (1987), 120–127.
[16] Sun D., An iterative method for solving variational inequality problems and Complementarity Problems, *Numer. Math. A J. Chinese Universities,* **16** (1994), 145–153.
[17] Sun D., A new step-size skill for solving a class of nonlinear projection equations, *J. Comput. Math.,* **13** : 4 (1995), 357–368.
[18] Sun D., Algorithms and convergence analysis for nonsmooth optimization and nonsmooth equations, Ph.D. Thesis, Institute of Applied Math., Academia Sinica, Beijing, 1994.
[19] Pang J.S., A posteriori error bounds for the linearly constrained variational inequality problem, *Math. Opera. Research,* **12 :** 3 (1987), 474–484.
[20] Wu S.Q., Convergence properties of descent methods for unconstrained minimization, *Optimization,* **26** (1992), 229–237.