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MIXED FINITE ELEMENT METHODS FOR A STRONGLY NONLINEAR PARABOLIC PROBLEM*1)

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Abstract

A mixed finite element method is developed to approximate the solution of a strongly nonlinear second-order parabolic problem. The existence and uniqueness of the approximation are demonstrated and L^2-error estimates are established for both the scalar function and the flux. Results are given for the continuous-time case.

Key words: Finite element method, Nonlinear parabolic problem.

1. Introduction

For second order elliptic problems, the mixed method was described and analyzed by many authors[1–3] in the case of linear equations in divergence form, as well as in [4, 5] for quasilinear or nonlinear problems in divergence form. Johnson and Thomée[6] considered alternative proofs of the previously known error estimates for such methods in the elliptic case. They also analyzed the mixed finite element method for the parabolic equation given by $p_t - \Delta p = f$. Garcia[7] studied the convergence of mixed finite element approximations to quasilinear parabolic equations in the continuous-time case and derived the superconvergent estimates for the difference between the approximate solution and the projection.

In this paper we consider a mixed finite element for approximating the pair (u, p) satisfying second-order, strongly nonlinear parabolic equation

$$\begin{aligned} &u(x,t) = -a(x, \nabla p), \\ &c(x,p)p_t(x,t) + \mathrm{div} u(x,t) = f(x,p,t), \end{aligned} \qquad x \in \Omega, \quad t \in J, \tag{1.1}$$

subject to the following conditions:

$$\begin{aligned} &p(x,0) = p_0(x), &&x \in \Omega, \quad t = 0, \\ &p(x,t) = -g(x,t), &&(x,t) \in \partial\Omega \times J, \end{aligned} \tag{1.2}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded, convex domain with C^2-boundary $\partial\Omega$, and $J = [0, T]$, $a : \overline{\Omega} \times \mathbf{R}^2 \to \mathbf{R}^2$ is cubic continuously differentiable with bounded derivatives through

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third order and has a bounded positive definite Jacobian with respect to the second argument, which implies that ∇p can be locally represented as a function of the flux, say

$$\nabla p = -b(u). \tag{1.3}$$

We shall assume that this representation is global, and that $u \in H^{7/2+\varepsilon_0}(\Omega)^2 \cap C^{0,1}(\overline{\Omega})^2$, $\varepsilon_0 > 0$. Furthermore, assume that the domain of definition of b contains a ball $\mathcal{B}_0$ centered at u in $L^\infty(\Omega)$[5].

The functions $c(x,\nu)$, $f = f(x,\nu,t)$, and $g = g(x,t)$ are continuously differentiable with respect to ν and t. Moreover, there exist constants c_*, c^* and K such that, for all $x \in \overline{\Omega}$, $t \in J$, and $\nu \in \mathbf{R}$,

$$0 < c_* \le c(x,\nu) \le c^*, \tag{1.4}$$

$$|f|, |g|, \left|\frac{\partial c}{\partial \nu}\right|, \ \left|\frac{\partial f}{\partial \nu}\right|, \ \left|\frac{\partial f}{\partial t}\right|, \left|\frac{\partial g}{\partial t}\right| \le K. \tag{1.5}$$

We also assume that the solution $\{u,p\}$ for (1.1)–(1.2) has sufficiently smooth regularity.

2. Formulation of the Mixed Method

Now we let $V = H(\mathrm{div};\Omega) = \{v \in L^2(\Omega)^2\colon \mathrm{div}\, v \in L^2(\Omega)\}$, $W = L^2(\Omega)$. Combining (1.1), (1.2), and (1.3), we arrive at the mixed weak form of (1.1)–(1.2): $(u,p) \in V \times W$ is the solution of the system

$$(b(u),v) - (\mathrm{div} v, p) = \langle g, v\cdot n\rangle, \qquad v \in V, \tag{2.1}$$

$$(c(p)p_t, w) + (\mathrm{div} u, w) = (f(p), w), \qquad w \in W, \tag{2.2}$$

and $p(x,0) = p_0$, where n is the unit exterior normal vector on $\partial\Omega$, $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote, respectively, the $L^2(\Omega)$-inner product and the $L^2(\partial\Omega)$-inner product. We consider the Raviart-Thomas[1] space $V_h \times W_h \subset V \times W$ of index $k > 0$ associated with quasiregular partition T_h of Ω by triangles or quadrilaterals, with boundary elements allowed to have one curved side. The mixed finite element method we shall analyzed is the discrete form of (2.1)–(2.2) and is given by: Find $(u_h, p_h) \in V_h \times W_h$ such that $p_h(0) = P(0)$,

$$(b(u_h),v) - (\mathrm{div} v, p_h) = \langle g, v\cdot n\rangle, \qquad v \in V_h, \tag{2.3}$$

$$(c(p_h)p_{ht}, w) + (\mathrm{div} u_h, w) = (f(p_h), w), \qquad w \in W_h, \tag{2.4}$$

where $P(0)$ is the elliptic mixed method projection (to be defined below) into the finite dimensional space W_h of the inital data function p_0.

3. Mixed Method Projection

For introducing an elliptic projection[8], we shall assume that the following boundary value problem

$$\begin{aligned} &-\mathrm{div}(a(\nabla z)) = f(p) - c(p)p_t, \quad \text{in } \Omega, \\ &z = -g, \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

is solvable for all $p = p(x,t)$, $t \in J$, p being the solution of (1.1)–(1.2). For $t \in J$, define a strongly nonlinear, mixed method, elliptic projection of $V \times W$ onto $V_h \times W_h$ by the map $(u,p) \to (U,P)$ determined by the relations:

$$\begin{aligned} &(b(U),v) - (\operatorname{div} v, P) = \langle g, v\cdot n\rangle, && v \in V_h,\\ &(\operatorname{div} U, w) = (f(p) - c(p)p_t, w), && w \in W_h. \end{aligned} \tag{3.2}$$

Note that the solution p for the problem (1.1)–(1.2) is a solution for the elliptic problem (3.1) for each $t = \tau$.

Let

$$\begin{aligned} &\eta = p - P, \quad \xi = P - p_h,\\ &\rho = u - U, \quad \zeta = U - u_h. \end{aligned} \tag{3.3}$$

Estimates for η, ρ and divρ are given in [5] and are presented in Lemma 1 and 2 without proof.

Lemma 1. *For $t \in J$ and for h sufficiently small,*

$$\begin{aligned} &\|u - U\|_0 \le Ch^r \|u\|_r, \quad \text{for } 1/2 < r \le k+1,\\ &\|div(u-U)\|_0 \le Ch^r\|u\|_{r+1}, \quad \text{for } 0 \le r \le k+1,\\ &\|p - P\|_0 \le Ch^r(\|p\|_r + \|u\|_{r-1}), \quad \text{for } 2 \le r \le k+1. \end{aligned}$$

Lemma 2. *For $t \in J$ and for h, ε sufficiently small,*

$$\begin{aligned} &\|u - U\|_{0,\infty} \le Ch^{r-\frac12}|\ln h|^{\frac12}\|u\|_{r,\infty}, \quad \text{for } 1/2 < r \le k+1,\\ &\|p - P\|_{0,\infty} \le Ch^r(\|p\|_{r,\infty} + \|u\|_{r-\frac12+\varepsilon,\infty}), \quad \text{for } 1 \le r \le k+1, \end{aligned}$$

here and below C is a generic constant depending on $\|u\|_{C^{0,1}(\overline{\Omega})^2}$ or $\|u\|_{7/2+\varepsilon_0}$ [5].

We shall need the following relations, which are integral form of Taylor's formula: for $\mu \in \mathcal{B}_0$,

$$\begin{aligned} b(\mu) - b(u) &= -\,B(u)(u-\mu) + (u-\mu)^\top[\widetilde{H}_1(\mu), \widetilde{H}_2(\mu)](u-\mu)\\ &= -\,\widetilde{B}(\mu)(u-\mu), \end{aligned} \tag{3.4}$$

where $B(u) = \dfrac{\partial b}{\partial u} = \dfrac{\partial(b_1,b_2)}{\partial(u_1,u_2)}$ is the Jacobian of b, a positive define matrix, $H_j = \dfrac{\partial^2 b_j}{\partial u^2}$ $(j = 1,2)$ is the Hessian of b_j, $\zeta^\top[H_1,H_2]\zeta = (\zeta^\top H_1\zeta, \zeta^\top H_2\zeta) \in \mathbf{R}^2$, for $j = 1,2$,

$$\widetilde{H}_j(\mu) = \begin{bmatrix} \displaystyle\int_0^1 (1-s)\frac{\partial^2 b_j}{\partial u_1^2}(u + s[\mu - u])ds & \displaystyle\int_0^1 (1-s)\frac{\partial^2 b_j}{\partial u_1 \partial u_2}(u + s[\mu - u])ds\\ \displaystyle\int_0^1 (1-s)\frac{\partial^2 b_j}{\partial u_1 \partial u_2}(u + s[\mu - u])ds & \displaystyle\int_0^1 (1-s)\frac{\partial^2 b_j}{\partial u_2^2}(u + s[\mu - u])ds \end{bmatrix} \tag{3.5}$$

$$\widetilde{B}(\mu) = \begin{bmatrix} \displaystyle\int_0^1 \frac{\partial b_1}{\partial u_1}(\mu + s[u-\mu])ds & \displaystyle\int_0^1 \frac{\partial b_1}{\partial u_2}(\mu + s[u-\mu])ds\\ \displaystyle\int_0^1 \frac{\partial b_2}{\partial u_1}(\mu + s[u-\mu])ds & \displaystyle\int_0^1 \frac{\partial b_2}{\partial u_2}(\mu + s[u-\mu])ds \end{bmatrix} \tag{3.6}$$

Note $\widetilde{B}(\mu)$ and $\widetilde{H}_j(\mu)$, j=1, 2, are bounded (matrix) functions since a has two continuous and bounded derivatives, and its Jacobian is bounded away from 0. Let us introduce the L^2-projection P_h: $W \to W_h$, and the Raviart-Thomas projection[1] $\pi_h : H^1(\Omega)^2 \to V_h$, which have the following useful commuting property:

$$\text{div} \circ \pi_h = P_h \circ \text{div} : \qquad H^1(\Omega)^2 \to W_h, \tag{3.7}$$

and the following approximation properties[4,5,7]:

$$\begin{aligned}
&||v - \pi_h v||_0 \le Ch^r ||v||_r, \quad 1 \le r \le k+1,\\
&||\text{div}(v - \pi_h v)||_0 \le Ch^r ||\text{div} v||_r, \quad 0 \le r \le k+1,\\
&||w - P_h w||_{0,q} \le Ch^r ||w||_{r,q}, \quad 0 \le r \le k+1,\ 1 \le q \le +\infty.
\end{aligned} \tag{3.8}$$

Lemma 3. *For h sufficiently small,*

$$\begin{aligned}
&||u_t - U_t||_0 \le Ch^r(||u||_r + ||u_t||_r), \quad \textit{for } 1/2 < r \le k+1,\\
&||\mathit{div}(u_t - U_t)||_0 \le Ch^r ||\mathit{div} u_t||_r, \quad \textit{for } 0 \le r \le k+1,\\
&||p_t - P_t||_0 \le Ch^r(||p_t||_r + ||u||_r + ||u_t||_r), \quad \textit{for} \quad 1/2 \le r \le k+1.
\end{aligned}$$

Proof. Let

$$\begin{aligned}
&\theta = p - P_h p, \qquad \sigma = u - \pi_h u,\\
&\tau = P_h p - P, \qquad \delta = \pi_h u - U,
\end{aligned} \tag{3.9}$$

then it follows from (2.1)–(2.2) and (3.2) that

$$\begin{aligned}
&(b(u) - b(U), v) - (\text{div} v, p - P) = 0, \qquad v \in V_h,\\
&(\text{div}(u - U), w) = 0, \qquad w \in W_h.
\end{aligned} \tag{3.10}$$

and, using the mean value theorem, (3.4), we obtain

$$\begin{aligned}
&(\widetilde{B}(U)\rho, v) - (\text{div} v, \eta) = 0, \qquad v \in V_h,\\
&(\text{div}\rho, w) = 0, \qquad w \in W_h.
\end{aligned} \tag{3.11}$$

where $\widetilde{B}(U)$ is given by (3.6) with $\mu = U$. Now, differentiate the above equations with respect to time:

$$\begin{aligned}
&(\widetilde{B}(U)\rho_t, v) - (\text{div} v, \eta_t) = -\left(\frac{\partial \widetilde{B}}{\partial t}(U)\rho, v\right), \quad v \in V_h,\\
&(\text{div}\rho_t, w) = 0, \quad w \in W_h.
\end{aligned} \tag{3.12}$$

Using (3.7) and (3.9), we rewrite (3.12) as

$$\begin{aligned}
&(\widetilde{B}(U)\delta_t, v) - (\text{div} v, \tau_t) = -\left(\frac{\partial \widetilde{B}}{\partial t}(U)\rho, v\right) - (\widetilde{B}(U)\sigma_t, v), \quad v \in V_h,\\
&(\text{div } \delta_t, w) = 0, \quad w \in W_h,
\end{aligned} \tag{3.13}$$

where

$$\frac{\partial \widetilde{B}}{\partial t}(U)=\begin{bmatrix}\int_0^1\Big[\frac{\partial^2 b_1}{\partial u_1^2}(y)y_{1t}+\frac{\partial^2 b_1}{\partial u_1\partial u_2}(y)y_{2t}\Big]ds & \int_0^1\Big[\frac{\partial^2 b_1}{\partial u_1\partial u_2}(y)y_{1t}+\frac{\partial^2 b_1}{\partial u_2^2}(y)y_{2t}\Big]ds\\ \int_0^1\Big[\frac{\partial^2 b_2}{\partial u_1^2}(y)y_{1t}+\frac{\partial^2 b_2}{\partial u_1\partial u_2}(y)y_{2t}\Big]ds & \int_0^1\Big[\frac{\partial^2 b_2}{\partial u_1\partial u_2}(y)y_{1t}+\frac{\partial^2 b_2}{\partial u_2^2}(y)y_{2t}\Big]ds\end{bmatrix} \tag{3.14}$$

$$y=(y_1,y_2)=U+s(u-U),\qquad y_t=U_t+s(u_t-U_t).$$

From Lemma 2, for h sufficiently small, U and $y\in\mathcal{B}_0$ so that $\frac{\partial^2 b}{\partial u_i\partial u_j}(y)$, $i,j=1,2$, and $\widetilde{B}(U)$ are bounded functions and there exists a positive constant λ independent of h and v such that

$$\lambda||v||_0^2\le(\widetilde{B}(U)v,v),\qquad v\in V. \tag{3.15}$$

Take now $v=\delta_t$ and $w=\tau_t$ in (3.13) and add the two equations:

$$\begin{aligned}\lambda||\delta_t||_0^2\le&(\widetilde{B}(U)\delta_t,\delta_t)=-\Big(\frac{\partial\widetilde{B}}{\partial t}(U)\rho,\delta_t\Big)-(\widetilde{B}(U)\sigma_t,\delta_t)\\ \le&C(||u_t||_0+||\sigma_t||_0+||\delta_t||_0)||\rho||_0||\delta_t||_0+C||\sigma_t||_0||\delta_t||_0\\ \le&C(h^{2r}||u||_r^2+||\sigma_t||_0^2)+\varepsilon||\delta_t||_0^2,\end{aligned} \tag{3.16}$$

here, we have used Lemma 1. Note that $(\pi_h u)_t=\pi_h u_t$, since the projection π_h is defined by means of moments over the edges and interiors of the triangles or the rectangles of the partition T_h. So, σ_t can be bounded using (3.8). Thus

$$||\delta_t||_0\le Ch^r(||u||_r+||u_t||_r),\qquad \text{for } 1/2<r\le k+1. \tag{3.17}$$

To estimate τ_t, we apply Lemma 2.2 in [5] to (3.13) and obtain

$$||\tau_t||_0\le C(h||\delta_t||_0+h^2||\text{div}\delta_t||_0+||\rho||_0+||\sigma_t||_0), \tag{3.18}$$

Observing that $\text{div}\delta_t=0$ from (3.7)and (3.17), we have

$$||\tau_t||_0\le Ch^r(||u||_r+||u_t||_r),\qquad \text{for } 1/2<r\le k+1. \tag{3.19}$$

Now,

$$\begin{aligned}&||\eta_t||_0\le||\theta_t||_0+||\tau_t||_0=||p_t-P_hp_t||_0+||\tau_t||_0,\\ &||\rho_t||_0\le||\sigma_t||_0+||\delta_t||_0=||u_t-\pi_hu_t||_0+||\delta_t||_0,\\ &||\text{div}\,\rho_t||_0=||\text{div}\sigma_t||_0=||\text{div}(u_t-\pi_hu_t)||_0,\end{aligned} \tag{3.20}$$

where, we also have used $(P_hp)_t=P_hp_t$. These inequalities suffice to prove Lemma 3. □

Using (3.4), we can rewrite the relations (3.11) as

$$(B(u)\rho,v)-(\text{div}v,\tau)=\Big(\rho^\top[\widetilde{H_1}(U),\widetilde{H_2}(U)]\rho,v\Big),\quad v\in V_h,$$

$$(\text{div}\rho, w) = 0, \quad w \in W_h. \tag{3.21}$$

and differentiate (3.21) in time,

$$\begin{aligned}(B(U)\rho_t, v) - (\text{div}v, \tau_t) = &\left(\rho^\top \left[\frac{\partial \widetilde{H}_1(U)}{\partial t}, \frac{\partial \widetilde{H}_2(U)}{\partial t}\right]\rho, v\right) - \left(\frac{\partial B(u)}{\partial t}\rho,\right) \\ &+ \left(\rho_t^\top[\widetilde{H}_1(U), \widetilde{H}_2(U)]\rho, v\right) + \left(\rho^\top[\widetilde{H}_1(U), \widetilde{H}_2(U)]\rho_t, v\right), \quad v \in V_h, \\ (\text{div}\rho_t, w) = &0, \quad w \in W_h. \end{aligned} \tag{3.22}$$

We can obtain two improved results by using the argument similar to lemma 3.1 and 4.1 in [5]:

$$||\tau||_0 \le Ch^{r+1}||u||_r, \quad \text{for } 1 \le r \le k+1. \tag{3.23}$$

$$||\tau_t||_0 \le Ch^{r+1}(||u||_r + ||u_t||_r), \quad \text{for } 1 \le r \le k+1. \tag{3.24}$$

From (3.24), (3.8) and the inverse hypothesis [9], we have

$$||p_t - P_t||_{0,\infty} \le Ch^r(||u||_r + ||u_t||_r + ||p||_{0,\infty}), \quad \text{for } 1 \le r \le k+1. \tag{3.25}$$

4. Existence and Uniqueness of the Solution of Discrete Problem

The discrete form (2.3)–(2.4) is a nonlinear ordinary differential system in the components of (u_h, p_h), which we must prove is uniquely solvable. We shall follow some of the idea of [4, 5] to use a fixed point argument for the proof of existence. First, we derive from (2.1)–(2.2), (2.3)–(2.4), and (3.10) the following useful error equations:

$$\begin{aligned}&(b(U) - b(u_h), v) - (\text{div}v, P - p_h) = 0, \quad v \in V_h, \\ &(c(p)p_t - c(p_h)p_{ht}, w) + (\text{div}(U - u_h), w) = (f(p) - f(p_h), w), \quad w \in W_h,\end{aligned} \tag{4.1}$$

with $(P - p_h)|_{t=0} = 0$. By using (3.4)–(3.6) with $\mu = u_h$ and u replaced by U, we rewrite (4.1) as

$$\begin{aligned}&(B(U)(U - u_h), v) - (\text{div}v, P - p_h) \\ &\qquad =((U - u_h)^\top[\widetilde{H}_1(u_h), \widetilde{H}_2(u_h)](U - u_h), v), \quad v \in V_h, \\ &(c(p_h)(P - p_h)_t, w) + (\text{div}(U - u_h), w) + ([c(P) - c(p_h)]P_t - [f(P) - f(p_h)], w) \\ &\qquad =(f(p) - f(P) - c(p)(p - P)_t - [c(p) - c(P)]P_t, w), \ w \in W_h,\end{aligned} \tag{4.2}$$

with $(P - p_h)|_{t=0} = 0$. Let h be small enough that $U \in V_h \cap \mathcal{B}_0$, and choose a ball $\mathcal{B}_1$ centered at U such that $\mathcal{B}_1 \subset \mathcal{B}_0$ with respect to the L^∞-norm.

Define $(y, q) = \phi((\mu, \beta))$ as a map of $(\mathcal{B}_1 \cap V_h) \times W_h$ into $V_h \times W_h$ given by

$$\begin{aligned}&(B(U)(U - y), v) - (\text{div}v, P - q) = ((U - \mu)^\top[\widetilde{H}_1(\mu), \widetilde{H}_2(\mu)](U - \mu), v), \ v \in V_h, \\ &(c(q)(P - q)_t, w) + (\text{div}(U - y), w) + ([c(P) - c(q)]P_t - [f(P) - f(q)], w)\end{aligned}$$

$$=(f(p)-f(P)-c(p)(p-P)_t-[c(p)-c(P)]P_t,w),\ w\in W_h, \tag{4.3}$$

with $(P-q)|_{t=0}=0$. Note that, since the left-hand side of (4.3) corresponds to the mixed finite element for a quasilinear parabolic operator with $B(U)$ positive definite and $\widetilde{H}_1(\cdot)$ and $\widetilde{H}_2(\cdot)$ uniformly bounded on $\mathcal{B}_1$, the operator ϕ is well defined [7]. Clearly, in order to establish the solvability of (2.3)–(2.4), it suffices to prove the following theorem (compare (4.2) with (4.3)):

Theorem 1. *For h sufficiently small, ϕ has a fixed point.*

By Brouwer′s fixed-point theorem, Theorem 1 will be true if we prove the following result.

Theorem 2. *For $\delta(0<\delta<1)$ sufficiently small (depending on h via the inverse inequality* (4.6)*, and smaller than the radius of $\mathcal{B}_1$ so that ϕ is well defined on $\mathcal{B}_1$), let*

$$\mathcal{B}_\delta=\Big\{(\mu,\beta)\Big|||U-\mu||_{L^\infty(J;L^2)}+||P-\beta||_{L^\infty(J;L^2)}\le\delta\Big\},$$

then ϕ maps the ball $\mathcal{B}_\delta$ of radius δ of $V_h\times W_h$, centered at (U,P), into itself.

Proof. Let $(U,P)\in\mathcal{B}_\delta$. Setting $v=U-y$, $w=P-q$ in (4.3) and adding the two relations, we can get

$$\begin{aligned}(c(q)(P-q)_t,P-q)&+(B(U)(U-y),U-y)\\ &=((U-\mu)^\top[\widetilde{H}_1(\mu),\widetilde{H}_2(\mu)](U-\mu),U-y)\\ &\quad+(f(p)-f(P)-c(p)(p-P)_t-[c(p)-c(P)]P_t\\ &\quad+f(P)-f(q)-[c(P)-c(q)]P_t,P-q),\end{aligned} \tag{4.4}$$

Then, from (1.5), (3.6) and Lemma 1, 2, and 3 we bound the right-hand of (4.4):

$$\begin{aligned}|((U-\mu)^\top[\widetilde{H}_1(\mu),\widetilde{H}_2(\mu)](U-\mu),U-y)|&\le C||U-\mu||_{0,4}^2||U-y||_0\\ &\le Ch^{-2}||U-\mu||_0^4+\varepsilon_1||U-y||_0^2,\end{aligned} \tag{4.5}$$

here, we used the following "inverse-type" estimate[9]:

$$||v||_{0,\theta}\le Ch^{2/\theta-2/\nu}||v||_{0,\nu},\ \text{for } 2\le\theta,\nu\le+\infty, v\in V_h, \tag{4.6}$$

$$\begin{aligned}|(f(p)-f(P)&-c(p)(p-P)_t-[c(p)-c(P)]P_t+f(P)-f(q)-[c(P)-c(q)]P_t,P-q)|\\ &\le C\Big[||p-P||_0+||(p-P)_t||_0+||p-P||_0||P_t||_{0,\infty}\\ &\quad+(1+||P_t||_{0,\infty})||P-q||_0\Big]||P-q||_0\le C_1h^{2s}+C||P-q||_0^2,\end{aligned} \tag{4.7}$$

where $s=7/2+\varepsilon_0$,

$$C1=C_1\Big(||u||_{C^{0,1}(\overline{\Omega})^2},||u||_{7/2+\varepsilon_0},||u_t||_{7/2+\varepsilon_0},||p||_{7/2+\varepsilon_0},||p_t||_{7/2+\varepsilon_0}\Big). \tag{4.8}$$

In order to get an estimate for $(c(q)(P-q)_t,P-q)$, we use an argument due to Wheeler[8]. We note that

$$(c(q)(P-q)_t,P-q)=\frac{d}{dt}\int_\Omega R(q,P-q,x)dx$$

$$-\int_\Omega\Big(\int_0^{P-q} c_p(P-\alpha)P_t\alpha d\alpha\Big)dx, \tag{4.9}$$

where

$$R(q,P-q,x)=\int_0^{P-q} c(x,P-\alpha)\alpha d\alpha. \tag{4.10}$$

Since

$$\Big|\int_\Omega\Big(\int_0^{P-q} c_p(P-\alpha)P_t\alpha d\alpha\Big)dx\Big| \le C||P_t||_{0,\infty}||P-q||_0^2, \tag{4.11}$$

and according to (1.4), for each (x,t)

$$\frac{1}{2}c_*|P-q|^2 \le R(q,P-q,x) \le \frac{1}{2}c^*|P-q|^2. \tag{4.12}$$

Now, we can find an estimate for (4.4) using the estimate above and the positive definiteness of $B(U)$,

$$\lambda_0||v||_0^2 \le (B(U)v,v), \quad \lambda_0>0, \quad v\in V, \tag{4.13}$$

to obtain the evolution inequality

$$\begin{aligned}&\frac{d}{dt}\int_\Omega R(q,P-q,x)dx+\lambda_0||U-y||_0^2\\ \le& Ch^{2s}+Ch^{-2}||U-\mu||_0^4+C||P-q||_0^2+\varepsilon_1||U-y||_0^2,\end{aligned} \tag{4.14}$$

Integrate (4.14) in time and note that $(U,P)\in\mathcal{B}_\delta$, we have

$$\begin{aligned}R(q,P-q,x)+\lambda_0||U-y||^2_{L^2(J;L^2)} &\le Ch^{2s}+Ch^{-2}||U-\mu||^4_{L^4(J;L^2)}+C\int_0^t ||P-q||_0^2 d\tau\\ &\le Ch^{2s}+Ch^{-2}||U-\mu||^4_{L^\infty(J;L^2)}+C\int_0^t ||P-q||_0^2 d\tau\\ &\le C(h^{2s}+h^{-2}\delta^4)+C\int_0^t ||P-q||_0^2 d\tau,\end{aligned} \tag{4.15}$$

where, we used that $L^\infty \hookrightarrow L^4$. Apply (4.12) and Gronwall's lemma, we obtain the following estimate:

$$||U-y||_{L^2(J;L^2)}+||P-q||_{L^\infty(J;L^2)} \le C_2(h^s+h^{-1}\delta^2). \tag{4.16}$$

Now, we choose $v=U-y$ in the first relation of (4.3) and bound it as (4.18) by using (4.5), (4.13), (4.16) and the inverse estimate:

$$\|\mathrm{div}\cdot v\|_0 \le Ch^{-1}\|v\|_0, \quad \text{for } v\in V_h. \tag{4.17}$$

$$\|U-y\|_0 \le h^{-1}[\|P-q\|_0+\|U-\mu\|_0^2] \le C_3(h^{s-1}+h^{-2}\delta^2). \tag{4.18}$$

Let $K=\max(C_2,C_3)$. Since we want $Kh^{s-1}\le\dfrac{\delta}{2}$ and $Kh^{-2}\delta^2\le\dfrac{\delta}{2}$ in (4.16) and (4.18), we need

$$2Kh^{s-1}\le\delta\le\frac{1}{2K}h^2. \tag{4.19}$$

Note that $s = 7/2 + \varepsilon_0$. Let $h \leq (2K)^{-\frac{2}{s-3}} = (2K)^{\frac{4}{1+2\varepsilon_0}}$ and $\delta = 2Kh^{s-1}$. It follows that (4.19) holds, and then (4.16) and (4.18)–(4.19) yields

$$||U - y||_{L^\infty(J;L^2)} \leq \delta \quad \text{and} \quad ||P - q||_{L^\infty(J;L^2)} \leq \delta, \tag{4.20}$$

which concludes the proof. □

Remark 1. Note that Theorem 2 not only proves that (2.3)–(2.4) is solvable, but also that the solution is close to (u, p). Specifically, for small h,

$$||U - u_h||_{V_h} + ||P - p_h||_0 \leq Ch^{\frac{5}{2}+\varepsilon_0}.$$

By the inverse inequality(4.6), this implies that

$$||U - u_h||_{0,\infty} \leq Ch^{-1}||U - u_h||_0 \leq \overline{C}h^{\frac{3}{2}+\varepsilon_0}, \tag{4.21}$$

where $\overline{C}$ depends on $||u||_{C^{0,1}(\overline{\Omega})^2}$ and the norms of u, u_t, p, and p_t in space $H^{7/2+\varepsilon_0}(\Omega)$ (see (4.8)). We shall need this estimate in the argument below.

We can also show that the solution of (2.3)–(2.4) is unique (near (u, p)).

Theorem 3. *Let* (u_h, p_h) *and* (v_h, q_h) *be solution of* (2.3)–(2.4). *Then,* $u_h = v_h$ *and* $p_h = q_h$.

Proof. Let $\Gamma = u_h - v_h$ and $S = p_h - q_h$. Then, (2.3)–(2.4) implies that $(\Gamma, S) \in V_h \times W_h$ satisfies the relations

$$\begin{aligned}&(b(u_h) - b(v_h), v) - (\text{div} v, S) = 0, \quad v \in V_h,\\ &(c(p_h)p_{ht} - c(q_h)q_{ht}, w) + (\text{div}\,\Gamma, w) = (f(p_h) - f(q_h), w), \; w \in W_h,\end{aligned} \tag{4.22}$$

with $S|_{t=0} = 0$. Using the mean value theorem (3.4), we rewrite (4.22) as

$$\begin{aligned}&(\widetilde{B}(u_h)\Gamma, v) - (\text{div} v, S) = 0, \quad v \in V_h,\\ &(c(p_h)S_t, w) + (\text{div}\Gamma, w) = (f(p_h) - f(q_h), w) + ([c(q_h) - c(p_h)]q_{ht}, w), \quad w \in W_h.\end{aligned} \tag{4.23}$$

Now, if the choices $v = \Gamma$ and $w = S$ are made in (4.23), the following equation is obtained after these two equations are added:

$$(c(p_h)S_t, S) + (\widetilde{B}(u_h)\Gamma, \Gamma) = (f(p_h) - f(q_h), S) + ([c(q_h) - c(p_h)]q_{ht}, S). \tag{4.24}$$

For h sufficiently small, the positive definiteness of $B(u)$ together with (4.21) imply the positive definiteness of $\widetilde{B}(u_h)$. That is,

$$\tilde{\lambda}||v||_0^2 \leq (\widetilde{B}(u_h)v, v), \quad v \in V_h, \tag{4.25}$$

where $\tilde{\lambda} > 0$ is independent of h and v. Using the same argument in (4.9)-(4.12), we can rewrite (4.24) as

$$\frac{d}{dt}\int_\Omega R(q_h, -S, x)dx + \tilde{\lambda}||\Gamma||_0^2 \leq K(1 + ||q_{ht}||_{0,\infty})||S||_0^2, \tag{4.26}$$

where

$$\frac{1}{2}c_*|S|^2 \leq R(q_h,-S,x) \leq \frac{1}{2}c^*|S|^2. \tag{4.27}$$

Using (4.26) and a Gronwall argument we have

$$||S(t)||_0 + ||\Gamma||_{L^2(J;L^2)} \leq C(t)||S(0)||_0. \tag{4.28}$$

so uniqueness is established.

5. Superconvergence for the Difference Between the Approximate Solution and the Projection

In this section we derive superconvergence results using an argument similar to that used by Garcia[7]. We show that if $(U,P)\in V_h\times W_h$ satisfies (3.2) and (u_h,p_h) satisfies (2.3) and (2.4), then under certain assumptions

$$||U-u_h||_{L^2(L^2)} + ||P-p_h||_{L^\infty(L^2)} \leq O(h^{k+2}).$$

This superconvergence result is useful to prove the following theorem.

Theorem 4. *There is a constant $C>0$, independent of h, such that*

$$\begin{aligned}&||u-u_h||_{L^2(J;L^2)} + ||p-p_h||_{L^\infty(J;L^2)} \leq C(u,p)h^r,\\ &C(u,p) = C\Big(||u||_{L^2(H^r)} + ||u_t||_{L^2(H^r)} + ||p||_{L^2(H^r)} + ||p_t||_{L^2(H^r)}\Big)\end{aligned} \tag{5.1}$$

for $2\leq r\leq k+1$, $k>0$.

In order to prove Theorem 4 we need to derive estimates for $||U-u_h||_{L^2(J;L^2)}$ and $||P-p_h||_{L^\infty(J;L^2)}$. The following results will often be used in the argument below.

Result 1. If $\overline{F}$ is the average value of $F(p)$ on each element of the partition T_h, then

$$\begin{aligned}(F(p)\rho,\beta) =&(\overline{F}\rho,\beta) + \Big((F(p)-\overline{F})\rho,\beta,\beta\Big)\\ \leq&(\overline{F}\rho,\beta) + ||F-\overline{F}||_{L^\infty}||\rho||_0||\beta||_0.\end{aligned}$$

Result 2. If $||\nabla g||_{L^\infty}\leq K$ and $\bar{g}$ is the average value of g on each element of T_h, then

$$|(g(p)\rho,\psi) - (\bar{g}\rho,\psi)| \leq CKh||\rho||_0||\psi||_0.$$

Lemma 4. *If $\zeta = U-u_h$ and $\xi = P-p_h$, then there is a constant C independent of h such that*

$$||\zeta||_{L^2(J;L^2)} + ||\xi||_{L^\infty(J;L^2)} \leq C(u,p)h^{r+1}, \tag{5.2}$$

for $1\leq r\leq k+1$, $k>0$, and $C(u,p)$ was given by (5.1).

Proof. First, we rewrite the error equations (4.1) as

$$\begin{aligned}&(\widetilde{B}(u_h)\zeta,v) - (\mathrm{div}v,\xi) = 0,\quad v\in V_h,\\ &(c(p_h)\xi_t,w) + (\mathrm{div}\zeta,w) = ([c(p_h)-c(p)]p_t,w) + (c(p_h)(P-p)_t,w)\\ &\qquad + (f(p)-f(p_h),w),\quad w\in W_h,\end{aligned} \tag{5.3}$$

choose $v = \zeta$ and $w = \xi$ as the test functions and add the two relations of (5.3):

$$(\widetilde{B}(u_h)\zeta, \zeta) + (c(p_h)\xi_t, \xi) \\ = ([c(p_h) - c(p)]p_t, \xi) + (c(p_h)(P - p)_t, \xi) + (f(p) - f(p_h), \xi). \tag{5.4}$$

We now bound each of the terms on the right-hand side of (5.4) using Result 1, 2 and Lemma 2, 3:

$$\begin{aligned}([c(p) - c(p_h)]p_t, \xi) = &([c(p) - c(P_hp)]p_t, \xi) + ([c(P_hp) - c(P)]p_t, \xi) \\ &+ ([c(P) - c(p_h)]p_t, \xi) \\ \leq &(c_p(p)(p - P_hp)p_t, \xi) + \Big(\frac{1}{2}\tilde{c}_{pp}(p - P_hp)^2 p_t, \xi\Big) \\ &+ C||p_t||_{0,\infty}||\tau||_0||\xi||_0 + C||p_t||_{0,\infty}||\xi||_0^2. \end{aligned} \tag{5.5}$$

Using Result 1 and 2 with $g(p) = c_p(p)p_t$, we obtain

$$\begin{aligned}([c(p) - c(p_h)]p_t, \xi) \leq &Ch||\theta||_0||\xi||_0 + C||\theta||_{0\infty}||\theta||_0||\xi||_0 \\ &+ C||p_t||_{0,\infty}||\tau||_0||\xi||_0 + C||p_t||_{0,,\infty}||\xi||_0^2 \end{aligned} \tag{5.6}$$

Next,

$$\begin{aligned}\Big|(f(p) - f(p_h), \xi)\Big| = &\Big|(([f(p) - f(P_hp)] + [(f(P_hp) - f(P)] \\ &+ [(f(P) - f(p_h)], \xi)\Big| \leq C\left\{h||\theta||_0||\xi||_0 + ||\tau||_0||\xi||_0 + ||\xi||_0^2\right\}, \end{aligned} \tag{5.7}$$

$$\begin{aligned}(c(p_h)(p - P)_t, \xi) = &(c(p)\eta_t, \xi) + ([c(p_h) - c(p)]\eta_t, \xi) \\ = &(c(p)\theta_t, \xi) + (c(p)\tau_t, \xi) + ([c(p_h) - c(P)]\eta_t, \xi) \\ &+ ([c(P) - c(p)]\eta_t, \xi) \\ \leq &Ch||\theta_t||_0||\xi||_0 + C||\tau_t||_0||\xi||_0 + C||\eta_t||_{0,\infty}||\xi||_0^2 \\ &+ C||\eta||_{0,\infty}||\eta_t||_0||\xi||_0. \end{aligned} \tag{5.8}$$

Now, we can find an estimate for (5.4) using the estimates above to obtain the evolution inequality

$$(c(p_h)\xi_t) + (\widetilde{B}(u_h)\zeta, \zeta) \leq C||\xi||_0^2 + C\mathcal{R}, \tag{5.9}$$

where we have used the estimate (3.25) and $\mathcal{R}$ was used to simplify

$$\begin{aligned}\mathcal{R}_1 = &\{h^2||\theta||_0^2 + ||\theta||_{0,\infty}^2||\theta||_0^2 + ||\tau||_0^2 + h^2||\theta_t||_0^2 \\ &+ ||\tau_t||_0^2 + ||\eta||_{0,\infty}^2||\eta_t||_0^2\} \end{aligned} \tag{5.10}$$

Using (3.8), (3.23)-(3.24) and Lemma 2 and 3, it is easy to see that

$$\mathcal{R}_1 \leq C\left(||u||_r + ||u_t||_r + ||p||_r + ||p_t||_r\right) h^{2r+2},\ 1 \leq r \leq k+1,\ k > 0.$$

We also note that

$$(c(p_h)\xi_t,\xi)=\frac{d}{dt}\int_\Omega R(p_h,\xi,x)dx-\int_\Omega\left(\int_0^\xi c_p(P-\alpha)P_t\alpha d\alpha\right)dx, \tag{5.11}$$

where

$$R(p_h,\xi,x)=\int_0^{P-p_h}c(x,P-\alpha)\alpha d\alpha. \tag{5.12}$$

Since

$$\left|\int_\Omega\left(\int_0^\xi c_p(P-\alpha)P_t\alpha d\alpha\right)dx\right|\le C||P_t||_{0,\infty}||\xi||_0^2, \tag{5.13}$$

and if $0<c_*\le c(x,t)\le c^*$ according to (1.5), then for each (x,t),

$$\frac{1}{2}c_*|P-p_h|^2\le R(p_h,\xi,x)\le\frac{1}{2}c^*|P-p_h|^2. \tag{5.14}$$

Using (5.11) and (5.13) we rewrite (5.9) as

$$\frac{d}{dt}\int_\Omega R(p_h,\xi,x)dx+(\widetilde{B}(u_h)\zeta,\zeta)\le C||\xi|_0^2+C\mathcal{R}_1. \tag{5.15}$$

Integrate (5.15) in time. Applying (4.25), Gronwall′s Lemma, and (5.14), we obtain the following estimate:

$$||\zeta||_{L^2(J;L^2)}+||\xi||_{L^\infty(J;L^2)}\le C_Th^{r+1}\int_0^T(||u||_r+||u_t||_r+||p||_r+||p_t||_r)d\tau\le C(u,p)h^{r+1}, \tag{5.16}$$

for $1\le r\le k+1$, $k>0$. So, Lemma 4 is established. □

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References

[1] P.A. Raviart, J.M. Thomas, A mixed finite element method for 2nd order elliptic problems, *Math. Aspects of the Finite Element Method, Lecture Notes in Math.,* 606 (Springer, Berlin, 1977), 292–315.

[2] J.Douglas, Jr., J.E. Roberts, Global estimates for mixed finite element methods for second order elliptic equations, *Math. Comp.,* **44**:169 (1985), 39–52.

[3] F.Brezzi, On the existence uniqueness, and approximation of saddle point problems arising from Lagrangian multipliers, *RAIRO Anal. Numér.,* **8** (1974), 129–151.

[4] F.A. Milner, Mixed finite element methods for quasilinear second-order elliptic problem, *Math. Comp.,* **44** (1985), 303–320.

[5] F.A. Milner, E.J. Park, A mixed finite element method for a strongly nonlinear second order elliptic problem, *Math. Comp.,* **64**:211 (1995), 9730-988.

[6] C. Johnson, V. Thomée, Error estimates for some mixed finite element methods for parabolic type problems, *RAIRO. Anal. Numér.,* **15** (1981), 41–78.

[7] S.M.F. Garcia, Improved error estimates for mixed finite-element approximations for non-linear parabolic equations: the continuous-time case, *Numer. Methods Partial Different, Equations* **10** (1994), 129–147.

[8] M.F. Wheeler, A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations, *SIAM J. Numer. Anal.,* **10** (1973), 723–759.

[9] Ph.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam. 1978.