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KAHAN'S INTEGRATOR AND THE FREE RIGID BODY DYNAMICS*

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Abstract

Kahan's integrator for the free rigid body dynamics is described and some of its properties are pointed out.

Key words: Kahan's integrator, Free rigid body.

1. Introduction

Among other unconventional numerical methods, Kahan[1] has suggested a discretization of a simple Lotka-Voltera system with the property that the computed points do not spiral. The motivation of this behaviour was given recently by Sanz-Serna[3] by showing that Kahan's integrator is a symplectic one.

The goal of our paper is to study some properties of this integrator in the particular case of the free rigid body.

2. The Free Rigid Body

Let $SO(3)$ be the Lie group of all 3×3 orthogonal matrices with determinant one and $so(3)$ its Lie algebra. It can be canonically identified with $\mathbf{R}^3$ via the map "$\wedge$" given by:

$$\wedge : \begin{bmatrix} 0 & -m_3 & m_1 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix} \in so(3) \mapsto \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \in \mathbf{R}^3.$$

Moreover, "$\wedge$" is a Lie algebra isomorphism between $(so(3), [\cdot,\cdot])$ and $(\mathbf{R}^3, \times)$.

Then the Euler angular momentum equations of the free rigid body can be written on $(so(3))^* \simeq (\mathbf{R}^3)^* \simeq \mathbf{R}^3$ in the following form:

$$\begin{cases} \dot{m}_1 = a_1 m_2 m_3 \\ \dot{m}_2 = a_2 m_1 m_3 \\ \dot{m}_3 = a_3 m_1 m_2 \end{cases} \tag{2.1}$$

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where

$$a_1 = \frac{1}{I_3} - \frac{1}{I_2};\ a_2 = \frac{1}{I_1} - \frac{1}{I_3};\ a_3 = \frac{1}{I_2} - \frac{1}{I_1},$$

I_1, I_2, I_3 being the components of the inertia tensor and we suppose as usually that

$$I_1 > I_2 > I_3.$$

It is not hard to see that the system (2.1) is an Hamilton-Lie-Poisson system with the phase space $\mathbf{R}^3$, the Poisson bracket given by the matrix:

$$\Pi = \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix}, \tag{2.2}$$

which is in fact the minus-Lie-Poisson structure on $so(3)^*$ and the Hamiltonian H given by

$$H = \frac{1}{2}\left[\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right]. \tag{2.3}$$

Moreover, a Casimir of our configuration $(\mathbf{R}^3, \Pi)$ is given by

$$C = \frac{1}{2}[m_1^2 + m_2^2 + m_3^2], \tag{2.4}$$

i.e. that the coadjoint orbits of $(so(3))^* \simeq \mathbf{R}^3$ are concentric spheres.

Since H and C are constants of motion it follows that the dynamics takes place at the intersection of the ellipsoid

$$H = \text{constant}$$

with the sphere

$$C = \text{constant}.$$

3. Kahan's Integrator

For the free rigid body equations (2.1), Kahan's integrator can be written in the following form:

$$\begin{cases} m_1^{n+1} - m_1^n = \dfrac{ha_1}{2}(m_2^{n+1}m_3^n + m_3^{n+1}m_2^n) \\ m_2^{n+1} - m_2^n = \dfrac{ha_2}{2}(m_1^{n+1}m_3^n + m_3^{n+1}m_1^n) \\ m_3^{n+1} - m_3^n = \dfrac{ha_3}{2}(m_2^{n+1}m_1^n + m_1^{n+1}m_2^n) \end{cases} \tag{3.1}$$

Now a long but straightforward computation or using eventually MAPLE V leads us to:

Theorem 3.1. *The following statements are equivalent:*
(i) *Kahan's integrator is a Poisson integrator;*
(ii) *Kahan's integrator is energy preserving.*

(iii) *Kahan's integrator preserves the coadjoint orbits of* $(so(3))^*$, *i.e. it is Casimir preserving.*

(iv) *The free rigid body is a symmetric one.*

Proof. Let

$$H_n = \frac{1}{2}\left[\frac{(m_1^n)^2}{I_1} + \frac{(m_2^n)^2}{I_2} + \frac{(m_3^n)^2}{I_3}\right]$$

and

$$C_n = \frac{1}{2}\left[(m_1^n)^2 + (m_2^n)^2 + (m_3^n)^2\right].$$

Then using eventually MAPLE V we can deduce that

$$H_{n+1} - H_n = \frac{-h^3 a_1 a_2 a_3 (2m_1^n + h a_1 m_2^n m_3^n)(2m_2^n + h a_2 m_1^n m_3^n)(2m_3^n + h a_3 m_1^n m_2^n) H_n}{\{4 - h^2[a_2 a_3 (m_1^n)^2 + a_1 a_3 (m_2^n)^2 + a_1 a_2 (m_3^n)^2] - h^2 a_1 a_2 a_3 m_1^n m_2^n m_3^n\}^2},$$

$$C_{n+1} - C_n = \frac{-h^3 a_1 a_2 a_3 (2m_1^n + h a_1 m_2^n m_3^n)(2m_2^n + h a_2 m_1^n m_3^n)(2m_3^n + h a_3 m_1^n m_2^n) C_n}{\{4 - h^3[a_2 a_3 (m_1^n)^2 + a_1 a_3 (m_2^n)^2 + a_1 a_2 (m_3^n)^2] - h^3 a_1 a_2 a_3 m_1^n m_2^n m_3^n\}^2},$$

for each $n \in \mathbf{N}^*$, and then our assertion follows easily. □

Remark 3.1 In the particular case of the symmetric free rigid body Kahan's integrator reduces to the midpoint rule and we refined some results from [3].

Remark 3.2 Similar results can be obtained for the more general case of the rigid body with three spinning rotors.

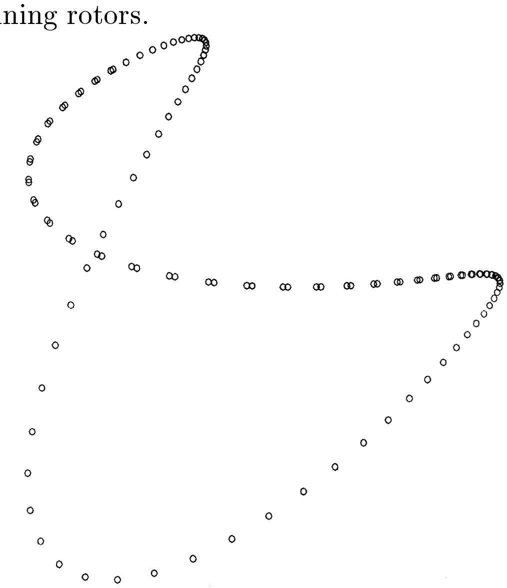

Fig.1. Kahan's integrator for the free rigid body with: $I_1 = 1$; $I_2 = \frac{1}{2}$; $I_3 = \frac{1}{3}$

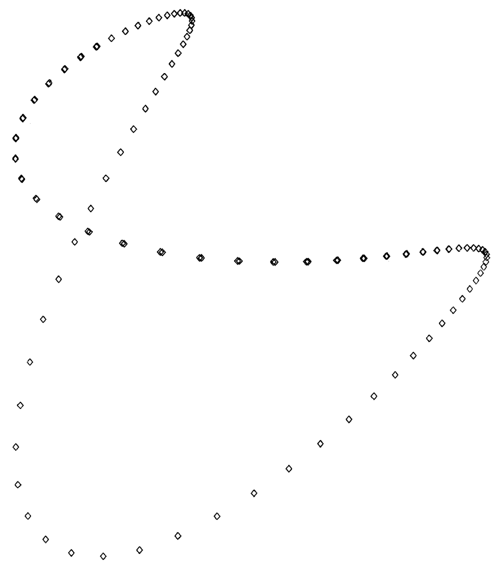

Fig.2. 4th order Runge-Kutta integrator for the free rigid body with: $I_1 = 1$; $I_2 = \frac{1}{2}; I_3 = \frac{1}{3}$

In this last section we want to make a comparison between Kahan's integrator and the 4th order Runge-Kutta integrator. The results can be found in Fig.1 and Fig.2. It is clear that both algorithms lead to almost the same picture. However, Kahan's integrator has the advantage that it is more convenient for implementation.

References

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