Journal of Computational Mathematics, Vol.17, No.3, 1999, 275–284.

# A KIND OF IMPLICIT ITERATIVE METHODS FOR ILL-POSED OPERATOR EQUATIONS*[1)]

Guo-qiang He Lin-xian Liu
(*Department of Mathematics, Shanghai University, Jiading Campus, Shanghai 201800, China*)

**Abstract**

In this paper we propose a kind of implicit iterative methods for solving ill-posed operator equations and discuss the properties of the methods in the case that the control parameter is fixed. The theoretical results show that the new methods have certain important features and can overcome some disadvantages of Tikhonov-type regularization and explicit iterative methods. Numerical examples are also given in the paper, which coincide well with theoretical results.

*Key words*: Ill-posed equation, Implicit iterative method, Control parameter, Discrepamcy principle, Optimal convergence rate

## 1. Introduction

Let $X, Y$ be two real Hilbert spaces and let $A : X \to Y$ be a bounded linear operator. Consider the operator equation

$$Ax = y. \tag{1.1}$$

If $R(A)$, i.e., the range of $A$, is nonclosed in $Y$, equation (1.1) is ill-posed[1]. Many important problems in applied sciences result in this kind of equations[2,3]. In this paper we consider the Moore-Penrose generalized solution $x^+ = A^+y$ to equation(1.1), where $A^+$ is the Moore-Penrose generalized inverse of operator $A$[1]. $A^+y$ exists if and only if y $\in D(A^+) = R(A) + R(A)^\perp$. In practice, instead of (1.1) we usually only have a perturbed version of equation

$$Ax = y_\delta, \tag{1.2}$$

where the perturbed right-hand term $y_\delta \in B_\delta(y) = \{z \in Y | \|Q(z-y)\| \le \delta\}$ with $\delta > 0$ being a known error level and $Q$ being the orthogonal projective operator from $Y$ onto $\overline{R(A)}$. A well-known kind of methods to obtain a suitable approximation of $x^+$ by using the perturbed equation (1.2) are regularization methods which can be constructed by variation methods or the spectrum of operator $A^*A$[1,2]. Another usually used approach is iterative method. In 1951, Landweber[4] proposed the first iterative method to solve ill-posed operator equations, though the convergence rate of the method is very slow. The next breakthrough was made by Nemirovskii and Palyak[5] and Brakhage[6] who developed independently iterative procedures of so-called $\nu$-method. In [7], Hanke

* Received April 18, 1996.
1) This work was supported by The National Natural Science Foundation of China

analysed all above-mentioned methods and some others, and established a framework for explicit iterative methods. However, the explicit iterations discussed in [7] still have some disadvantages.

We will discuss the following kind of implict iterative methods for equation (1.1)

$$(A^*A+\alpha_k I)x_k = A^*y+\alpha_k x_{k-1}, \quad k=1,2,\cdots \tag{1.3}$$
$$x_0 \text{ given,}$$

where $\alpha_k > 0$ are some parameters and $A^*: Y \to X$ is the adjoint operator of $A$. In this paper, we assume all $\alpha_k$ are equal and hence (1.3) becomes

$$(A^*A+\alpha I)x_k = A^*y+\alpha x_{k-1}, \quad k=1,2,\cdots \tag{1.4}$$

In a relative paper we will consider the general case.

## 2. Convergence Properties for Nonperturbed Equation(1.1)

Iteration (1.4) may be rewritten as

$$(A^*A+\alpha I)(x_k - x_{k-1}) = A^*(y-Ax_{k-1}) \tag{2.1}$$

Let $r_k = A^*(y-Ax_k)$, and (2.1) becomes $x_k = x_{k-1}+(A^*A+\alpha I)^{-1}r_{k-1}$. Repeat use of this formula gives $r_k = \alpha^k(A^*A+\alpha I)^{-k}r_0$ and

$$x_k = U_{k,\alpha}(A^*A)A^*y + P_{k,\alpha}(A^*A)x_0 \tag{2.2}$$

with

$$P_{k,\alpha}(\lambda) = \Big(\frac{\alpha}{\lambda+\alpha}\Big)^k \tag{2.3}$$

$$U_{k,\alpha}(\lambda) = (1-P_{k,\alpha}(\lambda))/\lambda \tag{2.4}$$

In the paper we will use following notations:

$I_o \;:= (0, \|A^*A\|]$

$S \;:= \{x\in X | A^*Ax = A^*y\}$

$P_s$ : the orthogonal projection $X \to S$

$\{E_\lambda\}$ and $\{F_\lambda\}$: the spectrum families of self-adjoint operators $A^*A$ and $AA^*$, respectively.

$S$ is the set of the least squares solutions to equation (1.1) and $S \neq \emptyset$ if $y \in D(A^+)$. In the sequel we will always assume the case and Qy≠0 as well.

**Lemma 2.1.** *For any fixed $\alpha > 0$ and $x \in N(A)^\perp$,*

$$\|P_{k,\alpha}(A^*A)\| \le 1, \; P_{k,\alpha}(A^*A)x \to 0, \; \textit{as } k\to\infty.$$

**Theorem 2.2.** *Let $\{x_k\}$ be the iterates of* (2.1), *then $x_k \to P_s x_0$, as $k \to \infty$. Especially if $x_0 = 0, x_k \to A^+y$.*

*Proof.* Since $P_s x_0 \in S$, $A^*y = A^*AP_s x_0$ and $x_0 - P_s x_0 \in N(A)^\perp$. By (2.2) and Lemma 2.1,

$$x_k - P_s x_0 = P_{k,\alpha}(A^*A)(x_0 - P_s x_0) \to 0, \quad \text{as} \quad k\to\infty$$

It is clear that $P_s x_0 = x^+ = A^+ y$ if $x_0 = 0$.

According to the theorem , we always take in the paper $x_0 = 0$, and hence

$$x_k - x^+ = -P_{k,\alpha}(A^*A)x^+ \tag{2.7}$$

As usual, to consider the convergence rate of $\|x_k - x^+\|$ one needs some "smoothness" property of $x^+$[1]. Let $X_\nu = R((A^*A)^\nu) \subset N(A)^\perp, \nu \geq 0$, and

$$\|x\|_\nu = \Big(\int_{I_0} \lambda^{-2\nu} d\|E_\lambda x\|^2\Big)^{1/2}, \text{ for } x \in X_\nu.$$

$x \in X_\nu$ if and only if there exists a unique $f \in N(A)^\perp$ such that $x = (A^*A)^\nu f$ and in this case $\|x\|_\nu = \|f\|$.

**Lemma 2.3.** *Let $x \in N(A)^\perp$. Then for any fixed $\nu \geq 0$,*

$$\|(A^*A)^\nu P_{k,\alpha}(A^*A)x\| = \varepsilon_k(x)\Big(\frac{\alpha}{k}\Big)^\nu, \ k \geq 1, \tag{2.8}$$

*where*

$$\varepsilon_k(x) = \varepsilon_k^{\nu,\alpha}(x) = \Big[\int_{I_0} \Big(\frac{k}{\alpha}\Big)^{2\nu} \lambda^{2\nu} P_{k,\alpha}(\lambda)^2 d\|E_\lambda x\|^2\Big]^{1/2} \tag{2.9}$$

*with the properties*

$$\varepsilon_k(x) \leq C_1 \nu^\nu \|x\|, \tag{2.10}$$

$$\varepsilon_k(x) \to 0, \ \textit{as} \ k \to \infty. \tag{2.11}$$

*The constant $C_1 = C_1(k,\nu)$ is given by the formula*

$$C_1 = \begin{cases} \|A^*A\|^{\nu-k}, & \textit{if } k \leq \nu \\ 1, & \textit{if } k > \nu \end{cases} \tag{2.12}$$

*Proof.* (2.8) is obvious. It can be verified by differentiation that

$$\sup_{\lambda \in I_0} \lambda^\nu P_{k,\alpha}(\lambda) \leq C_1 \nu^\nu \Big(\frac{\alpha}{k}\Big)^\nu \tag{2.13}$$

and hence (2.10) follows. Let $\xi_k(\lambda) = (k/\alpha)^\nu \lambda^\nu P_{k,\lambda}(\lambda)$, and (2.13) shows $\xi_k(\lambda) \leq C_1\nu^\nu$ for $\lambda \in I_0$ and $k \geq 1$. Let $\eta$ be any positive number such that $0 < \eta < \|A^*A\|$. It is not difficult to verify that $\xi_k(\lambda) \to 0$ uniformly in $[\eta, \|A^*A\|]$ as $k \to \infty$. These properties of $\xi_k(\lambda)$ result in (2.11).

As a consequence, we have

**Theorem 2.4.** *Suppose there exists $f \in N(A)^\perp$ such that $x^+ = (A^*A)^\nu f \in X_\nu$ for some $\nu > 0$, then*

$$\|x_k - x^+\| = \varepsilon_k(f)\Big(\frac{\alpha}{k}\Big)^\nu \tag{2.14}$$

*where $\varepsilon_k(f) = \varepsilon_k^{\nu,\alpha}(f)$ is given by* (2.9).

*Proof.* The result follows from Lemma 2.3 and the equality $x_k - x^+ = -P_{k,\alpha}(A^*A)x^+ = -P_{k,\alpha}(A^*A)(A^*A)^\nu f$.

**Remark 2.5.** (2.14) and (2.11) show that for any individual $x^+ \in X_\nu$,

$$\|x_k - x^+\| = o(k^{-\nu}), \ \text{as } k \to \infty. \tag{2.15}$$

The factor $\varepsilon_k(f)$ in (2.14) goes to zero, however the rate can be arbitrarily slow. In fact, the following result is valid: let $\{\eta_k\}$ be any positive sequence with $\eta_k \to 0$, then there exists a certain $x^+ = A^+y \in X_\nu$ for which

$$\overline{\lim_{k\to\infty}} \frac{\|x_k - x^+\|}{\eta_k k^{-\nu}} = \infty$$

We omit the details.

## 3. Convergence Properties for Perturbed Equation (1.2)

For equation (1.2), the iteration is

$$\begin{aligned}&(A^*A + \alpha I)(x_k^\delta - x_{k-1}^\delta) = A^*(y_\delta - Ax_{k-1}^\delta),\ k \geq 1,\\ &x_0^\delta = 0,\end{aligned} \tag{3.1}$$

and the iterates $x_k^\delta$ may be given by

$$x_k^\delta = U_{k,\alpha}(A^*A)A^*y_\delta \tag{3.2}$$

Because of ill-posedness of the equation, the quantity $\|A^+y_\delta - A^+y\|$ can be arbitrarily large eventhough $y_\delta \in D(A^+)$ and $\|y_\delta - y\| \leq \delta$.

**Lemma 3.1.** *Let $y_\delta \in B_\delta(y)$ and $\{x_k^\delta\}$ and $\{x_k\}$ be the iterates of* (3.1) *and* (2.1), *respectively. Then*

$$\|x_k^\delta - x_k\| \leq (\frac{k}{\alpha})^{1/2}\delta \tag{3.3}$$

*Proof.* Since $x_k^\delta - x_k = U_{k,\alpha}(A^*A)A^*(y_\delta - y) = U_{k,\alpha}(A^*A)A^*Q(y_\delta - y)$ and $y_\delta \in B_\delta(y)$, it is valid that

$$\begin{aligned}\|x_k^\delta - x_k\|^2 =&(U_{k,\alpha}(A^*A)A^*Q(y_\delta - y), U_{k,\alpha}(A^*A)A^*Q(y_\delta - y))\\ =&(AA^*U_{k,\alpha}^2(AA^*)Q(y_\delta - y), Q(y_\delta - y))\\ =&\int_{I_0} \lambda U_{k,\alpha}^2(\lambda)d\|F_\lambda Q(y_\delta - y)\|^2 \leq \delta^2 \sup_{\lambda\in I_0} \lambda U_{k,\alpha}^2(\lambda) \leq \delta^2\frac{k}{\alpha}.\end{aligned}$$

As a consequence, we derive

**Theorem 3.2.** *Let $y_\delta \in B_\delta(y)$ and the natural numbers k=k ($\delta$) be chosen such that $k(\delta) \to \infty$ and $\delta k(\delta)^{1/2} \to 0$ as $\delta \to 0$. Then $\|x_{k(\delta)}^\delta - x^+\| \to 0$, as $\delta \to 0$.*

*Proof.* The result is based on Theorem 2.2, Lemma 3.1 and the following inequality

$$\|x_k^\delta - x^+\| \leq \|x_k^\delta - x_k\| + \|x_k - x^+\| \tag{3.4}$$

Now turn to the consideration of convergence rate of $\{x_k^\delta\}$.

**Theorem 3.3.** *Let $y_\delta \in B_\delta(y)$ and $x^+ = (A^*A)^\nu f \in X_\nu$ for some $\nu > 0$ and $f \in N(A)^\perp$. Then with*

$$k(\delta) = C_2\alpha(\varepsilon_{k(\delta)}(f)/\delta)^{2/(2\nu+1)} \tag{3.5}$$

*where $\varepsilon_k(f) = \varepsilon_k^{\nu,\alpha}(f)$ is given by* (2.9), *we have*

$$\|x_{k(\delta)}^\delta - x^+\| \leq C_3\varepsilon_{k(\delta)}(f)^{1/(2\nu+1)}\delta^{2\nu/(2\nu+1)} \tag{3.6}$$

*where* $C_3 = C_2^{-\nu} + C_2^{1/2}$.

*Proof.* By (3.4), (3.3) and (2.14) it is valid with such $k = k(\delta)$ that

$$\|x_k^\delta - x^+\| \le \varepsilon_k(f)\Big(\frac{\alpha}{k}\Big)^\nu + \Big(\frac{k}{\alpha}\Big)^{1/2}\delta = C_3\varepsilon_k(f)^{1/(2\nu+1)}\delta^{2\nu/(2\nu+1)}.$$

If there exist two constant $\overline{C} > \underline{C} > 0$ such that the number $C_2 = C_2(\delta, k)$ can be chosen with the following condition

$$\underline{C} \le C_2 \le \overline{C}, \text{ for } \delta \le \delta_0, \tag{3.7}$$

(3.6) shows

$$\|x_{k(\delta)}^\delta - x^+\| = o(\delta^{2\nu/(2\nu+1)}) \tag{3.8}$$

It was proven by the first author[8] that for any regularization algorithm solving linear ill-posed equations, the convergence rate (3.8) is optimal in the sense like that mentioned in Remark 2.5. We do not take the trouble to discuss whether condition (3.7) can be satisfied or not, since (3.5) is not implemental in practice. Applicable strategies of choosing $k = k(\delta)$ in practice are a posteriori ones. In the next section we will consider such a strategy and prove the rate (3.8) can be retained.

## 4. The Morozov's Discrepancy Principle

As usual, we suppose in this section that $y_\delta$ *and* $\delta > 0$ satisfy

$$\|Q(y_\delta - y)\| \le \delta < \|Qy_\delta\|/\gamma_0 \tag{4.1}$$

with a certain constant $\gamma_0 > 1$. Since $Qy \ne 0$, all the elements of $B_\delta(y)$ satisfy (4.1) when $\delta$ is suitably small. By using Morozov discrepancy principle[9], which is widely used in practice, we derive the following regularization algorithm.

**Algorithm 4.1.** Let (4.1) be satisfied and $\{x_k^\delta\}$ be the iterates of (3.1). Suppose $k = k(\delta) = k(\delta, y_\delta)$ is the first natural number for which

$$\|Ax_{k(\delta)}^\delta - Qy_\delta\| \le \gamma_0\delta, \tag{4.2}$$

then stop the iteration and take $x_{k(\delta)}^\delta$ as the solution of (1.2).

**Theorem 4.2.** *Algorithm* 4.1 *can be terminated after a finite number of steps and*

$$k(\delta) \to \infty,\ x_{k(\delta)}^\delta \to x^+,\ \text{as } \delta \to 0 \tag{4.3}$$

*Proof.* By (3.2),

$$Ax_k^\delta - Qy_\delta = AU_{k,\alpha}(A^*A)A^*y_\delta - Qy_\delta = -P_{k,\alpha}(AA^*)Qy_\delta \tag{4.4}$$

Since $\overline{R(A)} = N(A^*)^\perp$ and hence $Qy_\delta \in N(A^*)^\perp$, lemma 2.1 shows $P_{k,\alpha}(AA^*)Qy_\delta \to 0$ as $k \to \infty$. In fact, by (4.4) $\|Ax_k^\delta - Qy_\delta\|$ converges monotoniclly to zero as $k \to \infty$ and thus Algorithm 4.1 can be terminated after a finit number of steps. It is also clear $k(\delta) \to \infty$ as $\delta \to 0$. For such $k(\delta)$, (4.1), (4.4) and Lemma 2.1 lead to

$$\|Ax_{k(\delta)} - Qy\| \le \|Ax_{k(\delta)}^\delta - Qy_\delta\| + \|A(x_{k(\delta)}^\delta - x_{k(\delta)}) - Q(y_\delta - y)\|$$

$$\leq \gamma_0\delta + \|P_{k(\delta),\alpha}(AA^*)Q(y_\delta - y)\| \leq (\gamma_0+1)\delta \tag{4.5}$$

and

$$\|Ax_{k(\delta)-1} - Qy\| \geq \|Ax^\delta_{k(\delta)-1} - Qy_\delta\| - \|P_{k(\delta)-1,\alpha}(AA^*)Q(y_\delta - y)\| \geq (\gamma_0 - 1)\delta. \tag{4.6}$$

On the other hand, since $A^*y = A^*Ax^+$ and $Qy = Ax^+$ we derive $Ax_k - Qy = AU_{k,\alpha}(A^*A)A^*y - Qy = -AP_{k,\alpha}(A^*A)x^+$. This with Lemma 2.3 gives

$$\begin{aligned}\|Ax_{k(\delta)-1} - Qy\| =& \|AP_{k(\delta)-1,\alpha}(A^*A)x^+\| = \|(A^*A)^{1/2}P_{k(\delta)-1,\alpha}(A^*A)x^+\| \\ =& \varepsilon^{1/2,\alpha}_{k(\delta)-1}(x^+)\Big(\frac{\alpha}{k(\delta)-1}\Big)^{1/2}\end{aligned} \tag{4.7}$$

By (4.6), (4.7) and Lemma 3.1,

$$\begin{aligned}\|x^\delta_{k(\delta)} - x_{k(\delta)}\| \leq& \Big(\frac{k(\delta)}{\alpha}\Big)^{1/2}\delta \leq \frac{1}{\gamma_0 - 1}\Big(\frac{k(\delta)}{\alpha}\Big)^{1/2}\|Ax_{k(\delta)-1} - Qy\| \\ =& \frac{1}{\gamma_0 - 1}\Big(\frac{k(\delta)}{k(\delta)-1}\Big)^{1/2}\varepsilon^{1/2,\alpha}_{k(\delta)-1}(x^+)\end{aligned}$$

Thus the theorem follows from (3.4), (2.11) and the property $k(\delta) \to \infty$ as $\delta \to 0$.

We turn now to the convergence rate of Algorithm 4.1.

**Theorem 4.3.** *Suppose $x^+ = (A^*A)^\nu f \in X_\nu$ for some $\nu > 0$ and $f \in N(A)^\perp$ and let $x^\delta_{k(\delta)}$ be the solution of* (1.2) *obtained by Algorithm* 4.1. *Then*

$$\|x^\delta_{k(\delta)} - x^+\| \leq C^{\nu,\alpha}_{k(\delta)}(\delta, f)\delta^{2\nu/(2\nu+1)} \tag{4.8}$$

*where*

$$\begin{aligned}C^{\nu,\alpha}_{k(\delta)}(\delta, f) =& \varepsilon^{0,\alpha}_{k(\delta)}(f)^{1/(2\nu+1)}(\gamma_0+1)^{2\nu/(2\nu+1)} \\ &+ \Big[\frac{\delta^{2/(2\nu+1)}}{\alpha} + \Big(\frac{\varepsilon^{\nu+1/2,\alpha}_{k(\delta)-1}(f)}{\gamma_0 - 1}\Big)^{2/(2\nu+1)}\Big]^{1/2}\end{aligned} \tag{4.9}$$

*Proof.* By (2.7), (4.7), (4.5), Lemma 2.3 and Hölder inequality, we have

$$\begin{aligned}\|x_{k(\delta)} - x^+\| =& \|P_{k(\delta),\alpha}(A^*A)(A^*A)^\nu f\| \\ \leq& \|P_{k(\delta),\alpha}(A^*A)f\|^{1/(2\nu+1)}\|(A^*A)^{1/2}P_{k(\delta),\alpha}(A^*A)(A^*A)^\nu f\|^{2\nu/(2\nu+1)} \\ =& \|P_{k(\delta),\alpha}(A^*A)f\|^{1/(2\nu+1)}\|Ax_{k(\delta)} - Qy\|^{2\nu/(2\nu+1)} \\ \leq& \varepsilon^{0,\alpha}_{k(\delta)}(f)^{1/(2\nu+1)}(\gamma_0+1)^{2\nu/(2\nu+1)}\delta^{2\nu/2\nu+1)}\end{aligned} \tag{4.10}$$

Assume now $k(\delta) \geq 2$. By (4.6) and Lemma 2.3 we derive

$$\begin{aligned}(\gamma_0 - 1)\delta \leq& \|Ax_{k(\delta)-1} - Qy\| = \|(A^*A)^{1/2}P_{k(\delta)-1,\alpha}(A^*A)(A^*A)^\nu f\| \\ =& \varepsilon^{\nu+1/2,\alpha}_{k(\delta)-1}(f)\Big(\frac{\alpha}{k(\delta)-1}\Big)^{\nu+1/2}\end{aligned}$$

and hence

$$k(\delta) \le 1 + \alpha\Big(\frac{\varepsilon_{k(\delta)-1}^{\nu+1/2,\alpha}}{\gamma_0 - 1}\Big)^{2/(2\nu+1)}\delta^{-2/(2\nu+1)}. \tag{4.11}$$

If $\nu > 0$, in(2.9) we may reasonably define $\varepsilon_0^{\nu,\alpha}(f) = 0$. Therefore (4.11) is valid for $k(\delta) = 1$ as well. We finally derive by (4.10), (4.11) and Lemma 3.1 that

$$\begin{aligned}\|x_{k(\delta)}^{\delta} - x^+\| \le & \|x_{k(\delta)} - x^+\| + \|x_{k(\delta)}^{\delta} - x_{k(\delta)}\| \\ \le & \varepsilon_{k(\delta)}^{0,\alpha}(f)^{1/(2\nu+1)}(\gamma_0+1)^{2\nu/(2\nu+1)}\delta^{2\nu/(2\nu+1)} \\ & + \Big[\frac{\delta^{2/(2\nu+1)}}{\alpha} + \Big(\frac{\varepsilon_{k(\delta)-1}^{\nu+1/2,\alpha}(f)}{\gamma_0 - 1}\Big)^{2/(2\nu+1)}\Big]^{1/2}\delta^{2\nu/(2\nu+1)} \\ = & C_{k(\delta)}^{\nu,\alpha}(\delta, f)\delta^{2\nu/(2\nu+1)}.\end{aligned}$$

For any fixed $\alpha > 0$, (2.11) and (4.3) show $C_{k(\alpha)}^{\nu,\alpha}(\delta, f) \to 0$ as $\delta \to 0$, and thus we have

**Corollary 4.4.** *Suppose $x^+ = A^+y \in X_\nu$ for some $\nu > 0$ and let $x_{k(\delta)}^{\delta}$ be the solution of* (1.2) *obtained by Algorithm* 4.1*. Then* (3.8) *is valid.*

Theorem 4.3 not only claims the optimal convergence rate of $x_{k(\delta)}^{\delta}$, but gives a clue how to choose suitable parameter $\alpha$. (4.11) shows that $k(\delta)$ may discreases when $\alpha$ discreases, while by (4.9) the error $\|x_{k(\delta)}^{\delta} - x^+\|$ may increases. This means we can change the number of iterations and somehow the error of the approximate solution by varying $\alpha$. Due to this property we will call $\alpha$ the control parameter of (2.1). Let $\mu$, $0 < \mu \le \infty$, be such a number that $x^+ \in X_\nu$, for any $\nu < \mu$, while $x^+ \overline{\in} X_\mu$. (4.9) shows the strategy of choosing $\alpha$ by

$$\alpha \sim \delta^{2/(2\mu+1)} \tag{4.12}$$

may be reasonable. In this case we have the estimate for the error

$$\|x_{k(\delta)}^{\delta} - x^+\| = O(\delta^{2\nu/(2\nu+1)}). \tag{4.13}$$

## 5. Numerical Implementation and Examples

Let $W_m \subset R(A)$ be an $m$-dimensional subspace and $Q_m$ be the orthogonal projector $Y \to W_m$. Let $A_m = Q_mA$, $V_m = A^*(W_m) \subset N(A)^{\perp}$. Then discrete versions of (2.1) and (3.1) have the forms [10,11]

$$(A_m^*A_m + \alpha I)(x_{k,m} - x_{k-1,m}) = A_m^*(y - A_m x_{k-1,m}) \tag{5.1}$$

$$(A_m^*A_m + \alpha I)(x_{k,m}^{\delta} - x_{k-1,m}^{\delta}) = A_m^*(y_\delta - A_m x_{k-1,m}^{\delta}) \tag{5.2}$$

with $x_{0,m} = x_{0,m}^{\delta} = 0$. Let $x_m^+$ be the limit of iterates $\{x_{k,m}\}$ obtained by (5.1), then $x_m^+ = P_m x^+$, where $P_m$ is the orthogonal projector $X \to V_m$. Consider Volterra integral equation of the first kind

$$(Ax)(s) = \int_0^s x(t)dt = y(s), \ \ s \in [0,1]. \tag{5.3}$$

To solve (5.3) for $x$ by given $y$ is equivalent to find $x = A^+y = \frac{d}{ds}y(s)$, which is a typical ill-posed problem. Let $X = Y = L^2[0,1]$ and $W_m$ be the $m$-dimensional linear splines subspace with evenly spaced nodes. In the computation we take $m = 65$ and $\gamma_0 = 1.01$ in (4.2).

**Example 5.1.** Consider equation (5.3) with

$$x(t) = t,\ y(t) = \frac{t^2}{2} \tag{5.4}$$

Using the singular value decomposition (SVD) of $A$ ,one can show $A^+y = x \in X_\nu$ for $\nu < 1/2$ but $x \overline{\in} X_{1/2}$. The perturbations used in computation are

$$\begin{array}{ll} \text{Pert.1} & y_\delta - y = \sqrt{2}\delta \sin 10\pi t \\ \text{Pert.2} & y_\delta - y = \sqrt{2}\delta \sin 100\pi t \end{array} \tag{5.5}$$

for several values of $\delta$. In this case, (4.12) suggests the control parameter $\alpha$ be taken such as $\alpha \sim \delta$ and the numerical results in Table 5.1 with $\alpha = \delta$ are underlined

**Table 5.1 Numerical results for Ex 5.1**

| $\delta$ | $\alpha$ | pert.1 | | | pert.2 | | |
|---|---|---|---|---|---|---|---|
| | | $k(\delta)$ | $\|x^\delta_{m,k} - x^+\|$ | $\|x^\delta_{m,k} - x^+\|/\delta^{1/2}$ | $k(\delta)$ | $\|x^\delta_{m,k} - x^+\|$ | $\|x^\delta_{m,k} - x^+\|/\delta^{1/2}$ |
| 1.E-1 | 1 | 20 | 0.345349 | | 4 | 0.435629 | |
| | <u>0.1</u> | <u>3</u> | <u>0.336833</u> | 1.07 | <u>1</u> | <u>0.388876</u> | 1.23 |
| | 0.01 | 1 | 0.416409 | | 1 | 0.234946 | |
| 1.E-2 | 1 | 168 | 0.191585 | | 98 | 0.226528 | |
| | 0.1 | 18 | 0.190365 | | 11 | 0.198019 | |
| | <u>0.01</u> | <u>3</u> | <u>0.184072</u> | 1.84 | <u>2</u> | <u>0.178698</u> | 1.79 |
| | 0.001 | 1 | 0.200516 | | 1 | 0.131895 | |
| 1.E-3 | 0.1 | 234 | 0.0925765 | | 204 | 0.0951964 | |
| | 0.01 | 24 | 0.0923209 | | 21 | 0.0951448 | |
| | <u>0.001</u> | <u>3</u> | <u>0.090044</u> | 2.85 | <u>3</u> | <u>0.0897274</u> | 2.84 |
| | 0.0001 | 1 | 0.0746789 | | 1 | 0.0743447 | |
| 1.E-4 | 0.1 | 3211 | 0.0514845 | | 3140 | 0.0518935 | |
| | 0.01 | 321 | 0.0514816 | | 314 | 0.0518892 | |
| | 0.001 | 33 | 0.0514126 | | 33 | 0.0516766 | |
| | <u>0.0001</u> | <u>4</u> | <u>0.0485582</u> | 4.86 | <u>4</u> | <u>0.0513982</u> | 5.14 |
| | 0.00001 | 1 | 0.0485005 | | 2 | 0.0486051 | |
| 0 | 0.1 | 6318 | 0.0488675 | | | | |
| | 0.01 | 631 | 0.0488669 | | | | |
| | 0.001 | 64 | 0.0488657 | | | | |
| | 0.0001 | 8 | 0.0487074 | | | | |

The last group of results, i.e., with $\delta = 0$, are those for unperturbed equation (5.1), and the corresponding $k = k(\delta)$ are determined by the condition

$$\frac{\|x_{k,m} - x^+\|}{\|x^+_m - x^+\|} \leq 1.01. \tag{5.6}$$

**Example 5.2.** Consider equation (5.3) again but with

$$x(t) = \cos\frac{\pi t}{2},\ y(t) = \frac{2}{\pi}\sin\frac{\pi t}{2} \tag{5.7}$$

Then $x \in X_\nu$ for any $\nu > 0$. The perturbations used in computation are the same as in Ex 5.1.

In this case, (4.12) suggest that $\alpha \sim 1$ and the corresponding numerical results are underlined in Table 5.2.

The numerical results of Examples 5.1 and 5.2 coincide quite well with the theoritical analysis in previous sections. Especially these results show the key role of parameter $\alpha$ in controlling the numbers of iteration.

**Table 5.2 Numerical results for Ex 5.2**

| $\delta$ | $\alpha$ | pert.1 | | | pert.2 | | |
|---|---|---|---|---|---|---|---|
| | | $k(\delta)$ | $\|x^\delta_{m,k} - x^+\|$ | $\|x^\delta_{m,k} - x^+\|/\delta$ | $k(\delta)$ | $\|x^\delta_{m,k} - x^+\|$ | $\|x^\delta_{m,k} - x^+\|/\delta$ |
| 1.E-1 | 1 | 9 | 5.35372E-2 | 0.535 | 5 | 1.29844E-1 | 1.30 |
| | 0.1 | 2 | 7.83470E-2 | | 1 | 1.40757E-1 | |
| | 0.01 | 1 | 2.99041E-1 | | 1 | 2.27441E-2 | |
| 1.E-2 | 1 | 16 | 6.95028E-3 | 0.695 | 12 | 1.20215E-2 | 1.20 |
| | 0.1 | 4 | 1.35180E-2 | | 3 | 5.60373E-3 | |
| | 0.01 | 2 | 5.60419E-2 | | 2 | 2.65899E-3 | |
| | 0.001 | 1 | 1.55571E-1 | | 1 | 1.06857E-2 | |
| 1.E-3 | 1 | 22 | 9.67112E-4 | 0.967 | 18 | 1.55617E-3 | 1.56 |
| | 0.1 | 5 | 1.67115E-3 | | 4 | 1.09980E-3 | |
| | 0.01 | 2 | 5.62067E-3 | | 2 | 5.23116E-4 | |
| | 0.001 | 2 | 2.31765E-2 | | 2 | 1.95685E-3 | |
| | 0.0001 | 1 | 2.82141E-2 | | 1 | 5.99294E-3 | |
| 1.E-4 | 1 | 29 | 1.56561E-4 | 1.57 | 25 | 1.89798E-4 | 1.90 |
| | 0.1 | 6 | 2.57750E-4 | | 6 | 3.44274E-4 | |
| | 0.01 | 3 | 7.82692E-4 | | 3 | 1.95665E-4 | |
| | 0.001 | 2 | 2.30535E-3 | | 2 | 3.47650E-4 | |
| | 0.0001 | 2 | 3.08864E-3 | | 2 | 9.88501E-4 | |

## 6. Comparison and conclusions

In this paper we propose a kind of implicit iterative methods to solve linear ill-posed operator equations, and discuss properties of the methods in which the control parameter $\alpha$ is fixed. The results presented in the paper show that for any fixed $\alpha > 0$, Algorithm 4.1, which is constructed together with Morozov discrepancy principle, always leads to the optimal convergence rate (3.8). Thus Algorithm 4.1 is a robust regularization algorithm[8]. Another remarkable feature of the new method is that one can efficiently control the number of iteration by varying the parameter $\alpha$. This property is valuable in practical applications. Each one of $\nu$-methods discussed in [5–7] forms for all $\nu \geq 0$ a family of sequent robust regularization algorithms[8]. A trouble with a family of such algorithms is how to choose a suitable value $\nu$ in practical computation, since the smoothness property of the generalized solution $x^+$ is usually unknown. Another difficulty with a $\nu$-method is that one has to estimate accurately the norm $\|A^*A\|$, otherwise the method may not lead to satisfactory result (recall the same situation for the Chebeshev acceleration in finite-dimensional cases[12]). Tikhonov-type regularization methods are another important approaches to solve ill-posed problems. For example, the $n$th Tikhonov iterative method[13,14] and the Tikhonov regularization in Hilbert scales[15,16] are also sequent robust families of regularization algorithms. Therefore, use of these methods still faces the trouble of choosing a suitable individual one in the families. Besides, when any a posteriori principle is used to determine the regularization parameter, one has to solve a supplementary nonlinear equation. This will result in extra difficulty. The implicit iterative methods discussed in the paper, however, have no such troubles, and may become an efficient approach to solve linear ill-posed problems. Furthermore, if the control parameter $\alpha$ varies suitably in the iteration procedure, we may obtain even more efficient algorithms. The further work is now underway.

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