Journal of Computational Mathematics, Vol.17, No.5, 1999, 533–544.

A V-CYCLE MULTIGRID METHOD FOR THE PLATE BENDING PROBLEM DISCRETIZED BY NONCONFORMING FINITE ELEMENTS*

Xue-jun Xu
(*ICMSEC, Chinese Academy of Sciences, Beijing 100080, China*)
Li-kang Li
(*Department of Mathematics, Fudan University, Shanghai 200433, China*)

Abstract

In this paper, an optimal V-cycle multigrid algorithm for some famous nonconforming plate elements is established.

Key words: The plate problem, V-cycle multigrid, Nonconforming elements.

1. Introduction

Multigrid methods have become some of the most powerful methods for solving partial differential equations discretized by the finite element and finite difference methods. (cf. [7][11][14] and reference therein). Multigrid methods for the nonconforming finite elements have been studied by some reseachers recently. For the second order problems, some optimal multigrid methods for the P1 nonconforming element and the Wilson nonconforming element have been established.(cf.[5][18][22]). Mutilgrid methods for biharmonic problem have also attracted many reseachers attention, in [9][12][17], the authors presented some optimal order multigrid methods for the Morely element, but only considered W-cycle multigrid. In [21], Zhang proposed a V-cycle multigrid for Bonger-Fox-Schmit (BFS) conforming plate element, the convergence of the method rests on the nestness of the mesh spaces. But until now effictive V-cycle multigrids for the nonconforming plate elements have not been constructed.

The purpose of this paper is to develop an optimal and effective V-cycle multigrid method for some well-known nonconforming finite elements such as the Morley element, the Adini elemet. The basic idea is that no matter how finite element spaces we deal with, we insist on using the Powell-Sabin (PS) finite element space as correction space on the level l ($l = 1, ..., L-1$). The V-cycle multigrid method for the nonconforming plate elements needs smooth enough steps on the last level L, but on the coarse mesh l ($l = 1, ..., L-1$) only needs smooth one step. Moreover, because we use the PS finite element as coarse mesh spaces($l = 1..., L-1$), the intergrid transfer operator only choose the most simple interpolation opertor, the computation become very cheap.

* Received March 12, 1996.

2. Plate Bending Problem and Nonconforming Elements

Let Ω be a convex polygonal domain in R^2, the variational form of the plate bending problem is defined as follows: Find $u \in H_0^2(\Omega)$(cf. [10] for Sobolev space notations) such that

$$a(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega), \tag{2.1}$$

where f is a function on $L^2(\Omega)$ and

$$a(u,v) = \int_\Omega \triangle u \triangle v + (1-\sigma)(2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2})dx,$$

$$(f,v) = \int_\Omega fv dx,$$

and $0 < \sigma < \frac{1}{2}$ is the Possion ratio. It is well-known that (2.1) has a unique solution $u \in H_0^2(\Omega)$, and

$$a(u,v) \leq C|u|_2|v|_2, \quad \forall u,v \in H_0^2(\Omega), \tag{2.2}$$

$$a(v,v) \geq C|v|_2^2, \quad \forall v \in H_0^2(\Omega), \tag{2.3}$$

where $|\cdot|_2$ is seminorm over space $H^2(\Omega)$.

Throughout this paper, c, C always denote strictly positive constant independent of h and L.

We assume the following elliptic regularity for the problem (2.1). For any $f \in H^{-1}(\Omega) = (H_0^1(\Omega))'$, there exists a solution $u \in H^3(\Omega) \cap H_0^2(\Omega)$ and

$$\|u\|_3 \leq C\|f\|_{-1}.$$

It was proved in [2] that the above assumption is true if Ω is a convex polygonal domain.

We assume that Γ_h is a quasiuniform triangular or rectangular partition of Ω, let $V_h \subset L^2(\Omega)$ be a finite element space with respect to Γ_h. Define

$$a_h(u,v) = \sum_{K\in\Gamma_h}\int_K (\triangle u \triangle v + (1-\sigma)(2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}))dx,$$

and

$$|v|_{i,h}^2 = \sum_{K\in\Gamma_h} |v|_{i,K}^2, \quad (i = 0,1,2).$$

We assume that the above definitions satisfy:

(H1) (1). $a_h(u,v) \leq C|u|_{2,h}|v|_{2,h}, \quad \forall u,v \in V_h$,

(2). $a_h(v,v) \geq C|v|_{2,h}^2, \forall v \in V_h$,

(3). $|u|_{2,h}$ *is a norm over* V_h.

(4). $D_h(u,v) \leq Ch|u|_3|v|_{2,h}$, $\forall u \in H^3(\Omega)$, $v \in V_h$, *and*

$$D_h(u,v) = \sum_K \int_{\partial K}((-\triangle u + (1-\sigma)\frac{\partial^2 u}{\partial \tau^2})\frac{\partial v}{\partial n} - (1-\sigma)\frac{\partial^2 u}{\partial n \partial \tau}\frac{\partial v}{\partial \tau})ds$$

where τ *and* n *denote the unit tangential and outward normal vector along* ∂K.

The discretization form of (2.1) is as follows: Find $u_h \in V_h$, such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \tag{2.4}$$

By (H1), it is easy to check that (2.4) has a unique solution $u_h \in V_h$. Define an operator $A_h : V_h \to V_h$ as follows

$$(A_h u, v) = a_h(u, v), \quad \forall u, v \in V_h.$$

Then (2.4) can be expressed as:

$$A_h u_h = f_h, \tag{2.5}$$

where $f_h \in V_h$, $(f_h, v) = (f, v), \quad v \in V_h$.

3. An Abstract Theory

In order to construct our V-cycle multigrid algorithm, we must choose suitable coarse mesh finite element spaces. Define S_l as the following Powell-Sabin finite element space (cf.[15][16] for details).

Let $\Gamma_l (l = 1, ..., L-1)$ be triangular partition of Ω with mesh size $h_l = 2^{L-l}h$. Γ_l is obtained by linking the minpoints of two side of all triangle of Γ_{l-1}. Assume $h_{L-1} = h/2$. Now we define the following PS finite element spaces on triangulation Γ_l $(1 \le l \le L-1)$.

The shape function space $P(K)$ is a piecewise quadratic polynomail space, the degrees of freedom of this element are given by

$$\{p(a_i), \ \frac{\partial p}{\partial x_1}(a_i), \ \frac{\partial p}{\partial x_2}(a_i), \ i = 1, 2, 3\},$$

where a_i is the vertex of triangle K. Obviously, we have

$$S_1 \subset S_2 \subset \cdots \subset S_{L-1}.$$

Now we can define operators A_{S_l}, Q_{S_l} as follows

$$A_{S_l} : S_l \to S_l, \quad (A_{S_l} u, v) = a(u, v), \quad \forall u, v \in S_l \quad (l = 1, ..., L-1).$$

$$Q_{S_l} : S_{L-1} \to S_l, \quad (Q_{S_l} u, v) = (u, v), \quad \forall u \in S_{L-1}, v \in S_l \quad (l = 1, ..., L-1).$$

Let operator $J_{L-1} : H^3(\Omega) \cap H_0^2(\Omega) \to S_{L-1}$ be the standard interpolation operator with respect to space S_{L-1}.

(H2). *Assume there exists an opertor $\pi_h : S_{L-1} \to V_h$ such that*

$$(1). |\pi_h v - v|_{r,h} \le C h^{2-r} |v|_2, \quad \forall v \in S_{L-1} \quad (r = 0, 1, 2)$$

$$(2). |v - \pi_h J_{L-1} v|_{2,h} \le C h |v|_3, \quad \forall v \in H^3(\Omega) \cap H_0^2(\Omega).$$

Remark 3.1. In what follows, we can find that the operator π_h will be appeared in the following multigrid algorithm, so its implementation should be cheap, because we choose S_l as PS finite element space, we can find that the construction of operator π_h is very simple and cheap for any nonconforming plate element spaces.

Define operators Q_{L-1} and P_{L-1}:$V_h \to S_{L-1}$ as follows:

$$(Q_{L-1}u, v) = (u, \pi_h v), \quad \forall u \in V_h, \quad v \in S_{L-1}; \tag{3.1}$$

$$a(P_{L-1}u, v) = a_h(u, \pi_h v), \quad \forall u \in V_h, \quad v \in S_{L-1} \tag{3.2}$$

By the definition of P_{L-1} and (H1),(H2), it is easy to check that

$$|P_{L-1}u|_2 \le C|u|_{2,h}. \tag{3.3}$$

In order to define our V-cycle multigrid algorithm, we must introduce some smoothing operators. For simplicity, we only consider the following symmetric smoothers.

(H3). *Assume that there exists a symmetric smoother $R_h : V_h \to V_h$ such that*

$$(1). \quad C\frac{1}{\lambda_h}(v, v) \le (R_h v, v), \quad \forall v \in V_h,$$

$$(2). \quad a_h(R_h A_h v, v) \le \theta a_h(v, v), \quad \forall v \in V_h,$$

where $\theta \in (0, 2)$ and λ_h is the maximum eigenvalue of A_h.

By (H3) and the similar arguments of Theorem 3.6, Theorem 5.1 of [7], we have

Lemma 3.1. *For all $v \in V_h$, we have*

$$c\frac{\|A_h K_h^m v\|^2}{\lambda_h} \le a_h((I - K_h^2)K_h^m v, K_h^m v) \le C\frac{1}{m}a_h(v, v).$$

where $K_h = I - R_h A_h$ and m is the number of smoothing.

On the coarse mesh space $S_l (l = 0, ..., L-1)$, using the similar idea of chapter 5 in [7], we can construct symmetric smoothers $R_{S_l} : S_l \to S_l$ which satisfy

$$C\frac{1}{\lambda_l}(v, v) \le (R_{S_l} v, v),$$

and

$$a(R_{S_l} A_{S_l} v, v) \le \theta a(v, v), \quad \forall v \in V_h,$$

where $\theta \in (0, 2)$ and λ_l is the maximum eigenvalue of A_l.

In fact, the Richardson, Jacobi and symmtric Gauss-Seidel smoother satisty the above smoothing condition.(cf.[7] for details)

Now we can define the following V-cycle multigrid algorithm.

V-cycle multigrid algorithm

For any given $g \in V_h$, define $B_L g$ as follows

(1). Set $x_0 = 0$, $\quad x^n = x^{n-1} + R_h(g - A_h x^{n-1}), \quad n = 1, ..., m.$

(2). Define $x^{m+1} = x^m + \pi_h q$, where

$$q = M_{L-1}Q_{L-1}(g - A_h x^m).$$

(3). Let $y_0 = x^{m+1}$ and

$$y^n = y^{n-1} + R_h(g - A_h y^{n-1}), \quad n = 1, ..., m.$$

(4). Set $B_L g = y^m$.

Moreover the operater M_{L-1} is defined as follows: Set $M_1 = A_{S_1}^{-1}$, for any given $g_l \in S_l$, $M_l g_l$ $(l = 2, ..., L-1)$ is defined by

(i). Set $x_1 = R_l g_l$.

(ii). Define $M_l g_l = x_1 + p$, where $p \in S_{l-1}$ is given by

$$p = M_{l-1} Q_{l-1} (g_l - A_{S_l} x_1).$$

It is easy to check that

$$I - B_L A_h = K_h^m (I - \pi_h P_{L-1} + \pi_h (I - M_{L-1} A_{S_{L-1}}) P_{L-1}) K_h^m. \tag{3.4}$$

In what follows, we give an estimate of operator $I - M_{L-1} A_{S_{L-1}}$.

Lemma 3.2. *Let $Q_{S_0} = 0$, then for any $u \in S_{L-1}$, we have*

$$\sum_{l=1}^{L-1} h_l^{-4} \|Q_{S_l} u - Q_{S_{l-1}} u\|_0^2 \leq C |u|_2^2.$$

we refer the proof of this lemma to Theorem 5.1 in [6].

In order to obtain the so-called strengthen Cauchy-Schwarz inequality for the PS finite element space S_l, we first introduce the fractional Sobolev space $H_0^{m+\sigma}(\Omega)$ ($m = 0, 1, 2$ and $0 < \sigma < 1$), which is defined by the completion of $C_0^\infty(\Omega)$ in the following norm

$$\|v\|_{m+\sigma} = (\|v\|_m^2 + |v|_{m+\sigma}^2)^{\frac{1}{2}},$$

where

$$|v|_{m+\sigma}^2 = \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha v(x) - D^\alpha v(y)|^2}{|x-y|^{2+2\sigma}} dxdy.$$

For $\alpha \in (0, 2]$, we define $H^{-\alpha}(\Omega) = (H_0^\alpha(\Omega))'$, the dual of $H_0^\alpha(\Omega)$. The following inverse inequality (cf. [20] for details) holds

$$|v|_{2+\sigma} \leq h_l^{-\sigma} |v|_2, \quad \forall v \in S_l,$$

$$|v|_s \leq h_l^{-s} \|v\|_0, \quad \forall v \in S_l,$$

where $\sigma \in (0, \frac{1}{2})$ and $s \in [0, 2]$.

Lemma 3.3. *There exists $\gamma \in (0,1)$ such that for any $u_i \in S_i$, $u_j \in S_j$, $(i < j)$, we have*

$$a(u_i, u_j) \leq C \gamma^{j-i} h_j^{-2} a(u_i, u_i)^{\frac{1}{2}} \|u_j\|_0,$$

Proof. Obviously, by Green formulation, we know that $\triangle : H_0^2(\Omega) \to L^2(\Omega)$ is continuous, and by duality $\triangle : L^2(\Omega) \to H^{-2}(\Omega)$ is continuous. By interpolation, for $\alpha \in (0, \frac{1}{2})$, $\triangle : H^{2-\alpha}(\Omega) \to H^{-\alpha}(\Omega)$ is also continuous. Hence $|\triangle v|_{-\alpha} \leq C|v|_{2-\alpha}, \quad \forall v \in H_0^2(\Omega)$. Therefore

$$\begin{aligned} a(u_i, u_j) &\leq |\triangle u_i|_\alpha |\triangle u_j|_{-\alpha} \\ &\leq C |u_i|_{2+\alpha} |u_j|_{2-\alpha}. \end{aligned}$$

By the above inverse inequality, we have

$$\begin{aligned} a(u_i,u_j) &\le Ch_i^{-\alpha}|u_i|_2 \cdot h_j^{-2+\alpha}\|u_j\|_0 \\ &\le C((\frac{1}{2})^{\alpha})^{j-i}h_j^{-2}a(u_i,u_i)^{\frac{1}{2}}\|u_j\|_0, \end{aligned}$$

which yields Lemma 3.3.

Based on Lemma 3.2, 3.3 and the similar arguments of Theorem 4.4, Lemma 6.3 in [19], we have

Theorem 3.1. *For the operator $I - M_{L-1}A_{S_{L-1}}$, we have*

$$|a((I - M_{L-1}A_{S_{L-1}})u,u)| \le \delta_0 a(u,u), \quad \forall u \in S_{L-1}.$$

where $\delta_0 \in (0,1)$ is independent of L and h. In fact, Theorem 3.1 provides a V-cycle convergence analysis for the PS finite element.

Let $\{\lambda_j\}_{j=1}^{N_h}$ and $\{\varphi_j\}_{j=1}^{N_h}$ be the eigenvalues and corresponding normalized eigenfunctions of A_h, i.e.

$$A_h\varphi_j = \lambda_j\varphi_j, \quad j = 1,...,N_h,$$

and

$$(\varphi_i,\varphi_j) = \delta_{ij},$$

where δ_{ij} is Kronecker symbol.

For any $v_h \in V_h$, $v_h = \sum_j^{N_h} c_j\varphi_j$, let $A_h^{\frac{s}{2}}v = \sum_j^{N_h} \lambda_j^{\frac{s}{2}}c_j\varphi_j$, then we can define the following discrete norm on the space V_h

$$\|v\|_{s,h} \hat{=} (A_h^{\frac{s}{2}}v,v)^{\frac{1}{2}}.$$

Note that $\|v\|_{2,h} = a_h(v,v)^{\frac{1}{2}}$, $\|v\|_{0,h} = \|v\|_0$ and for any $u,v \in V_h$ the following inequalities are not difficult to check

$$\|v\|_{s,h} \le Ch^{t-s}\|v\|_{t,h}, \quad s > t, \tag{3.5}$$

$$a_h(u,v) \le \|u\|_{2+t,h}\|v\|_{2-t,h}, t \in R, \tag{3.6}$$

$$\|v\|_{3,h} \le \|v\|_{4,h}^{\frac{1}{2}}\|v\|_{2,h}^{\frac{1}{2}}. \tag{3.7}$$

Let I_h denote the linear or bilinear finite element interpolation operator with respect to Γ_h.

(H4).*For the operator I_h, we assume that $I_h : V_h \to H_0^1(\Omega)$ and*

$$|v - I_h v|_{1,h} \le Ch|v|_{2,h}, \quad \forall v \in V_h.$$

Lemma 3.4. *Under the assumptions (H3), (H4), we have*

$$\|v\|_{1,h} \le C|v|_{1,h}.$$

Proof. By an argument completely similar to the one used to prove proposition 8.1 of [17], we can show that

$$\|v\|_{1,h} \le C(|I_h v|_1 + h|v|_{2,h}), \quad \forall v \in V_h.$$

Hence, it follows that the inverse inequality and (H4), we have

$$\begin{aligned}\|v\|_{1,h} &\le C(|I_h v - v|_{1,h} + |v|_{1,h} + h|v|_{2,h})\\ &\le C(h|v|_{2,h} + |v|_{1,h}) \le C|v|_{1,h}.\end{aligned}$$

So this Lemma is true.

Lemma 3.5. *Under the assumptions (H1), (H2), (H4), for the operator P_{L-1} defined by (3.2), we have*

$$|u - P_{L-1}u|_{1,h} \le Ch\|u\|_{2,h}, \quad \forall u \in V_h$$

Proof. By the triangle inequality, we get

$$|u - P_{L-1}u|_{1,h} \le |u - I_h u|_{1,h} + |I_h u - P_{L-1}u|_{1,h},$$

where I_h is defined by (H4).

By (H1), (H4), we have

$$|u - I_h u|_{1,h} \le Ch|u|_{2,h} \le Ch\|u\|_{2,h}.$$

Note that $I_h u - P_{L-1}u \in H_0^1(\Omega)$, then $\phi = \triangle(I_h u - P_{L-1}u) \in H_0^{-1}(\Omega)$. Consider the following auxiliary problem

$$\begin{cases} \triangle^2\theta = \phi, \quad in \quad \Omega,\\ \theta = \dfrac{\partial\theta}{\partial n} = 0, \quad on \quad \partial\Omega, \end{cases} \tag{3.8}$$

By regularity assumption, we know that

$$\|\theta\|_3 \le C|\phi|_{-1} \le C|I_h u - P_{L-1}u|_1. \tag{3.9}$$

Furthermore, there holds that (cf.[10] for details)

$$\begin{aligned}-\int_\Omega \nabla(\triangle\theta)\nabla\eta dx &= (\triangle(I_h u - P_{L-1}u), \eta),\\ &= (\nabla(I_h u - P_{L-1}u), \nabla\eta), \quad \forall\eta \in H_0^1(\Omega).\end{aligned}$$

Then, we have

$$\begin{aligned}|I_h u - P_{L-1}u|_1^2 &= -\int_\Omega \nabla(\triangle\theta)\nabla(I_h u - P_{L-1}u)dx\\ &= -\sum_K \int_K \nabla(\triangle\theta)\nabla(I_h u - u)dx\\ &\quad -\sum_K \int_K \nabla(\triangle\theta)\nabla(u - P_{L-1}u)dx \hat{=} R_1 + R_2.\end{aligned}$$

By (H1), (H4), (3.9), we get

$$\begin{aligned}|R_1| &\le |-\sum_K \int_K \nabla(\triangle\theta)\nabla(I_h u - u)dx| \le C\|\theta\|_3|I_h u - u|_{1,h}\\ &\le Ch|I_h u - P_{L-1}u|_1\|u\|_{2,h}.\end{aligned}$$

On the other hand

$$\begin{aligned} R_2 &= -\sum_K \int_K \nabla(\triangle\theta)\nabla(u - P_{L-1}u)dx \\ &= -\sum_K \int_K \nabla(\triangle\theta)\nabla u dx + \sum_K \int_K \nabla(\triangle\theta)\nabla P_{L-1}u dx \\ &= a_h(\theta, u) - D_h(\theta, u) - a(\theta, P_{L-1}u) \\ &= a_h(\theta - \pi_h J_{L-1}\theta, u) + a(J_{L-1}\theta - \theta, P_{L-1}u) - D_h(\theta, u) \\ &\hat{=} I_1 + I_2 + I_3. \end{aligned}$$

By (H1) and the interpolation error estimate of the PS finite element space(cf.[15]), we have

$$\begin{aligned} |I_2| &\leq |\theta - J_{L-1}\theta|_2|u|_{2,h} \leq Ch\|\theta\|_3|u|_{2,h} \\ &\leq Ch|I_h u - P_{L-1}|_{1,h}\|u\|_{2,h}. \end{aligned}$$

By (H1),(H2), we have

$$\begin{aligned} |I_1| &\leq |\theta - \pi_h J_{L-1}\theta|_{2,h}|u|_{2,h} \leq Ch\|\theta\|_3|u|_{2,h} \\ &\leq Ch|I_h u - P_{L-1}|_{1,h}\|u\|_{2,h}. \end{aligned}$$

By (H1), we have

$$|I_3| \leq Ch\|\theta\|_3|u|_{2,h} \leq Ch|I_h u - P_{L-1}|_{1,h}\|u\|_{2,h}.$$

Combined above inequalities, we obtain Lemma 3.5.

Lemma 3.6. *Under the assumptions (H1),(H2),(H4), we have*

$$\|v - \pi_h P_{L-1}v\|_{2,h} \leq Ch\|v\|_{3,h}, \quad \forall v \in V_h.$$

Proof. By (3.3), (H1), Lemma 3.4 and the inverse inequality, we have

$$\begin{aligned} \|v - \pi_h P_{L-1}v\|_{1,h} &\leq |v - \pi_h P_{L-1}v|_{1,h} \\ &\leq |v - P_{L-1}v|_{1,h} + |(I - \pi_h)P_{L-1}v|_{1,h} \\ &\leq Ch\|v\|_{2,h} + Ch|P_{L-1}v|_2 \\ &\leq Ch\|v\|_{2,h}. \end{aligned}$$

On the other hand, for any fixed $v \in V_h$, by (3.6) and above inequatily, we have

$$\begin{aligned} \|v - \pi_h P_{L-1}v\|_{2,h} &= \sup_{w \in V_h, \|w\|_{2,h}=1} a_h(v - \pi_h P_{L-1}v, w) \\ &= \sup_{w \in V_h, \|w\|_{2,h}=1} a_h(v, w - \pi_h P_{L-1}w) \\ &\leq \sup_{w \in V_h, \|w\|_{2,h}=1} \|v\|_{3,h}\|w - \pi_h P_{L-1}w\|_{1,h} \\ &\leq Ch\|v\|_{3,h}. \end{aligned}$$

So this Lemma is true.

Finally, we can obtain the main result of this paper.

Theorem 3.2. *Under the assumptions (H1)-(H4), then for any* $\delta \in (\delta_0, 1)$*, if* m *is large enough, then we have*

$$|a_h((I - B_L A_h)u, u)| \leq \delta a_h(u, u), \quad \forall u \in V_h.$$

Proof. Let $\tilde{u} = K_h^m u$, then by (3.2), (3.4) and Theorem 3.1, we have

$$\begin{aligned}
&|a_h((I - B_L A_h)u, u)| \\
&\quad \leq |a_h((I - \pi_h P_{L-1})\tilde{u}, \tilde{u})| + |a((I - M_{L-1} A_{S_{L-1}})P_{L-1}\tilde{u}, P_{L-1}\tilde{u})| \\
&\quad \leq |a_h((I - \pi_h P_{L-1})\tilde{u}, \tilde{u})| + \delta_0 |a(\pi_h P_{L-1}\tilde{u}, \tilde{u})| \\
&\quad \leq (1 + \delta_0)|a_h((I - \pi_h P_{L-1})\tilde{u}, \tilde{u})| + \delta_0 |a(\tilde{u}, \tilde{u})|.
\end{aligned}$$

On the other hand, by (3.7), Lemma 3.1, lemma 3.6 and (H3), we get

$$\begin{aligned}
|a_h((I - \pi_h P_{L-1})\tilde{u}, \tilde{u})| &\leq Ch\|\tilde{u}\|_{3,h}\|\tilde{u}\|_{2,h} \\
&\leq Ch\|\tilde{u}\|_{4,h}^{\frac{1}{2}}\|\tilde{u}\|_{2,h}^{\frac{3}{2}} = C(\frac{\|A_h\tilde{u}\|_0^2}{\lambda_h})^{\frac{1}{4}}\|\tilde{u}\|_{2,h}^{\frac{3}{2}} \\
&\leq C(a_h((I - K_h^2)K_h^m u, K_h^m u))^{\frac{1}{4}}\|u\|_{2,h}^{\frac{3}{2}} \\
&\leq C\frac{1}{\sqrt[4]{m}} a_h(u, u).
\end{aligned}$$

Hence if m large enough, we obtain

$$\begin{aligned}
|a_h((I - B_L A_h)u, u)| &\leq (\frac{C(1 + \delta_0)}{\sqrt[4]{m}} + \delta_0)a_h(u, u) \\
&\leq \delta a_h(u, u),
\end{aligned}$$

which is Theorem 3.2.

4. Applications

In this section, we will apply the abstract theory developed and analyzed in section 3 to Morely element, Adini element. From section 2, it can be seen that we only need verify them to satisfy the assumptions (H1)-(H4).

(1). Morley element

In this subsection, we will consider the well-known Morley element [10] [13]. whose shape function is a quadratic polynomial and nodal parameters are function values at the vertices of the triangle and first nomal derivatives at midpoints of the sides of the triangle. Morley element is a strong nonconforming element. Let V_h^m be the Morley finite element space. We can see that (H1) holds for Morley finite element spaces (cf. [10][13] for details).

It is easy to show

Lemma 4.1. *For any* $v \in V_h^m$*,* $K \in \Gamma_h$*, we have*

$$c\|v\|_{0,K}^2 \leq h_K^2 \sum_{i=1}^{3} v^2(a_i) + h_K^4 \sum_{i=1}^{3} (\frac{\partial v}{\partial n}(m_i))^2 \leq C\|v\|_{0,K}^2,$$

where a_i is the vertex of K, m_i is the midpoint of the edges of K, n is the unit outward normal vector along ∂K. For the Morley finite element space V_h^m, we define $\pi_h : S_{L-1} \to V_h^m$ as the standard nodal interpolation operator with respect to Γ_h.

Lemma 4.2. *For any $v \in S_{L-1}$, we have*

$$|\pi_h v - v|_{r,h} \le Ch^{2-r}|v|_2, \quad \forall v \in S_{L-1}, \quad r = 0,1,2.$$

Proof. For any $K \in \Gamma_h$, define $\triangle K = \{K_{L-1} \in \Gamma_{L-1} | K_{L-1} \cap \bar{K} \neq 0\}$. By the definition of π_h, we have

$$\pi_h v|_K = \sum_{i=1}^{3} v(a_i)\phi_i + \sum_{i=1}^{3} \frac{\partial v}{\partial n}(m_i)\psi_i,$$

where a_i and m_i are the vertex and midpoint of the edge of K respectively, ϕ_i and ψ_i are the associated basis functions.

Because $|\phi_i|_{r,K} \le ch^{1-r}$ and $|\psi|_{r,K} \le ch^{2-r}$ ([8]), we have

$$\begin{aligned} |\pi_h v|_{r,K} &\le C(|v|_{0,\infty,\Delta K} h^{1-r} + |v|_{1,\infty,\Delta K} h^{2-r}) \\ &\le \sum_{s=0}^{1} Ch^{1+s-r}|v|_{s,\infty,\Delta K}. \end{aligned}$$

Using inverse ineqality, we have

$$|v|_{s,\infty,\Delta K} \le Ch_L^{-1}|v|_{s,\Delta K} \le Ch^{-1}|v|_{s,\Delta K}.$$

By the approximation theory (cf. [10]) and the fact $\pi_h p|_K = p$ for $p \in P_1$, where P_1 is the set of linear polynomials, we get

$$\begin{aligned} |v - \pi_h v|_{r,K} &\le C \inf_{p \in P_1(\Delta K)} \sum_{s=0}^{2} h^{s-r}|v + p|_{s,\Delta K} \\ &\le C \sum_{s=0}^{2} h^{s-r} h^{2-s} |v|_{2,\Delta K} \le Ch^{2-r}|v|_{2,\Delta K}. \end{aligned}$$

Squaring and summing up $K \in \Gamma_h$, we can see that Lemma 4.2 holds.

Lemma 4.3. *Let J_{L-1} be a standard interpolation operator with respect to space S_{L-1}, then we have*

$$|v - \pi_h J_{L-1} v|_{2,h} \le Ch|v|_3, \quad \forall v \in H^3(\Omega) \cap H_0^2(\Omega).$$

Proof. In fact, we have

$$|v - \pi_h J_{L-1} v|_{2,h} \le |v - \pi_h v|_{2,h} + |\pi_h(v - J_{L-1} v)|_{2,h}.$$

By the standard interpolation error estimate theory (cf. [10]), we obtain

$$|v - \pi_h v|_{2,h} \le Ch|v|_3.$$

On the other hand, for any $K \in \Gamma_h$, by Lemma 4.1, inverse inequality and interpolation theory (cf.[15]), we have

$$
\begin{aligned}
&|\pi_h(v - J_{L-1}v)|^2_{2,K} \\
&\quad \le Ch_K^{-4}\|\pi_h(v - J_{L-1}v)\|^2_{0,h} \\
&\quad \le Ch_K^{-4}(h_K^2\sum_{i=1}^{3}(v - J_{L-1}v)^2(a_i) + h_K^4\sum_{i=1}^{3}(\frac{\partial(v - J_{L-1}v)}{\partial n}(m_i)^2) \\
&\quad \le Ch_K^{-2}|v - J_{L-1}v|^2_{0,\infty,\triangle K} + C|v - J_{L-1}v|^2_{1,\infty,\triangle K} \\
&\quad \le Ch|v|_{3,\triangle K},
\end{aligned}
$$

Summing up all $K \in \Gamma_h$ and combining above inequalities, we obtain this Lemma.

Based on Lemma 4.2-4.3, we can see that (H2) is true for Morley element.

Using standard interpolation theory, we can see that (H4) is trivial.

Using a similar argument of Chapter 5 in [7], we can easily construct the smoother R_h for Morley element which satisfies (H2).

Finally, we have

Theorem 4.1. *Theorem 3.2 holds for space* V_h^m.

(2). Adini element In this subsection, we will apply the abstract theory to a nonconforming rectangle element, i.e., Adini element (cf. [10]), whose shape function is a incomplete bicubic polynomial and the nodal parameters are function values and two first partial derivatives at the vertices. It is a weak nonconforming element (i.e., C^0-element). Let V_h^a be Adini finite element space. We assume that $\partial\Omega$ and all sides of the rectangles in Γ_h are parallel to the coordinate axes.

we can see that (H1) holds for Adini finite element space. For the Adini finite element space V_h^a, we define $\pi_h : S_L \to V_h^a$ as standard nodal interpolation operator, then, by a similar argument of Lemma 4.2 and Lemma 4.3, we can show that (H2) holds for the Adini finite element space V_h^a. (H4) is trivial.

Using similar argument of Chapter 5 in [7], we can easily obtain the smoother R_h for Adini element which satisfies (H2).

Theorem 4.2. *Theorem 3.2 holds for Adini space* V_h^a.

References

[1] R.E. Bank, T.Dupont, An optimal order process for solving finite element equations, *Math. comp.,* **36** (1980), 35-51.

[2] H. Blum, R. Rannacher, On the boundery value problem of biharmonic operator on domains with angular corners, *Math. Methods Appl.Sci.,***2** (1980), 556-581.

[3] F.K. Bonger, R.L. Fox, L.A. Schmit, The generation of interelement compatible stiffness and mass matrices by the use of interpolation formulas, *In Proceedings of the Conference on Matrix Methods in Structural Mechanics* (1965), Wright Patterson A.F.B., Ohio.

[4] D. Braess, W. Hackbusch, A new convergence proof for the multigrid method including the V-cycle, *SIAM J.Numer. Anal.,* **20** (1983), 967-975.

[5] D. Braess, R. Verfurth, Multigrid methods for nonconforming finite element methods, *SIAM J. Numer. Anal.,* **27** (1990), 979-986.

[6] J.H. Bramble, Interpolation between Sobolev spaces in Lipschitz domains with an application to multigrid theory, *Math. Comp.,* **64** (1995), 1359-1365.

[7] J.H. Bramble, Multigrid Methods, Pitman, 1993.

[8] S.C. Brenner, Two-level additive Schwarz preconditioner for nonconforming plate element, to appear Numer. Math..

[9] S.C. Brenner, An optimal-order nonconforming multigrid method for the biharmonic equation, *SIAM J.Numer.Anal.,* **26** (1989), 1124-1138.

[10] P.G. Ciarlet, The Finite Element Method for Elliptic Problem, North-Holland, Amsterdam, 1978.

[11] W. Hackbusch, U. Trottenberg (ads), Multigrid Methods, Lecture Notes in Math, 960, Spring-verlag, 1982.

[12] M.R. Harnish, Multigrid preconditioning for the biharmonic dirichlet problem, *SIAM J. Numer. Anal.,* **30** (1993), 184-214.

[13] P. Lascaux, P. Lesaint, Some nonconforming finite element for plate bending problem, *RAIRO (R-1)*(1975), 9-53.

[14] J. Mandel, S. McCormick, R. Bank, Variational multigrid theory, in Multigrid Methods (S.McCormick, ed.), *SIAM Frontiers in Applied Mathematics 3, SIAM, Philadephia,* 1987, 131-177.

[15] P. Oswald, Hierarchical conforming finite element methods for biharmonic equation,*SIAM J. Numer. Anal.,* **29** (1992), 1610-1625.

[16] M.J.D. Powell, M.A. Sabin, Piecewise quadratic approximations on triangles, *ACM Trans. Mat. Software,* **3** (1977), 316-325.

[17] P. Peisker, D. Braess, A conjugate gradient method and a multigrid algorithm for Morley's finite element approximation of the biharmonic equations, *Numer. Math.,* **50** (1987), 567-587.

[18] P.S. Vassilevski, J. Wang, An application of the abstract multilevel theory to nonconforming finite element methods, *SIAM J. Numer. Anal.,* **32** (1995), 235-248.

[19] J. Xu, Iterative methods by space decomposition and subspace correction, *SIAM Review.,* **34** (1992), 581-613.

[20] J. Xu, Theory of Multilevel Methods, PhD Thesis of Cornell University, 1989.

[21] X. Zhang, Multilevel Schwarz methods for the biharmonic Dirichilet problem, *SIAM J. Sci. Comput.,* **15** (1994), 621-644.

[22] S. Yu, Multigrid method of nonconforming Wilson finite element, *Math. Numer. Sinica.,* **13** (1993), 346-351. (in chinese)