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FINITE ELEMENT NONLINEAR GALERKIN COUPLING METHOD FOR THE EXTERIOR STEADY NAVIER-STOKES PROBLEM[*1)]

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Abstract

In this paper we represent a new numerical method for solving the steady Navier-Stokes equations in three dimensional unbounded domain. The method consists in coupling the boundary integral and the finite element nonlinear Galerkin methods. An artificial smooth boundary is intrdouced seperating an interior inhomogeneous region from an exterior one. The Navier-Stokes equations in the exterior region are approximated by the Oseen equations and the approximate solution is represented by an integral equation over the artificial boundary. Moreover, a finite element nonlinear Galerkin method is used to approximate the resulting variational problem. Finally, the existence and error estimates are derived.

Key words: Navier-Stokes equations, Oseen equations, Boundary integral, Finite element, Nonlinear Galerkin method.

1. Introduction

Nonlinear Galerkin methods are multilevel schemes for the dissipative evolution partial differential equations. They correspond to the splittings of the unknown u : $u = y + z$, where the components are of different order of magnitude with respect to a parameter related to the spatial discretization. The numerical procedure consists of introducing an approximate inertial manifold which is a simplified approximation for the small component z. In particular, z is often obtained as a nonlinear functional of y. These methods have mainly been studied in the case of Fourier spectral discretizations (see [1-4]). The Finite elements approximations are considered in [5-8]. However, these works do not apply to the steady exterior Navier-Stokes equations.

Our purpose here is to present a new numerical method for solving the steady exterior Navier-Stokes equations. First, we introduce an artificial smooth boundary Γ_2 separating an unbounded part Ω_2 from a bounded part Ω_1. Then the Navier-Stokes equations in Ω_2 are approximated by the Oseen equations. By use of the Green

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formula, we derive the coupling problem of the Navier-Stokes equations in Ω_1 combining the boundary integral equation over Γ_2. Next, we present the coupling method of the boundary integral method and the finite element nonlinear Galerkin method for solving the coupling problem. Finally, we prove the well-posedness of the approximate problem and analyse the couvergence rate of the approximate solution. Our result show that the finite element nonlinear Galerkin coupling method is superior to the usual finite element Galerkin coupling method presented in the paper [9].

2. Continuous Coupling Problem

Let Ω_0 be a simply connected bounded open set of R^3 with smooth boundary Γ and let Ω denote the complement of $\Omega_0 \cup \Gamma$. The steady Navier-Stokes problem for a fluid occupying Ω consists in finding the velocity vector u of the fluid and its pressure p^* sach that

$$(N-S)\begin{cases} -\nu\Delta u^* + (u^*\cdot D)u^* + \nabla p^* = f & \text{in } \Omega \\ \text{div } u^* = 0 & \text{in } \Omega \\ u^* = \phi & \text{on } \Gamma \\ u^*(x) \to w_0 & \text{as } x \to \infty \end{cases}$$

Here the coefficient $\nu > 0$ is the dynamic viscosity of the fluids, f represents a density vector of external forces and ϕ is the velocity vector of the flow on Γ satisfying the condition $\int_\Gamma \phi\cdot n ds = 0$, where n denotes the unit vector normal to Γ, exterior to Ω, and w_0 is a constant vector. Moreover, we assume that f has a compact support in Ω.

For simplicity, we deal with the homogeneous boundary condition case of $\phi = 0$ in the sequel, but all the results stated here will still hold if the trace ϕ on Ω is any given sufficient smooth function that admits a solenoidel extension (div $u = 0$) in Ω.

For some sufficient large real number R, we introduce an artificial boundary $\Gamma_2 = \{x \in \Omega; |x| = R\}$ embedded in Ω, separating an unbounded region Ω_2 from a bounded region Ω_1 such that Ω_1 contains the support of f and $((u - w_0)\cdot\nabla)u$ is sufficiently small in Ω_2. We shall also denete by n the unit vector normal (from Ω_2) to Γ_2.

With above assumptions, we introduce an approximation (u, p) of (u^*, p^*) such that (u, p) satisfies the following coupling problem

$$(N-S')\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega_1 \\ \text{div } u = 0 & \text{in } \Omega_1 \\ u|_\Gamma = 0, \sigma(u,p)\cdot n|_{\Gamma_2} = \lambda^+ & \\ -\nu\Delta u + (w_0\cdot\nabla)u + \nabla p = 0 & \text{in } \Omega_2 \\ \text{div } u = 0 & \text{in } \Omega_2 \\ u|_{\Gamma_2} = u^-, \lim\limits_{|x|\to\infty} u(x) = w_0 & \end{cases}$$

where

$$\sigma(u,p)\cdot n|_{\Gamma_2} = -pn|_{\Gamma_2} + \nu\frac{\partial u}{\partial n}|_{\Gamma_2}, \lambda^+ = \sigma(u^+,p^+)\cdot n|_{\Gamma_2}$$
$$(u^-,p^-) = \lim_{x\to\Gamma_2}(u|_{\Omega_1}, p|_{\Omega_1}), (u^+,p^+) = \lim_{x\to\Gamma_2}(u|_{\Omega_2}, p|_{\Omega_2}).$$

We are now ready to give an integral representation formula for the solution (u,p) of the Oseen equations in Ω_2. Referring to [9], we have that for $k=1,2,3$

$$u_k(x) = -\int_{\Gamma_2} u(y)\cdot\sigma(U_k,P_k)\cdot n(y)ds_y - \int_{\Gamma_2}(u(y)\cdot U_k(x-y))(w_0\cdot n(y)ds_y$$
$$+\int_{\Gamma_2} U_k(x-y)\cdot\lambda(y)ds_y + w_0 \quad \forall x\in\Omega_2, \tag{2.1}$$
$$p(x) = -\int_{\Gamma_2}\nu u(y)\cdot n(y)(w_0\cdot\nabla\frac{1}{|x-y|}ds_y - \int_{\Gamma_2} u(y)\cdot P(x-y)(w_0\cdot n(y))ds_y$$
$$+\int_{\Gamma_2} P(x-y)\cdot\lambda(y)ds_y + C \quad \forall x\in\Omega_2, \tag{2.2}$$
$$\frac{1}{2}u_k(x) = -\int_{\Gamma_2} u(y)\cdot\sigma(U_k,P_k)(x-y)\cdot n(y)ds_y - \int_{\Gamma_2}(u(y)\cdot U_k(x-y))(w_0\cdot n(y))ds_y$$
$$+\int_{\Gamma_2} U_k(x-y)\cdot\lambda(y)ds_y + w_0 \quad \forall x\in\Gamma_2, \tag{2.3}$$

where $P=(P_1,P_2,P_3)$, (U_k,P_k) is the fundamental solution of the Oseen system:

$$-\nu\Delta U_k(x-y) + (w_0\cdot\nabla)U_k(x-y) + \nabla P_k(x-y) = \delta(x-y)e_k,$$
$$\text{div } U_k(x-y) = 0,$$

and (U_k,P_k) is given by

$$\nu U_k = \delta_{ki}\Delta\phi - \frac{\partial^2\phi}{\partial x_k\partial x_i}, P_k = -\frac{\partial}{\partial x_k}(\frac{1}{4\pi|x-y|}),$$
$$\phi = \frac{1}{8\pi\alpha}\int_0^{\alpha s}\frac{1-e^{-t}}{t}dt,$$
$$\alpha = \frac{|w_0|}{2\nu}, s = |x-y| - \frac{w_0\cdot(x-y)}{|w_0|}.$$

By introducing the following Sobolev spaces (see [9]):

$$X = \{v\in H^1(\Omega_1)^3; v=0 \quad \text{on } \Gamma\}.$$
$$X_0 = \{v\in X; \text{div} \ \ v=0 \quad \text{in } \Omega_1\},$$
$$M = L_0^2(\Omega_1) = \{q\in L^2(\Omega_1);\ \int_{\Omega_1} qdx = 0\},$$
$$T = \{\mu\in H^{-1/2}(\Gamma_2)^3;\ \int_{\Gamma_2}\mu\cdot ndx = 0\},$$

we obtain the continuous coupling variational problem corresponding to problem (N-S') and the integral equation (2.3):

$$
(Q)\begin{cases}
\text{Find } \ (u,\lambda,p)\in X\times M\times T \ \text{ such that} & \\
a(u,v)+a_1(u,u,v)-(p,\text{div } v)+<\gamma_0 v,\lambda>=(f,v) & \forall v\in X\\
b(\lambda,\mu)-\dfrac{1}{2}<\gamma_0 u,\mu>-<G(\gamma_0 u),\mu>=0 & \forall \mu\in T\\
(q,\text{div } \ u)=0 & \forall q\in M
\end{cases}
$$

where

$$(u,v)=\int_{\Omega_1} u\cdot v dx, \langle\gamma_0 v,\lambda\rangle=\int_{\Gamma_2} v\cdot\lambda ds_x,$$

$$a(u,v)=\nu((u,v)), ((u,v))=\int_{\Omega_1}\nabla u\cdot\nabla v dx,$$

$$a_1(u,v,w)=\int_{\Omega_1}(u\cdot\nabla)v\cdot w dx,$$

$$b(\lambda,\mu)=\int_{\Gamma_2}\int_{\Gamma_2}\mu(x)\cdot U(x-y)\cdot\lambda(y)ds_y ds_x,$$

$$G_k(\gamma_0 u)=\int_{\Gamma_2}u(y)\cdot\sigma(U_k,P_k)(x-y)\cdot n(y)ds_y, +\int_{\Gamma_2}(u(y)\cdot U_k(x-y))(w_0\cdot n(y))ds_y,$$

where G is a linear operator with respect to u.

The following estimates are classical (see [9, 11-15]):

$$|a(u,v)|\le\nu|u|_1|v|_1, a(u,u)=\nu|u|_1^2 \quad \forall u,v\in X, \tag{2.4}$$

$$|a_1(u,v,w)|\le c_0|u|_1|v|_1|w|_1 \quad \forall u,v,w\in X, \tag{2.5}$$

$$|a_1(u,v,w)|\le c_1|u|_0^{1/4}|u|_1^{3/4}|v|_1|w|_0^{1/4}|w|_1^{3/4} \quad \forall u,v,w\in X, \tag{2.6}$$

$$|b(\lambda,\mu)|\le c_2\|\lambda\|_{-1/2,\Gamma_2}\|\mu\|_{-1/2,\Gamma_2},$$

$$b(\mu,\mu)\ge c_3\|\mu\|_{-1/2,\Gamma_2}^2 \quad \forall\mu,\lambda\in T, \tag{2.7}$$

where $c=c(\Omega_1)$, $c_i=c_i(\Omega_1)$ $(i=0,1,\cdots)$ are positive constants dependent of Ω_1,

$$|u|_1=|u|_{1,\Omega_1}=\|\nabla u\|_{L^2(\Omega_1)^4}, |u|_0=|u|_{0,\Omega_1}=\|u\|_{L^2(\Omega_1)^2},$$

$$\|\lambda\|_{-1/2,\Gamma_2}=\|\lambda\|_{H^{-1/2}(\Gamma_2)^3}, \|\lambda\|_{1/2,\Gamma_2}=\|\lambda\|_{H^{1/2}(\Gamma_2)^3}.$$

Remark. In (2.4)–(2.7) we using following fact: Friedrichs inequality in X is still valid. Therefore, In X the seminorm $|\cdot|_1$ and full norm $\|\cdot\|_1$ are equivalent.

Theorem 2.1. *Suppose that* $f|_{\Omega_1}\in X'$ *and*

$$4c_0\nu^{-2}\|f\|_*<1 \tag{2.8}$$

Then the variational problem (Q) *admits a unique solution* $(u,\lambda,p)\in X\times T\times M$.

Moreover, if $f|_{\Omega_1} \in L^2(\Omega_1)^3$, then $(u, \lambda, p) \in (H^2(\Omega_1)^3 \cap X) \times (H^{1/2}(\Gamma_2)^3 \cap T) \times (H^1(\Omega_1) \cap M)$ satisfies

$$\|u\|_2 + \|\lambda\|_{1/2,\Gamma_2} + \|p\|_1 \leq c_4 |f|_{0,\Omega_1}, \tag{2.9}$$

where $\|u\|_2 = \|u\|_{H^2(\Omega_1)^3}$, $\|p\|_1 = \|p\|_{H^1(\Omega_1)}$, $\|f\|_* = \sup\limits_{v \in X} \dfrac{(f, v)}{|v|_1}$.

This proof can be found in the paper [9].

3. Finite Element Galerkin Coupling Approximation

For simplicity we restrict the discussion here to the case where Ω_1 has polyhedral boundary, but the results can be easily extended to a general curved domain, by introducing an approximate boundary $\Gamma_h \cup \Gamma_{2h}$. For further details we refer to [14].

From now on, h will be a real positive parameter tending to zero. First, we introduce three finite-dimensional subspaces X_h, T_h and M_h of X, T and M as followis. For each $h > 0$, let τ_h be a triangulation of Ω_1 made of tetrahedra K with diameters bounded by h. We suppose that $\{\tau_h\}$ is an affine family of class C^0, regular in the sense that there exists a constant $\gamma_1 > 0$ independent of h such that

$$h_K \leq \gamma_1 \rho_K \quad \forall K \in \tau_h,$$

where $h_K \leq h$ is the diameter of K and ρ_K is the diameter of the inscribed sphere in K. Let us denote by $s_i, 1 \leq i \leq n$ the finite number of triangular composing the boundary Γ_2. We take the following finite element spaces:

$$\begin{aligned}
X_h =& \{v_h \in C(\bar{\Omega}_1)^3 \cap X; v_h|_K \in P_2^3(K), \forall K \in \tau_h\},\\
T_h =& \{\mu_h \in C^0(\Gamma_2)^3 \cap T; \mu_h|_{s_i} \in P_1^3(s_i), 1 \leq i \leq n\},\\
M_h =& \{q_h \in M; q_h|_K \in P_0(K), \forall K \in \tau_h\},
\end{aligned}$$

where P_l denotes the space at all palynomials in three variables of degree $\leq l, 0 \leq l$. Moreover, we define the subspace X_{0h} of X_0 given by

$$X_{0h} = \{v_h \in X_h;\ (q_h, \mathrm{div}\ \ v_h) = 0, \quad \forall q_h \in M_h\}.$$

According to the literatures [9, 13-14], there hold the following approsimate properties:

(H_1) There exists an operator $\pi_h \in \mathcal{L}(H^2(\Omega_1)^3; X_h)$ such that

$$\begin{aligned}
&(q_h, \mathrm{div}\ \ (v - \pi_h v)) = 0 \quad \forall q_h \in M_h, \forall v \in H^2(\Omega_1)^3,\\
&|v - \pi_h v|_1 \leq ch\|v\|_2.
\end{aligned}$$

(H_2) The orthogonal projection operator $S_h : L^2(\Gamma_2)^3 \to T_h$ satisfies

$$\|\mu - S_h\mu\|_{-1/2,\Gamma_2} \leq ch\|\mu\|_{1/2,\Gamma_2}, \quad \forall \mu \in H^{1/2}(\Gamma_2)^3 \cap T.$$

(H_3) The orthogonal projection operator $\rho_h : L_0^2(\Omega_1) \to M_h$ verifies

$$|q - \rho_h q|_0 \le ch\|q\|_1, \quad \forall q \in H^1(\Omega_1)^3 \cap M.$$

(H_4) There exists a cons tant β, independent of h such that

$$\sup_{v_h \in X_{0h}} \frac{(q_h, \text{div } v)}{|v_h|_1} \ge \beta |q_h| \quad \forall q_h \in M_h.$$

with these finite element spaces, problems (Q) and (P) are approximated by

$$(Q_h) \begin{cases} \text{Find} \quad (u_h, \lambda_h, p_h) \in X_h \times T_h \times M_h \text{ such that} \\ a(u_h, v) + a_1(u_h, u_h, v) + \langle \gamma_0 v, \lambda_h \rangle - (p_h, \text{div } v) = (f, v) & \forall v \in X_h \\ b(\lambda_h, \mu) - \frac{1}{2}\langle \gamma_0 u_h, \mu \rangle - \langle G(\gamma_0 u_h), \mu \rangle = 0 & \forall \mu \in T_h \\ (q, \text{div } u_h) = 0 & \forall q \in M_h \end{cases}$$

and

$$(P_h) \begin{cases} \text{Find} \quad (u_h, \lambda_h) \in X_{0h} \times T_h \text{ such that} \\ a(u_h, v) + a_1(u_h, u_h, v) + \langle \gamma_0 v, \lambda_h \rangle = (f, v) & \forall v \in X_{0h} \\ b(\lambda_h, \mu) - \frac{1}{2}\langle \gamma_0 u_h, \mu \rangle - \langle G(\gamma_0 u_h), \mu \rangle = 0 & \forall \mu \in T_h \end{cases}$$

Theorem 3.1. *Assume that* $f|_{\Omega_1} \in X'$ *and*

$$4\nu^{-2} c_0 \|f\|_* < 1. \tag{3.1}$$

Then problem (Q_h) *has exactly one solution* $(u_h, \lambda_h, p_h) \in X_h \times T_h \times M_h$, *where* $(u_h, \lambda_h) \in X_{0h} \times T_h$ *is the unique solution of problem* (P_h). *Moreover, if* $f|_{\Omega_1} \in L^2(\Omega_1)^3$ *then*

$$\|u_h\|_2 + \|\lambda_h\|_{1/2,\Gamma_2} \le c|f|_{0,\Omega_1}. \tag{3.2}$$

and

$$|u - u_h|_1 + \|\lambda - \lambda_h\|_{1/2,\Gamma_2} + |p - p_h|_0 \le ch. \tag{3.3}$$

For the proof of Theorem 3.1, the readers can see the paper [9].

4. Finite Element Nonlinear Galerkin Coupling Approximation

In this section, we are given two parameters h and H, tending to 0, with $H > h > 0$. We consider four finite element spaces X_h, X_H, T_h and M_h with $X_H \subset X_h$ and we write

$$X_h = X_H + W_h, W_h = (I - R_H)X_h,$$

where $R_H : X \to X_H$ denote the L^2-orthogonal projections defined by

$$(R_H v, v_H) = (v, v_H) \quad \forall v \in X, v_H \in X_H.$$

The modified nonlineaar Galerkin method associated with (X_H, X_h, T_h, M_h) consists of looking for an approximate solution $(u^h, \lambda^h, p^h) \in X_h \times T_h \times M_h$ such that

$$u^h = y + z, y \in X_H, z \in W_h$$

and

$$(Q^h)\begin{cases} a(y+z,v) + a_1(y+z,y+z,v) + \langle \gamma_0 v, \lambda^h \rangle - (p^h, \text{div } v) = (f,v) & \forall v \in X_H \\ a(y+z,w) + a_1(y+z,y,w) + a_1(y,z,w) + \langle \gamma_0 w, \lambda^h \rangle & \\ \qquad -(p^h, \text{div } w) = (f,w) & \forall w \in W_h \\ b(\lambda^h, \mu) - \dfrac{1}{2}\langle \gamma_0 u^h, \mu \rangle - \langle G(\gamma_0 u^h), \mu \rangle = 0 & \forall \mu \in T_h \\ (q, \text{div } u^h) = 0 & \forall q \in M_h \end{cases}$$

or finding $(u^h, \lambda^h) \in X_{0h} \times T_h$ such that

$$(P^h)\begin{cases} a(y+z,v) + a_1(y+z,y+z,v) + \langle \gamma_0 v, \lambda^h \rangle = (f,v) & \forall v \in X_H \cap X_{0h} \\ a(y+z,w) + a_1(y+z,y,w) + a_1(y,z,w) + \langle \gamma_0 w, \lambda^h \rangle = (f,w) & \forall w \in W_h \cap X_{0h} \\ b(\lambda^h, \mu) - \dfrac{1}{2}\langle \gamma_0 u^h, \mu \rangle - \langle G(\gamma_0 u^h), \mu \rangle = 0 & \forall \mu \in T_h \end{cases}$$

Recalling again [6–7, 16], the following properties are classical, namely

(H_5) There exists a constant H_0 such that for $0 < h < H \le H_0$, $X_H \cap X_{0h} \ne \{0\}$

(H_6) There exists a constant $0 < \delta < 1$ such that

$$\delta(|v|_1^2 + |w|_1^2) \le |v+w|_1^2 \quad \forall v \in X_H, \quad w \in W_h.$$

$$(H_7) \qquad |w|_0 \le \gamma H |w|_1 \quad \forall w \in W_h.$$

In order to consider the well-posedness of problem (Q^h), we introduce the following lemma.

Lemma 4.1. *For any $u^h \in X_{0h}$, the variational formulation*

$$b(\lambda^h, \mu) - \frac{1}{2}\langle \gamma_0 u^h, \mu \rangle - \langle G(\gamma_0 u^h), \mu \rangle = 0 \quad \forall \mu \in T_h$$

admits a unique solution $\lambda^h = \lambda(\gamma_0 u^h)$ such that

$$\langle \gamma_0 u^h, \lambda(\gamma_0 u^h) \rangle \ge 0$$
$$\|\lambda(\gamma_0 u^h)\|_{-1/2,\Gamma_2} \le c|u^h|_1$$

This proof can refer to [9].

Thanks to Lemma 4.1, problems (Q^h) and (P^h) can be rewritten as

$$(Q^h)\begin{cases} \text{Find } \ (u^h = y+z, p^h) \in X_h \times M_h \ \text{ such that} & \\ a(u^h, v) + a_1(u^h, u^h, v) - a_1(z, z, r_H v) + \langle \gamma_0 v, \lambda(\gamma_0 u^h) \rangle & \\ \qquad -(p^h, \text{div } v) = (f, v) & \forall v \in X_h \\ (q, \text{div } u^h) = 0 & \forall q \in M_h \end{cases}$$

and

$$(P^h)\begin{cases} \text{Find } \ u^h = y + z \in X_{0h} \ \text{ such that} \\ a(u^h, v) + a_1(u^h, u^h, v) - a_1(z, z, r_H v) \\ +\langle \gamma_0 v, \lambda(\gamma_0 u^h)\rangle = (f, v) \qquad\qquad \forall v \in X_{0h} \end{cases}$$

where $r_H = I - R_H$.

In order to consider the well-posedness of problem (Q^h), we first consider ones of problem (P^h). To do this, we study the following problem:

Given $g \in K_h$, find $v^h \in X_{0h}$ such that

$$\begin{aligned} a(v^h, v) &+ \langle \gamma_0 v, \lambda(\gamma_0 u^h)\rangle \\ &= (f, v) - a_1(g, g, v) + a_1(r_H g, r_H g, r_H v) \quad \forall v \in X_{0h} \end{aligned} \tag{4.1}$$

where $K_h = \{g \in X_{0h};\, |g|_1 \le \dfrac{3}{\nu}\|f\|_*\}$.

In view of Lemma 4.1, the bilinear form: $a(\cdot,\cdot) + \langle \gamma_0 \cdot, \lambda(\gamma_0 \cdot)\rangle$ is continuous and coercive on $X_{0h} \times X_{0h}$. Hence, there exists a unique solution $v^h \in X_{0h}$ for (4.1) according to the Lax-Milgram theorem. So, (4.1) defines a mapping $E : K_h \to X_{0h}$. Thus, the problem (P^h) is equivalent to the operator equation

$$u^h = E u^h \tag{4.2}$$

In other words, u^h is a solution of (P^h) if and only if u^h is a fixed point of E.

Theorem 4.2. *Suppose that $\nu, c_0(\Omega_1), f|_{\Omega_1} \in X'$ and $H \le H_0$ satisfy the unique condition:*

$$8c_0\nu^{-2}\|f\|_* < 1,\, 6\nu^{-2} c_1 \gamma^{1/2} H^{1/2} \delta^{-3/2}\|f\|_* < \frac{1}{5}. \tag{4.3}$$

Then there exists a unique fixed point u^h of E in the set K_h.

Proof. First, we prove that E is mapping of K_h into K_h. Let $g \in K_h$, then $v^h = Eg$ satisfies (4.1). Taking $v = v^h$ in (4.1) and using (2.5)–(2.6), $(H_6) - (H_7)$ and the following estimates:

$$a(v^h, v^h) + \langle \gamma_0 v^h, \lambda(\gamma_0 v^h)\rangle \ge \nu |v^h|_1^2, \tag{4.4}$$

we obtain

$$\nu |v^h|_1 \le \|f\|_* + c_0 |g|_1^2 + c_1 \gamma^{1/2} H^{1/2} \delta^{-3/2} |g|_1^2. \tag{4.5}$$

Thanks to the uniqueness condition (4.3) and $|g|_1 \le \dfrac{3}{\nu}\|f\|_*$, (4.5) yields

$$\nu |v^h|_1 \le 3\|f\|_*. \tag{4.6}$$

So, $v^h \in K_h$, namely, $E : K_h \to K_h$.

Secondly, E is a contraction mapping in K_h. In fact, if $g_1, g_2, \in K_h$, then $v_1^h = Eg_1, v_2^h = Eg_2$ satisfy

$$a(v_1^h - v_2^h, v) + \langle \gamma_0 v, \lambda(\gamma_0(v_1^h - v_2^h))\rangle = a_1(g_2 - g_1, g_2, v) + a_1(g_1, g_2 - g_1, v)$$

$$- a_1(r_H(g_2 - g_1), r_H g_2, r_H v) - a_1(r_H g_1, r_H(g_2 - g_1), r_H v) \quad \forall v \in X_{0h}. \tag{4.7}$$

Taking $v = v_1^h - v_2^h$ in (4.7) and using (4.4), (2.5)–(2.6) and $(H_6) - (H_7)$, we obtain

$$\nu |v_1^h - v_2^h|_1 \le c_0 |g_1 - g_2|_1 (|g_1|_1 + |g_2|_1 + c_1 \gamma^{1/2} H^{1/2} \delta^{-3/2} (|g_1|_1 + |g_2|_1) |g_1 - g_2|_1. \tag{4.8}$$

Due to (4.3), we derive from (4.8) that

$$|v_1^h - v_2^h|_1 \le (6c_0 \nu^{-2} \|f\|_* + 6c_1 \gamma^{1/2} H^{1/2} \delta^{-3/2} \|f\|_*) |g_1 - g_2|_1 \le \frac{19}{20} |g_1 - g_2|_1. \tag{4.9}$$

So, E is a contraction mapping of K_h into K_h. By the fixed point theorem, Theorem 4.2 is proven.

Once u^h is obtained as the solution of problem (P^h), there remains to solve: find $p^h \in M_h$ such that

$$\begin{aligned}(p^h, \text{div } v) =& a(u^h, v) + a_1(u^h, u^h, v) - a_1(z, z, r_H v) \\ &+ \langle \gamma_0 v, \lambda(\gamma_0 u^h) \rangle - (f, v) \quad \forall v \in X_h.\end{aligned} \tag{4.10}$$

Here, the right hand-side of (4.10) is a functional on X_h which, due to the definition of u^h, vanishes on X_{0h}. It is classical that the inf-sup condition (H_4) guarantees that (4.10) is uniquely solvable in the space M_h. This give the following existence and uniqueness of the solution (u^h, p^h) of problem (Q^h).

Theorem 4.3. *With the above finite element spaces X_H, X_h, M_h and T_h and the uniqueness condition* (4.3)*, the problem (Q^h) admits a unique solution $(u^h, \lambda^h, p^h) \in X_h \times T_h \times M_h$, where $(u^h, \lambda^h) \in X_{0h} \times T_h$ is the unique solution of problem (P^h).*

Moreover, if $f|_{\Omega_1} \in L^2(\Omega_1)^3$ then

$$\|u^h\|_2 + \|\lambda^h\|_{1/2,\Gamma_2} \le c|f|_{0,\Omega_1}. \tag{4.11}$$

The proof of (4.10) is classical, it can be omitted.

5. Error Estimaates

In this section, we aim to derive the error estimates for the finite element nonlinear Galerkin coupling method in terms of the three parameters R, H and h.

First, we shall give the estimate $|u^* - u|_{1,\Omega}$. According to problem $(N - S')$, u satisfies

$$(N-S')\begin{cases} -\nu \Delta u + X_{\Omega_1}(u \cdot \nabla) u + (1 - X_{\Omega_1})(w_0 \cdot \nabla) u + \nabla p = f & \text{in } \Omega \\ \text{div } u = 0 & \text{in } \Omega \\ u|_\Gamma = 0, \lim\limits_{|x| \to \infty} u(x) = w_0 \end{cases}$$

where

$$X_{\Omega_1}(x) = \begin{cases} 1 & x \in \bar{\Omega_1} \\ 0 & x \bar{\in} \bar{\Omega_1}. \end{cases}$$

Hence, $w = u^* - u$ and $\eta = p^* - p$ satisfy

$$\begin{aligned} &-\nu \Delta w + x_{\Omega_1}((w \cdot \nabla)u^* + (u \cdot \nabla)w) + \nabla \eta \\ &+ (1 - X_{\Omega_1})((u^* - w_0) \cdot \nabla)(u^* - w_0) + (w_0 \cdot \nabla)w) = 0, \qquad (5.1) \\ &\text{div} \;\; w = 0, \qquad (5.2) \\ &w|_\Gamma = 0, \lim_{|x| \to \infty} w(x) = 0. \qquad (5.3) \end{aligned}$$

According to the literatures [10–12], there hold

$$\begin{aligned} &u^*(x) - w_0 = O(|x|^{-1}), u(x) - w_0 = O(|x|^{-1}) \quad \forall x \geq R, \qquad (5.4) \\ &\int_\Omega (u^* \cdot \nabla)w \cdot w dx = 0. \qquad (5.5) \end{aligned}$$

Equation (5.1) formally multiplied by w and integrated in Ω yields

$$\begin{aligned} \nu|w|^2_{1,\Omega} + a_1(w, u^*, w) + a_1(u, w, w) + \int_{\Omega_2} ((u^* - w_0) \cdot \nabla)(u^* - w_0) \cdot w dx \\ + + \int_{\Omega_2} (w_0 \cdot \nabla)w \cdot w dx = 0, \qquad (5.6) \end{aligned}$$

where (5.2) is used. Thanks to (5.2) and (5.5), we have

$$\begin{aligned} &\int_{\Omega_2} ((u^* - w_0) \cdot \nabla)(u^* - w_0) \cdot w dx + \int_{\Omega_2} ((u^* - w_0) \cdot \nabla)w \cdot (u^* - w_0) dx \\ &\qquad = \int_{\Gamma_2} (u^* - w_0) \cdot n((u^* - w_0) \cdot w) ds_x, \qquad (5.7) \\ &\int_{\Omega_2} (w_0 \cdot \nabla)w \cdot w dx = \int_{\Omega_2} ((w_0 - u^*) \cdot \nabla)w \cdot w dx - a_1(u^*, w, w), \qquad (5.8) \\ &\int_{\Omega_2} ((w_0 - u^*) \cdot \nabla)w \cdot w dx = \frac{1}{2} \int_{\Gamma_2} (w_0 - u^*) \cdot n|w|^2 ds_x. \qquad (5.9) \end{aligned}$$

Moreover, due to (5.4) there hold

$$\begin{aligned} &\int_{\Gamma_2} (u^* - w_0) \cdot n(u^* - w_0) \cdot w ds_x = O(R^{-1}), \qquad (5.10) \\ &\frac{1}{2} \int_{\Gamma_2} (w_0 - u^*) \cdot n|w|^2 ds_x = O(R^{-1}). \qquad (5.11) \end{aligned}$$

Combining (5.6) with (5.7)–(5.11) yields

$$\begin{aligned} &\nu|w|^2_{1,\Omega_1} + a_1(w, u^*, w) + a_1(u^*, w, w) + a_1(u, w, w) \\ &\qquad + \int_{\Omega_2} ((u^* - w_0) \cdot \nabla)w \cdot (u^* - w_0) dx = O(R^{-1}). \qquad (5.12) \end{aligned}$$

However, due to (2.5) and (5.5) we imply

$$a_1(w,u^*,w)+a_1(u^*,w,w)=2a_1(w,w,w)+a_1(w,u,w)+a_1(u,w,w), \tag{5.13}$$

$$|a_1(u,w,w)|\le c_0|u|_1|w|_1^2, \tag{4.14}$$

$$|a_1(w,u,w)|\le c_0|u|_1|w|_1^2, \tag{4.15}$$

$$\Big|\int_{\Omega_2}((u^*-w_0)\cdot\nabla)w\cdot(u^*-w_0)dx\Big|\le\Big(\int_{\Omega_2}|\nabla w|^2dx\Big)^{1/2}\Big(\int_{\Omega_2}|u^*-w_0|^4dx\Big)^{1/2}$$
$$\le\frac{\nu}{4}|w|_{1,\Omega}^2+\nu^{-1}\int_{\Omega_2}|u^*-w_0|^4dx, \tag{5.16}$$

$$\int_{\Omega_2}|u^*-w_0|^4dx=\int_{\Omega_2}O(|x|^{-4})dx=O(R^{-1}), \tag{5.17}$$

$$2a_1(w,w,w)=\int_{\Omega_2}w\cdot n|w|^2dx=O(R^{-1}). \tag{5.18}$$

So, (5.12) amd (5.13)–(5.18) yield

$$3\nu|w|_{1,\Omega}^2-12c_0|u|_1|w|_{1,\Omega}^2=O(R^{-1}). \tag{5.19}$$

According to the paper [9], there holds

$$|u|_1\le\frac{2}{\nu}\|f\|_*. \tag{5.20}$$

namely, we have

$$3\nu\Big(1-\frac{8}{\nu^2}c_0\|f\|_*\Big)|w|_{1,\Omega}^2=O(R^{-1}). \tag{5.21}$$

Using again the uniqueness condition (4.3), we derive

$$1-\frac{8}{\nu^2}c_0\|f\|_*=\alpha>0. \tag{5.22}$$

Therefore, (5.21) and (5.22) yield

$$|u^*-u|_{1,\Omega}=O(R^{-1/2}). \tag{5.23}$$

Recalling again the discussions given in section 3, we obtain the approximate accuracy of u_h.

Theorem 5.1. *Assume that $\nu, c_0(\Omega_1)$ and f satisfy the uniqueness condition* (5.22), *then*

$$|u^*-u_h|_1=O(R^{-1/2}+h). \tag{5.24}$$

Next, it remains now to derive the convergence rate of u^h.

Theorem 5.2. *Assume that $\nu, c_0(\Omega_1), f$ and H satisfy the uniqueness condition* (4.3), *then*

$$|u^*-u_h|_1=O(R^{-1/2}+h+H^{5/2}). \tag{5.25}$$

Proof. We set

$$E = u_h - u^h = e + \varepsilon,\ e = R_H u_h - y, \varepsilon = (I - R_H)u_h - z$$
$$\eta = p_h - p^h,\ \xi = \lambda_h - \lambda^h.$$

Then problem (Q_h) and problem (Q^h) yield

$$a(E,v) + a_1(u_h,E,v) + a_1(E,u^h,v) + a_1(z,z,r_H v) + \langle \gamma_0 v, \xi\rangle - \langle \eta, \mathrm{div}\ \ v\rangle = 0 \quad \forall v \in X_h, \tag{5.26}$$
$$b(\xi,\mu) - \frac{1}{2}\langle \gamma_0 E, \mu\rangle - \langle G(\gamma_0 E), \mu\rangle = 0 \quad \forall \mu \in T_h, \tag{5.27}$$
$$(q\mathrm{div}\ \ E) = 0 \quad \forall q \in M_h. \tag{5.28}$$

According to Lemma 4.1, (5.27) implies that $\xi = \lambda(\gamma_0 E)$ satisfies

$$\langle \gamma_0 E, \lambda(\gamma_0 E)\rangle \geq 0, \tag{5.29}$$
$$\|\xi\|_{-1/2,\Gamma_2} \leq c|E|_1. \tag{5.30}$$

Thus, taking $v = E$ in (5.26) and using (5.28)–(5.29), we derive

$$\nu|E|_1^2 + a_1(u_h,E,E) + a_1(E,u^h,E) + a_1(z,z,\varepsilon) = 0. \tag{5.31}$$

Thanks to (2.5)-(2.6), we have

$$|a_1(u_h,E,E) + a_1(E,u^h,E) \leq c_0(|u^h|_1 + |u_h|_1)|E|_1^2, \tag{5.32}$$
$$|a_1(z,z,\varepsilon)| \leq c_1|z|_0^{1/4}|z|_1^{1/4}|\varepsilon|_0^{1/4}|\varepsilon|_1^{1/4}. \tag{5.33}$$

Recalling the paper [9], there holds

$$|u_h|_1 \leq \frac{2}{\nu}\|f\|_*. \tag{5.34}$$

Referring again to the proof of Theorem 4.2, u^h satisfies

$$|u^h|_1 \leq \frac{3}{\nu}\|f\|_*. \tag{5.35}$$

Thus, (5.32) and (5.34)-(5.35) give

$$|a_1(u_h,E,E)| + a_1|(u^h,E,E) \leq \frac{5}{\nu}c_0\|f\|_*|E|_{1,\Omega}^2. \tag{5.36}$$

Using again $(H_6) - (H_7)$, (5.33) yields

$$|a_1(z,z,\varepsilon)| \leq c_1\gamma^{1/2}H^{1/2}|z|_1^2|\varepsilon|_1 \leq c_1\delta^{-1/2}\gamma^{1/2}H^{1/2}|z|_1^2|E|_1 \leq \frac{3}{\nu}c_0|f\|_*|E|_1^2 + cH|z|_1^4. \tag{5.37}$$

Combining (5.31) with (5.36)–(5.37) yields

$$\alpha|E|_1^2 \le cH|z|_1^4, \tag{5.38}$$

where $\alpha = \nu - \frac{8}{\nu}c_0\|f\|_* > 0$.

Referring again to Ait Ou Ammi [7], there holds

$$|z|_0 + H|z|_1 = |u^h - R_H u^h|_0 + H|u^h - R_H u^h|_1 \le cH^2\|u^h\|_2.$$

Hence, we imply

$$|z|_1^4 \le cH^4\|u^h\|_2^4. \tag{5.39}$$

This and (5.38) imply

$$|E|_1 \le cH^{5/2}. \tag{5.40}$$

Combining (5.40) with (5.24) implies (5.25). The proof ends.

Remark According to Theorem 5.1 and Theorem 5.2, the nonlineaar Galerkin scheme provides the same order of approximation as the classical Galerkin scheme if we choose $H = O(h^{2/5})$. However, in the nonlinear Galerkin scheme, the nonlinearity is treated on the coarse grid finite element space X_H and only the linear problem needs to be solved on the fine grid finite element increment space W_h. For the classical Galerkin scheme, the nonliearity needs to be treated in the fine grid finite element space X_h. Hence, the nonlinear Galerkin scheme is superior to the classical Galerkin scheme.

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