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ON THE CONVERGENCE OF NONCONFORMING FINITE ELEMENT METHODS FOR THE 2ND ORDER ELLIPTIC PROBLEM WITH THE LOWEST REGULARITY*1)

Lie-heng Wang
(*LSEC, Institute of Computational Mathematics, Academia Sinica P.O.Box 2719, Beijing 100080, China*)

Abstract

The convergences ununiformly and uniformly are established for the nonconforming finite element methods for the second order elliptic problem with the lowest regularity, i.e., in the case that the solution $u \in H_0^1(\Omega)$ only.

Key words: Nonconforming finite element methods, Lowest regularity.

1. Introduction

The aim of this note is to establish the convergence of the nonconforming finite element methods for the second order elliptic problem with the lowest regularity. The proof of the convergence is not trivial, although the convergence results for the conforming finite element methods were known ([2], [3]).

Consider the following boundary value problem on a polygonal domain $\Omega \subset R^2$:

$$\begin{cases} Au = \sum_{i,j=1}^2 -\partial_j(a_{ij}(x)\partial_i u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

We assume that the coefficients $a_{ij}(x) \in L^\infty(\Omega)$ and the A is uniformly elliptic on Ω, i.e., there exists a constant $\alpha > 0$ such that for all real vectors $\xi = (\xi_1, \xi_2)$ and all $x \in \Omega$

$$\sum_{i,j=1}^2 a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^2 \xi_i. \tag{1.2}$$

The weak formulation of (1.1) is: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) \equiv \int_\Omega a_{ij}\partial_i u \partial_j v dx = \int_\Omega f v dx \equiv f(v), \quad \forall \ v \in H_0^1(\Omega). \tag{1.3}$$

It is well known that for any given $f \in H^{-1}(\Omega)$, there exists an unique solution $u \in H_0^1(\Omega)$ of the problem (1.3), by the Lax-Milgram Lemma, and the conforming finite element approximation u_h converges to u in $H^1(\Omega)$ space (c.f.[2]).

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We now consider the nonconforming finite element methods for the problem (1.3). For each $h \in (0,1)$, let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω, and V_h be a nonconforming finite element space with respect to the triangulation $\mathcal{T}_h$. In this case it should be noted that $V_h \not\subset H^1(\Omega)$, and assume that $f \in L^2(\Omega)$, while it can be assumed that $f \in H^{-1}(\Omega)$ for the conforming finite element methods, since the functional $f \in H^{-1}(\Omega)$ is defined on the space $H_0^1(\Omega)$ only. And it is also noted that the solution u of the problem (1.3) is, in general, in $H_0^1(\Omega)$ space only, in tha case of that $f \in L^2(\Omega)$, since that it is not known in general whether $u \in H^s(\Omega)$ for some $s > 1$ even if $f \in C^\infty(\Omega)$. Finally it is assumed that the element of the nonconforming finite element space V_h passes the generalized patch test, which is the necessary and sufficient condition, assuming the approximation holding, for the convergence of nonconforming finite element methods in the case of the solution u of the problem (1.3) smoother enough (c.f.[5]).

Then the nonconforming finite element approximation to (1.3) is: Find $u_h \in V_h$, such that

$$a_h(u_h, v_h) \equiv \sum_K \int_K a_{ij} \partial_i u_h \partial_j v_h dx = \int_\Omega f \cdot v_h dx \equiv f(v_h) \quad \forall \quad v_h \in V_h. \tag{1.4}$$

2. Convergence

Theorem 2.1. *Assume that the solution of the problem* (1.3) $u \in H_0^1(\Omega)$, $f \in L^2(\Omega)$, *the triangulation* $\mathcal{T}_h$ *of the polygonal* Ω *is quasi-uniform and satisfies the inverse hypothesis* (*c.f.*[2]), *and the nonconforming finite element space* $V_h \not\subset H_0^1(\Omega)$ *possessing the following property, for any given* $\phi \in C_0^\infty$, *there exists* $C = Const. > 0$ *independent of* h, *such that*

$$\Big| \sum_K \int_{\partial K} \partial_\nu \phi \cdot w_h ds \Big| \leq Ch \|\phi\|_{2,\Omega} \cdot \|w_h\|_h, \quad \forall \quad w_h \in V_h, \tag{2.1}$$

where $K \in \mathcal{T}_h$ *is the element with the edge* $\partial K, \partial_\nu$ *denotes the conormal derivative operator associated with the operator* A *in* (1.1) *on* ∂K, *and*

$$\|w_h\|_h \equiv \Big\{ \sum_K |w_h|_{1,K}^2 \Big\}^{\frac{1}{2}}. \tag{2.2}$$

Then the solution of the problem (1.4) u_h *converges to the solution of the problem* (1.3) u *in the space* $H^1(\Omega)$ *as* $h \longrightarrow 0$. *Precisely, for any given* $\epsilon > 0$, *there exists* $h_0 = h_0(\epsilon, u, f) > 0$, *such that*

$$\|u - u_h\|_h < \epsilon, \quad as \quad 0 < h \leq h_0. \tag{2.3}$$

Proof. (i) By the second Strang Lemma (c.f.[4])

$$\|u - u_h\|_h \leq \Big\{ \inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{w_h \in V_h} \frac{E_h(u, w_h)}{\|w_h\|_h} \Big\}, \tag{2.4}$$

where

$$E_h(u,w_h) = a_h(u,w_h) - f(w_h). \tag{2.5}$$

In the same way as in [2] for the conforming finite element methods, we can find that there exists $h_0' = h_0'(\epsilon, u) > 0$, such that

$$\inf_{v_h \in V_h} \|u - v_h\|_h < \frac{\epsilon}{2}, \quad \text{as} \quad 0 < h \leq h_0'. \tag{2.6}$$

(ii) We now estimate the term $E_h(u, w_h)$. Since $u \in H_0^1(\Omega)$, then for any given $\epsilon' > 0$, there exists a function $\tilde{u} \in C_0^\infty(\Omega)$, such that

$$\|u - \tilde{u}\|_{1,\Omega} < \epsilon'. \tag{2.7}$$

So we have

$$E_h(u,w_h) = E_h(u - \tilde{u}, w_h) + E_h(\tilde{u}, w_h), \tag{2.8}$$

and

$$|E_h(u - \tilde{u}, w_h)| = |a_h(u - \tilde{u}, w_h)| \leq C\|u - \tilde{u}\|_{1,\Omega} \cdot \|w_h\|_h < C\epsilon'\|w_h\|_h. \tag{2.9}$$

With use of the Green formula, we have

$$\begin{aligned} E_h(\tilde{u}, w_h) &= a_h(\tilde{u}, w_h) - f(w_h) = \sum_K \int_K a_{ij}\partial_i\tilde{u}\partial_j w_h dx - \int_\Omega f w_h dx \\ &= \Big\{ -\sum_K \int_K \partial_j(a_{ij}\partial_i\tilde{u}) w_h dx - \int_\Omega f w_h dx \Big\} + \sum_K \int_{\partial K} \partial_\nu \tilde{u} w_h ds. \end{aligned} \tag{2.10}$$

By the assumption of the Theorem

$$\Big| \sum_K \int_{\partial K} \partial_\nu \tilde{u} w_h ds \Big| \leq Ch\|\tilde{u}\|_{2,\Omega} \cdot \|w_h\|_h. \tag{2.11}$$

(iii)We now turn to estimate the first two terms on the right hand side of (2.10). Let

$$-\partial_j(a_{ij}(x)\partial_i\tilde{u}) = \tilde{f} \quad \text{in} \quad \Omega, \quad \tilde{u} = 0 \quad \text{on} \quad \partial\Omega, \tag{2.12}$$

which is equivalent to the following problem: $\tilde{u} \in H_0^1(\Omega) \cap H^2(\Omega)$, such that

$$a(\tilde{u}, v) = \tilde{f}(v) \quad \forall \quad v \in H_0^1(\Omega). \tag{2.13}$$

And let the interpolation operator $\tilde{\Pi}_h : V_h \longrightarrow \tilde{V}_h, \tilde{V}_h$ be the corresponding conforming finite element space, $\tilde{V}_h \subset H_0^1(\Omega)$, and assume that

$$\|\tilde{\Pi}_h w_h - w_h\|_{0,\Omega} \leq Ch\|w_h\|_h, \tag{2.14}$$

we can find such interpolation operator for the nonconforming finite elements of Wilson, Crouzeit-Raviart (c.f.[1]). Then

$$-\sum_K \int_K \partial_j(a_{ij}\partial_i\tilde{u}) w_h dx - \int_\Omega f w_h dx = \int_\Omega (\tilde{f} - f) w_h dx$$

$$
\begin{aligned}
&= \int_\Omega (\tilde{f} - f)\tilde{\Pi}_h w_h dx + \int_\Omega (\tilde{f} - f)(w_h - \tilde{\Pi}_h w_h) dx \\
&\leq C\|\tilde{f} - f\|_{-1,\Omega} \cdot \|\tilde{\Pi}_h w_h\|_{1,\Omega} + \|\tilde{f} - f\|_{0,\Omega} \cdot \|w_h - \tilde{\Pi}_h w_h\|_{0,\Omega}.
\end{aligned} \tag{2.15}
$$

And by (1.3) and (2.13), we have

$$
\begin{aligned}
\|\tilde{f} - f\|_{-1,\Omega} &= \sup_{v \in H_0^1(\Omega)} \frac{\tilde{f}(v) - f(v)|}{\|v\|_{1,\Omega}} = \sup_{v \in H_0^1(\Omega)} \frac{|a(\tilde{u} - u, v)|}{\|v\|_{1,\Omega}} \\
&\leq C\|\tilde{u} - u\|_{1,\Omega} < C\epsilon'.
\end{aligned} \tag{2.16}
$$

Then from (2.14)–(2.16), and with the inverse inequality, we have

$$
-\sum_K \int_K \partial_j (a_{ij}(x)\partial_i \tilde{u}) w_h dx - \int_\Omega f w_h dx \leq C\{\epsilon' + h\|\tilde{f} - f\|_{0,\Omega}\} \cdot \|w_h\|_h. \tag{2.17}
$$

Finally, from (2.8), (2.11)and (2.17), it can be seen that

$$
|E_h(u, w_h)| \leq C(\epsilon' + h)\|w_h\|_h, \tag{2.18}
$$

where the constent C is dependent on f. Then there exists $h_0'' = h_0''(\epsilon, u, f) > 0$, such that

$$
Ch_0'' < \frac{\epsilon}{4},
$$

and choosing ϵ' such that

$$
C\epsilon' < \frac{\epsilon}{4},
$$

thus

$$
|E_h(u, w_h)| < \frac{\epsilon}{2}\|w_h\|_h, \quad \text{as} \quad 0 < h \leq h_0''. \tag{2.19}
$$

Summarizing (2.4), (2.6) and (2.19) implies the result (2.3) of the Theorem as $h_0 = min(h_0', h_0'')$.

3. Uniformly Convergence

In the previous section, it is investigated that the nonconforming finite element approximation u_h conveges to the solution u of the problem (1.3) as $h \longrightarrow 0$, but not uniformly, that means that in the Theorem 2.1, $h_0 = h_0(\epsilon, u, f)$ is dependent not only on ϵ, but also on u and f. For the situation of conforming finite element methods, the uniformly convergence has been considered by Schatz and Wang in [3]. By the similar way as [3], in this section we can also prove the uniformly convergence for the nonconforming finite element methods. Our result is the following

Theorem 3.1. *Under the hypotheses of the Theorem* 2.1*, then the following result holds: For any given* $\epsilon > 0$*, there exists an* $h_0 = h_0(\epsilon) > 0$*, such that for all* $0 < h \leq h_0$*,*

$$
\|u - u_h\|_h \leq \epsilon\|f\|_0. \tag{3.1}
$$

Before proving Theorem 3.1, we will statement some lemmas in [3].

Lemma 3.2. *Let $D = \{f : f \in L^2(\Omega),\ \|f\|_0 = 1\}$ be the unit sphere in $L^2(\Omega)$. Let $W = \{u : u = Tf, f \in D\}$ where $u = Tf \in H_0^1(\Omega)$ is the solution of* (1.3), *i.e.,*

$$a(Tf, v) = f(v) \quad \forall \quad v \in H_0^1(\Omega). \tag{3.2}$$

Then W is precompact in $H_0^1(\Omega)$.

Lemma 3.3. *Let V be a fixed compact subset of $H_0^1(\Omega)$. Then there exists a finite open cover: $S(\phi_1, \epsilon), \cdots, S(\phi_n, \epsilon)$, such that $V \subset \cup_{i=1}^n S(\phi_i, \epsilon)$, and $\phi_i \in C_0^\infty$ for $1 \le i \le n$, where $S(\phi, \epsilon)$ is an open ball with the center ϕ and the radius ϵ in the sense of $H^1(\Omega)-$norm.*

The proof of the Lemma 3.3 is due to that the space $C_0^\infty(\Omega)$ is dence in $H_0^1(\Omega)$.

Proof of Theorem 3.1. We prove the estimate (3.1) by the similar manner as in [3]. For $f \in L^2(\Omega)$ set

$$\bar{f} = \frac{f}{\|f\|_0}, \bar{u} = \frac{u}{\|f\|_0} \quad \text{and} \quad \bar{u}_h = \frac{u_h}{\|f\|_0}.$$

Then $a(\bar{u}, v) = \bar{f}(v)\ \forall v \in H_0^1(\Omega)$, and $a_h(\bar{u}_h, v_h) = \bar{f}(v_h)\ \forall\ v_h \in V_h$, and hence from (2.4) we have

$$\|\bar{u} - \bar{u}_h\|_h \le C\Big\{\inf_{v_h \in V_h} \|\bar{u} - v_h\|_h + \sup_{w_h \in V_h} \frac{E_h(\bar{u}, w_h)}{\|w_h\|_h}\Big\}. \tag{3.3}$$

It has been obtained in [3] that there exists $h_0' = h_0'(\frac{\epsilon}{2}, \bar{W})$ such that for $0 < h < h_0'\Big(\frac{\epsilon}{2}, \bar{W}\Big)$,

$$\inf_{v_h \in V_h} \|\bar{u} - v_h\|_h \le \frac{\epsilon}{2}, \tag{3.4}$$

where $\bar{W} = \{\bar{u} : a(\bar{u}, v) = \bar{f}(v), \|\bar{f}\|_0 = 1\}$.

As to estimate the second term on the right hand side of (3.3), from the steps (ii) and (iii) in the proof of Theorem 2.1 and taking account of Lemma 3.3, we can find that there exists $h_0''\Big(\frac{\epsilon}{2}, \bar{W}\Big) > 0$, such that for $0 < h < h_0''\Big(\frac{\epsilon}{2}, \bar{W}\Big)$,

$$|E_h(\bar{u}, w_h)| \le \frac{\epsilon}{2}\|w_h\|_h. \tag{3.5}$$

Thus we have, from (3.3)–(3.5),

$$\|\bar{u} - \bar{u}_h\|_h \le \epsilon \quad \text{as} \quad 0 < h \le h_0 = \min\Big\{h_0'\Big(\frac{\epsilon}{2}, \bar{W}\Big), h_0''\Big(\frac{\epsilon}{2}, \bar{W}\Big)\Big\}, \tag{3.6}$$

or

$$\|u - u_h\|_h \le \epsilon \|f\|_0,$$

which completes the proof of (3.1).

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