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ON THE DOMAIN DECOMPOSITION METHOD FOR MORLEY ELEMENT –FROM WEAK OVERLAP TO NONOVERLAP*

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Abstract

In this paper, following our original ideas[9], we first consider a weakly overlapping additive Schwarz preconditioner according to the framework of [2] for Morley element and show that its condition number is quasi-optimal; we then analyze in detail the structure of this preconditioner, and after proper choices of the inexact solvers, we obtain a quasi-optimal nonoverlapping domain decomposition preconditioner in the last. Compared with [12], [13], it seems that according to this paper's procedure we can make out more thoroughly the relationship between overlapping and nonoverlapping domain decomposition methods for nonconforming plate elements, and certainly, we have also proposed another formal and simple strategy to construct nonoverlapping domain decomposition preconditioners for nonconforming plate elements.

Key words: Morley element, Domain decomposition, Weak overlap.

1. Introduction

We consider, for simplicity, the following clamped plate bending problem:

$$\begin{cases} \Delta^2 u = f, & (\Omega), \\ u = \partial_n u = 0, & (\partial\Omega), \end{cases} \tag{1.1}$$

where Ω is a plane polygonal domain and n denotes the unit outward normal along the boundary $\partial\Omega$. The related variational form is

$$\begin{cases} u \in V \equiv H_0^2(\Omega), \\ a(u,v) = (f,v), & v \in V, \end{cases} \tag{1.2}$$

where $a(u,v) \equiv \int_\Omega [\Delta u \Delta v + (1-\nu)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v)]dx$, $(f,v) \equiv \int_\Omega fvdx$, $\nu \in (0, 0.5)$ is the Poisson ratio. Clearly, the above bilinear form $a(\cdot,\cdot)$ satisfies

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the boundedness and coercivity estimates:

$$\begin{cases} |a(v,w)| \le (1+\nu)|v|_{2,\Omega}|w|_{2,\Omega}, & v,w \in H^2(\Omega), \\ a(v,v) \ge (1-\nu)|v|_{2,\Omega}^2, & v \in H^2(\Omega). \end{cases} \tag{1.3}$$

Throughout this paper we adopt the standard conventions for Sobolev norms and semi-norms of a function v defined on an open set G:

$$\|v\|_{m,G} \equiv (\int_G \sum_{|\alpha|\le m} |\partial^\alpha v|^2 dx)^{1/2},$$
$$|v|_{m,G} \equiv (\int_G \sum_{|\alpha|=m} |\partial^\alpha v|^2 dx)^{1/2},$$
$$|v|_{m,\infty,G} \equiv \max_{|\alpha|=m} \|\partial^\alpha v\|_{L^\infty(G)}.$$

We shall also denote the space of polynomials of degree less than or equal to l on G by $P_l(G)$.

Let $\bar\Omega = \cup_{K\in T_h} \bar K$ be a quasi-uniform and regular triangulation of Ω[4], the diameter size of which is denoted by h, here each $K \in T_h$ is an open triangle. On this triangulation we construct the so-called Morley element[4],[11]:

$V^h \equiv \{v : v|_K \in P_2(K)$, v(respectively, $\partial_n v$) is continuous at each vertex p of K(respectively, each edge midpoint m of K), $\forall\ K \in T_h\}$,

$$V_0^h \equiv \{v \in V^h :\ v(p) = 0,\ p \in \partial\Omega,\ \partial_n v(m) = 0,\ m \in \partial\Omega\}. \tag{1.4}$$

Here and henceforth, p and m (with or without subscript) represent a vertex and an edge midpoint of the elements in T_h respectively. Then, based on (1.4), the discrete problem of (1.2) reads as follows:

$$\begin{cases} a_h(u_h, v_h) = (f, v_h), & v_h \in V_0^h, \\ u_h \in V_0^h, \end{cases} \tag{1.5}$$

where $a_h(v,w) \equiv \sum_{K\in T_h} \int_K [\Delta v \Delta w + (1-\nu)(2\partial_{12}v\partial_{12}w - \partial_{11}v\partial_{22}w - \partial_{22}v\partial_{11}w)]dx$.

It is well-known that the PCG is a proper method to solve (1.5), and the core step is how to design a well-preconditioned and easily invertible in parallel preconditioner, since the condition number of the discrete system (1.5) is $O(h^{-4})$. In [2], S.C.Brenner proposed a two-level additive Schwarz preconditioner for nonconforming plate elements; the main ingredient is the construction of proper intergrid transfer operators which build important bridges among nonconforming elements and their conforming relatives, and thus the difficulty that subspaces are not nested for nonconforming element case was overcome successfully. In [8], J.Gu and X.Hu presented some extension theorems for nonconforming plate elements with applications to two-subregions domain decomposition method, but their results can not show the equivalence between the energy of the discrete biharmonic function and the norm of its related boundary terms, which is crucial to the construction of nonoverlapping domain decomposition method for many subregions case. In [12], [13], using a modified intergrid transfer operator induced by [2] and the continuity of the Morley element, Z.Shi and Z.Xie obtained the extension theorems of that kind, they then followed the ideas of J.H.Bramble et al[1] and achieved a quasi-optimal nonoverlapping domain decomposition preconditioner.

In this paper, following our original ideas[9], we first consider a weakly overlapping additive Schwarz preconditioner according to the framework of [2] for Morley element and show that its condition number is quasi-optimal, equal to $O((1+\log \frac{H}{h})^2)$, where H, h denote the diameters of the coarse and the finite element triangulations respectively; we then analyze in detail the structure of this preconditioner, and after proper choices of the inexact solvers, we also obtain a quasi-optimal nonoverlapping domain decomposition preconditioner in the last. Compared with [12], [13], it seems that according to this paper's procedure we can make out more thoroughly the relationship between overlapping and nonoverlapping domain decomposition methods for nonconforming plate elements, and certainly, we have also given another formal and simple strategy to construct nonoverlapping domain decomposition preconditioners for nonconforming plate elements.

2. Algorithm Descriptions

In order to present the weakly overlapping domain decomposition preconditioner for (1.5), we need, at first, give an arbitrary coarse triangulation of Ω, $T_H = \{\Omega_i\}_{i=1}^M$, which is quasi-uniform and regular with the diameter size H, here each Ω_i is a coarse open triangle. As usual, we assume that $\partial\Omega_i$ is aligned with the finite element triangulation T_h [5],[6]. On this coarse triangulation, $a_H(\cdot,\cdot)$, $V^H(V_0^H)$ are defined as before. For the purpose of global communication among the local subspaces, which is necessary for a good preconditioner, we next introduce some intergrid transfer operators and some interpolation operators. Let AR^h denote the Argyris element associated with the triangulation T_h, which is defined below: For any $v \in AR^h$, $v|_K \in P_5(K)$, and it has the degrees of freedom $\{\ \partial^\alpha v(p_i),\ |\alpha| \le 2;\ \partial_n v(m_i)\ \}$ [4],[11] .

We then construct an intergrid transfer operator E_h [2],[12],[13] as follows, which builds an important bridge between the nonconforming element space V^h and its conforming relative AR^h. For arbitrary vertex p of T_h, we assign to it one of its adjacent edge midpoint e_p; we want that, if $p \in \cup_{i=1}^m \partial\Omega_i \setminus \partial\Omega$ (respectively, $\partial\Omega$), e_p should also belong to $\cup_{i=1}^m \partial\Omega_i \setminus \partial\Omega$ (respectively, $\partial\Omega$). Obviously, there is certain freedom for the choice of e_p. After that, for any $v \in V^h$, $E_h v \in AR^h$ is defined by

$$\begin{aligned}
&E_h v(p) = v(p), \qquad \forall \text{vertex } p, \\
&\partial^\alpha E_h v(p) = 0, \qquad |\alpha| = 2, \\
&\partial_n E_h v(m) = \partial_n v(m), \qquad \forall \text{midpoint } m, \\
&\partial_i E_h v(p) = \partial_i v(e_p), \qquad i = 1, 2.
\end{aligned} \tag{2.1}$$

The main ingredient of this construction is that for $v \in V^h$, the slope of v, i.e. $\partial_i v$, $i = 1, 2$, is continuous at any edge midpoint m. For the coarse spaces V^H, AR^H, the operator $E_H : V^H \to AR^H$ is defined in the same manner. We next denote by I_h the conventional interpolation operator from $C^1(\bar{\Omega})$ onto V^h [4]. Now we can define an intergrid transfer operator $I_H^h : V^H \to V^h$ as

$$I_H^h v = I_h E_H v, \qquad v \in V^H. \tag{2.2}$$

Clearly, for $v \in V_0^H$, $I_H^h v \in V_0^h$. To apply the abstract framework of [2], we proceed to construct some weakly overlapping subspaces. For arbitrary i, $j \in \{1, 2, \cdots, M\}$, $i \neq j$, we denote $(i, j) \in S$ if Ω_i and Ω_j have a common edge. For arbitrary $(i, j) \in S$, we introduce the following subspace:

$$V_{ij}^h = \{v \in V_0^h : \text{ nodal parameters of } v \text{ are zero outside } (\overline{\Omega_i \cup \Omega_j})^0\}$$

and the related orthogonal projection operator:

$$\begin{cases} a_h(P_{ij} v, w) = a_h(v, w), \qquad v \in V_0^h, \ w \in V_{ij}^h, \\ P_{ij} v \in V_{ij}^h, \end{cases}$$

here, as usual for any point set B, $\bar{B}$ and B^0 denote its closure and interior point set respectively.

Then we have a space decomposition of V_0^h:

$$V_0^h = I_H^h V_0^H + \sum_{(i,j) \in S} V_{ij}^h. \tag{2.3}$$

As in [2], let $(.,.)_h$ and $(.,.)_H$ denote the discrete inner products on V_0^h and V_0^H respectively, i.e., $\forall v, \ w \in V_0^h$,

$$(v, w)_h \equiv h^2 \sum_p v(p) w(p) + h^4 \sum_m \partial_n v(m) \partial_n w(m), \tag{2.4}$$

where the summation are taken over all vertices p and midpoints m of the triangulation T_h; the inner product $(.,.)_H$ is defined in the same way. Furthermore, we define A_h :

$V_0^h \to V_0^h$, $A_{ij} : V_{ij}^h \to V_{ij}^h$, $A_H : V_0^H \to V_0^H$, $I_h^H : V_0^h \to V_0^H$ and $Q_{ij} : V_0^h \to V_{ij}^h$ by

$$\begin{cases} (A_h v, w)_h = a_h(v, w), & v, \quad w \in V_0^h, \\ (A_{ij} v, w)_h = a_h(v, w), & v, \quad w \in V_{ij}^h, \\ (A_H v, w)_H = a_H(v, w), & v, \quad w \in V_0^H, \\ (I_h^H v, w)_H = (v, I_H^h w)_h, & v \in V_0^h, \quad w \in V_0^H. \\ (Q_{ij} v, w)_h = (v, w)_h, & v \in V_0^h, \quad w \in V_{ij}^h. \end{cases} \tag{2.5}$$

Then, based on (2.3), we have the following two-level additive Schwarz preconditioned problem:

$$\begin{cases} B \equiv I_H^h A_H^{-1} I_h^H + \sum\limits_{(i,j)\in S} A_{ij}^{-1} Q_{ij}, \\ B A_h u_h = B f_h, \end{cases} \tag{2.6}$$

where $f_h \in V_0^h$ and $(f, v) = (f_h, v)_h, \ \forall v \in V_0^h$.

We call (2.6) the weakly overlapping domain decomposition method for (1.5).

3. Estimates of the Condition Number

To begin with, we give some standard conventions. Let $G \subset \Omega$ be any open subset aligned with the finite element triangulation T_h. Then we define that $V^h(G) \equiv V^h|_G$, $V_0^h(G) \equiv \{v \in V^h(G), \ v(p) = \partial_n v(m) = 0, \ v, \ m \in \partial G\}$, and

$$\begin{cases} \|v\|_{k,h,G} \equiv (\sum\limits_{K \in T_h \cap G} \|v\|_{k,K}^2)^{1/2}, & v \in V^h(G), \\ |v|_{k,h,G} \equiv (\sum\limits_{K \in T_h \cap G} |v|_{k,K}^2)^{1/2}, & v \in V^h(G), \\ a_{h,G}(v, w) \equiv \sum\limits_{K \in T_h \cap G} \int_K [\Delta v \Delta w + (1-\nu)(2\partial_{12} v \partial_{12} w - \partial_{11} v \partial_{22} w - \partial_{22} v \partial_{11} w)] dx. \end{cases} \tag{3.1}$$

Note that in order to make the following interpretation more conveniently we also apply the above definition for $v \in H^2(G)$.

To go on with the analysis of the condition number of (2.6), we next recall some known results.

Lemma 3.1[2],[12],[13]. *For the intergrid transfer operator $E_h(E_H)$ defined in §2, we have*

$$\begin{cases} \sum\limits_{i=0}^{2} h^i |v - E_h v|_{i,K} \le C h^2 |v|_{2,\tilde{K}}, & v \in V^h, \ K \in T_h, \\ \sum\limits_{i=0}^{2} H^i |v - E_H v|_{i,e} \le C H^2 |v|_{2,\tilde{e}}, & e \in T_H, \ v \in V^H. \end{cases} \tag{3.2}$$

Here and in the following $\tilde{K}$ is the union of all elements in T_h, each of which has a common edge (or vertex) with K, $\tilde{e}$ is defined in the same way, and C(with or without *subscript) denotes a generic constant independent of H and h, it may be different in different places.*

Lemma 3.2[2],[4]. *For the operator I_h, the following estimate holds:*

$$\sum_{i=0}^{3} h^i |v - I_h v|_{i,K} \le Ch^3 |v|_{3,K}, \qquad v \in H^3(K).$$

Lemma 3.3. *Let E be a coarse triangle in $\{\Omega_i\}_{i=1}^m$, $AR^h(E) \equiv AR^h|_E$. Then for any $v \in AR^h(E)$,*

$$|v|_{0,\infty,E}^2 \le C[(1 + \log \frac{H}{h})|v|_{1,E}^2 + H^{-2}|v|_{0,E}^2]. \tag{3.3}$$

If there exists some point $q \in E$ such that $v(q) = 0$, then the lower term in the above estimate can be cancelled. Result (3.3) also holds for v replaced by $\partial_i v$.
The proof can be given by the arguments similar to those employed in the derivation of the Lemma 3.5 in [1].

Lemma 3.4. *Let E be any coarse triangle in $\{\Omega_i\}_{i=1}^m$, with E_k, E_{ij}, $i, j, k \in \{1, 2, 3\}$ as its vertices and edges respectively. Assume $v \in V^h(E)$ to be discrete $a_{h,E}$-biharmonic, i.e.*

$$a_{h,E}(v, w) = 0, \qquad w \in V_0^h(E).$$

Furthermore, the nodal parameters of v are zero on E_{23} and E_{13}(including endpoints of the two edges). As in §2, we introduce a special intergrid transfer operator $\bar{E}_h$: $V^h(E) \to AR^h(E)$ such that for $p \in E_{12}$(which is looked upon as an open set), $e_p \in E_{12}$; for $p \in \partial E \setminus E_{12}$, $e_p \in \partial E \setminus E_{12}$. Then we have the following estimates:

$$a_{h,E}(v, v) \le C_2[\|\partial_\tau \bar{E}_h v\|_{H_{00}^{1/2}(E_{12})}^2 + \|\partial_n \bar{E}_h v\|_{H_{00}^{1/2}(E_{12})}^2],$$

$$C_1[\|\partial_\tau \bar{E}_h v\|_{H_{00}^{1/2}(E_{12})}^2 + \|\partial_n \bar{E}_h v\|_{H_{00}^{1/2}(E_{12})}^2] \le a_{h,E}(v, v),$$

where τ is the unit tangent direction along E_{12}, and $\|.\|_{H_{00}^{1/2}(E_{12})}$ is the half norm[1],[10] *defined by*

$$\|w\|_{H_{00}^{1/2}(\gamma)}^2 \equiv \int_\gamma \int_\gamma \frac{|(w(x) - w(y)|^2}{|x - y|^2} ds(x)ds(y) + \int_\gamma (w(x))^2 (\frac{1}{|x - b_1|} + \frac{1}{|x - b_2|}) ds(x),$$

where b_i, $i = 1, 2$, are the two endpoints of the line segment γ respectively, $s(x)$ denotes the disc length parameter along γ. In other words, the above estimates state that the energy of the discrete $a_{h,E}$-biharmonic function is equivalent to the related norms for the boundary terms.

Proof. After the proper choice of the intergrid transfer operator $\bar{E}_h$, the proof of the above inequalities are formal in some sense[1],[9], here, for completeness of this paper, we will present a somewhat detailed deduction, please see also similar results given in [8],[12] and [13]. Without loss of generality, assume that $\mathrm{diam}(E) \equiv \min_{x,y\in E}|x-y| = 1$; results for general case can be obtained by the scale transformation $x = \mathrm{diam}(E)\hat{x}$ and standard scaling argument. Let us first consider the proof of the first inequality in the Lemma 3.4. It is clear from the construction of the interpolation operator $\bar{E}_h$ that $\tilde{v}|_{E_{12}} \in H_{00}^{5/2}(E_{12})$, $\tilde{v}|_{\partial E\backslash E_{12}} = 0$, $(\partial_n\tilde{v})|_{E_{12}} \in H_{00}^{3/2}(E_{12})$ and $(\partial_n\tilde{v})|_{\partial E\backslash E_{12}} = 0$, here $\tilde{v} \equiv \bar{E}_h v$, and as usual for any function w and any point set B, $w|_B$ means the restriction of the function w on B, and meanwhile, $H_{00}^{5/2}(E_{12})$ and $H_{00}^{3/2}(E_{12})$ denote the conventional fractional order Sobolev spaces, please see e.g. [7], [10] for details. Thus, from the trace theory on polygonal domain[7], we can find some function $\tilde{\tilde{v}} \in H^3(E)$ such that $\tilde{\tilde{v}}|_{\partial E} = \tilde{v}|_{\partial E}$, $(\partial_n\tilde{\tilde{v}})|_{\partial E} = (\partial_n\tilde{v})|_{\partial E}$ and

$$\|\tilde{\tilde{v}}\|_{3,E} \le C[\|\tilde{v}\|_{H_{00}^{5/2}(E_{12})} + \|\partial_n\tilde{v}\|_{H_{00}^{3/2}(E_{12})}]. \tag{3.4}$$

We next introduce the following auxiliary problem:

$$\begin{cases} \Delta^2 w = 0, \quad (E), \\ w|_{\partial E} = \tilde{v}|_{\partial E}, \\ (\partial_n w)|_{\partial E} = (\partial_n\tilde{v})|_{\partial E}. \end{cases} \tag{3.5}$$

Thus, thanks to (3.4) and the fundamental result that Δ^2 is an isomorphism from $H^3(E)\cap H_0^2(E)$ onto $H^{-1}(E)$ when E is a convex polygonal domain, we easily have that problem (3.5) has a unique weak solution $w \in H^3(E)$ and that the following estimate holds:

$$\|w\|_{3,E} \le C[\|\tilde{v}\|_{H_{00}^{5/2}(E_{12})} + \|\partial_n\tilde{v}\|_{H_{00}^{3/2}(E_{12})}]. \tag{3.6}$$

Pay attention to the fact that v is just the Morley element approximate solution of (3.5), then by the well-known error estimates for Morley element[8],[11] we obtain that

$$|w-v|_{2,h,E} \le Ch|w|_{3,E}. \tag{3.7}$$

Therefore, from (3.6), (3.7) and the inverse inequalities we see that

$$\begin{aligned} a_{h,E}(v,v) \le& C[|v-w|_{2,h,E}^2 + |w|_{2,E}^2] \\ \le& C[h^2(\|\tilde{v}\|_{H_{00}^{5/2}(E_{12})}^2 + \|\partial_n\tilde{v}\|_{H_{00}^{3/2}(E_{12})}^2) + |w|_{2,E}^2] \\ \le& C[\|\tilde{v}\|_{H_{00}^{3/2}(E_{12})}^2 + \|\partial_n\tilde{v}\|_{H_{00}^{1/2}(E_{12})}^2 + |w|_{2,E}^2]. \end{aligned} \tag{3.8}$$

Note also that from the minimal potential principle, w(solution of (3.5)) is the unique solution of the following variational problem:

$$J(w) = \min_{\bar{w}\in W} J(\bar{w}), \tag{3.9}$$

where $W \equiv \{\bar{w} \in H^2(E) : \ \bar{w}|_{\partial E} = \tilde{v}|_{\partial E}, \ (\partial_n \bar{w})|_{\partial E} = (\partial_n \tilde{v})|_{\partial E}\}$ and $J(\bar{w}) \equiv \frac{1}{2} a_{h,E}(\bar{w}, \bar{w})$. On the other hand, from the trace theory on polygonal domain[7] there exists some $\tilde{w} \in H^2(E)$ such that $\tilde{w}|_{\partial E} = \tilde{v}|_{\partial E}$, $(\partial_n \tilde{w})|_{\partial E} = (\partial_n \tilde{v})|_{\partial E}$ and

$$\|\tilde{w}\|_{2,E} \leq C[\|\tilde{v}\|_{H^{3/2}_{00}(E_{12})} + \|\partial_n \tilde{v}\|_{H^{1/2}_{00}(E_{12})}]. \tag{3.10}$$

Consequently, from (1.3), (3.9) and (3.10) we know

$$\begin{aligned} |w|^2_{2,E} \leq & C a_{h,E}(w,w) \leq C a_{h,E}(\tilde{w},\tilde{w}) \leq C \|\tilde{w}\|^2_{2,E} \\ \leq & C[\|\tilde{v}\|^2_{H^{3/2}_{00}(E_{12})} + \|\partial_n \tilde{v}\|^2_{H^{1/2}_{00}(E_{12})}]. \end{aligned} \tag{3.11}$$

Therefore, combining (3.8) and (3.11) we achieve the first inequality in the Lemma 3.4.

Now we proceed with the demonstration of the second inequality in the Lemma 3.4. It follows from the trace theory on polygonal domain [7], generalized Poincare inequality [10],[14] and the Lemma 3.1 that

$$\begin{aligned} [\|\partial_\tau \tilde{v}\|^2_{H^{1/2}_{00}(E_{12})} + \|\partial_n \tilde{v}\|^2_{H^{1/2}_{00}(E_{12})}] \leq & C\|\tilde{v}\|^2_{2,E} \leq C|\tilde{v}|^2_{2,E} \\ \leq & C|v|^2_{2,E} \leq C a_{h,E}(v,v). \end{aligned}$$

The desired result then follows in the last.

We also want to borrow the following important result:

Lemma 3.5[2]. *For the two-level additive Schwarz preconditioned problem(2.6) the following estimate holds:*

$$Cond_2(BA_h) \equiv \lambda_{max}(BA_h)/\lambda_{min}(BA_h) \leq C\beta,$$

where β is a generic constant such that, for any $v \in V_0^h$, there exist $v_0 \in V_0^H$, $v_{ij} \in V_{ij}^h$, such that, $v = I_H^h v_0 + \sum_{(i,j)\in S} v_{ij}$, and

$$a_H(v_0, v_0) + \sum_{(i,j)\in S} a_h(v_{ij}, v_{ij}) \leq C\beta a_h(v,v).$$

After the above preparations, now we can verify the following main theorem:

Theorem 3.6. *For the preconditioned problem (2.6), we have*

$$Cond_2(BA_h) \leq C(1 + \log \frac{H}{h})^2.$$

That is to say the condition number of the problem (2.6) is quasi-optimal.

Proof. According to the Lemma3.5 it suffices to bound the corresponding constant β in more details. For any $v \in V_0^h$, we choose $v_0 = I_H E_h v$; the construction of v_{ij} is complicated comparatively. Without loss of generality, we consider a coarse triangle e in $\{\Omega_i\}_{i=1}^m$, which has three adjacent coarse triangles $\{e_i\}_{i=1}^3$.

Obviously, the decomposition functions on the subdomains $(\overline{e \cup e_i})^0$, when restricted on e, should satisfy that 1. the sum is $\bar{v} \equiv v - I_H^h v_0$ (Note that $\bar{v}$ is zero at each vertex of e); 2. they satisfy the support restriction conditions. These are also the sufficient conditions. Thus we first decompose the function $\bar{v}|_e$ into the following three functions on e, which are associated with the three edges of e respectively:

$$\begin{cases} a_{h,e}(v_{e\cup e_i}, w) = 0, \quad w \in V_0^h(e), \\ v_{e\cup e_i}(p) = \bar{v}(p), \quad p \in \gamma_i\gamma_{i+1};\ \text{or } 0, \quad \text{for other cases}, \\ \partial_n v_{e\cup e_i}(m) = \partial_n \bar{v}(m), \quad m \in \gamma_i\gamma_{i+1};\ \text{or } 0, \quad \text{for other cases}, \end{cases} \tag{3.12a}$$

$$v_{e\cup e_3} = \bar{v} - v_{e\cup e_1} - v_{e\cup e_2}. \tag{3.12b}$$

Here, γ_i, $i = 1,2,3$, denote the three vertices of e respectively, and $\gamma_i\gamma_{i+1}$ represents the edge of e connecting γ_i and γ_{i+1}, and for simplicity of exposition we have also used the recursive index convention, e.g., $4 = 1$. Obviously, there is certain freedom of construction, e.g. we can also choose $v_{e\cup e_2}$, $v_{e\cup e_3}$ to be discrete $a_{h,e}$-biharmonic and $v_{e\cup e_1}$ is then constructed as (3.12b) similarly. It should be pointed out that if e is a boundary element of Ω, then the decomposition function associated with the edge belonging to the boundary $\partial\Omega$ should be discrete $a_{h,e}$-biharmonic function, at this time, this function is just the zero function. Notice also that $v_{e\cup e_i}$ denotes the related decomposition function on e, which is associated with the common edge of e and e_i. Furthermore, we can get the global decomposition functions by patching together those related local functions in a proper way to ensure the continuity restrictions. As a matter of fact, if $(i,j) \in S$, then we can define $v_{ij} \equiv \{ \begin{matrix} v_{\Omega_i\cup\Omega_j}, & \text{on } \bar{\Omega}_i, \\ v_{\Omega_j\cup\Omega_i}, & \text{on } \bar{\Omega}_j. \end{matrix}$ It is clear from the above construction that these functions satisfy the above restriction conditions.

Thus, in order to bound β, it suffices to give the estimates of $a_H(v_0, v_0)$, $a_{h,e}(v_{e\cup e_1}, v_{e\cup e_1})$, $a_{h,e}(v_{e\cup e_2}, v_{e\cup e_2})$. We first consider the term $a_H(v_0, v_0)$. In fact, from (1.3) and scaling argument we have that

$$\begin{cases} a_H(v_0, v_0) \le C|v_0|_{2,H,\Omega}^2 = C(\sum\limits_{e\in T_H} |v_0|_{2,e}^2)^{1/2}, \\ |v_0|_{2,e}^2 \le C \sum\limits_{i=1}^{3} |\partial_n(v_0 - I_1 v_0)(m_i)|^2, \end{cases} \tag{3.13}$$

where I_1 is the conventional piecewise linear conforming interpolation operator on the coarse triangulation T_H, and m_i, $i = 1,2,3$, denote the edge midpoints of e. Then, by the maximum norm estimates of the Lemma 3.3, and note that $\partial_n(v_0 - I_1 v_0)(m_i) =$

$\partial_n(E_h v - I_1 E_h v)(m_i)$, we know

$$\sum_{i=1}^{3} |\partial_n(v_0 - I_1 v_0)(m_i)|^2 \leq C[(1+\log\frac{H}{h})|w - I_1 w|_{2,e}^2 + H^{-2}|w - I_1 w|_{1,e}^2] \leq C(1+\log\frac{H}{h})|w|_{2,e}^2,$$

here and henceforth, $w \equiv E_h v$.

Therefore, it follows from the Lemma3.1 that

$$a_H(v_0, v_0) \leq C(1+\log\frac{H}{h})|E_h v|_{2,\Omega}^2 \leq C(1+\log\frac{H}{h})|v|_{2,h,\Omega}^2 \tag{3.14}$$

and

$$a_{h,e}(\bar{v}, \bar{v}) \leq C[|v|_{2,h,e}^2 + |I_H^h v_0|_{2,h,e}^2] \leq C[|v|_{2,h,e}^2 + |v_0|_{2,\tilde{e}}^2]. \tag{3.15}$$

Secondly, for the term $a_{h,e}(v_{e\cup e_1}, v_{e\cup e_1})$, it follows from the Lemma3.4 that

$$a_{h,e}(v_{e\cup e_1}, v_{e\cup e_1}) \leq C[\|\partial_\tau \bar{E}_h \bar{v}\|_{H_{00}^{1/2}(\gamma_1\gamma_2)}^2 + \|\partial_n \bar{E}_h \bar{v}\|_{H_{00}^{1/2}(\gamma_1\gamma_2)}^2]. \tag{3.16}$$

For the sake of brevity, we might as well put $\gamma_1\gamma_2 = [0, H] \times \{0\}$, $Z \equiv \bar{E}_h \bar{v}$, and $Z(x_1,0)$ is denoted by $Z(x_1)$. Then

$$\|\partial_\tau \bar{E}_h \bar{v}\|_{H_{00}^{1/2}(\gamma_1\gamma_2)}^2 = \int_0^H [\frac{1}{x_1} + \frac{1}{H - x_1}]|\partial_\tau Z|^2 dx_1 + \int_0^H \int_0^H \frac{|\partial_\tau Z(x_1) - \partial_\tau Z(y_1)|^2}{|x_1 - y_1|^2} dx_1 dy_1.$$

Thinking of the symmetry, it suffices to bound $\int_0^H \frac{|\partial_\tau Z|^2}{x_1} dx_1$ and

$$\int_0^H \int_0^H \frac{|\partial_\tau Z(x_1) - \partial_\tau Z(y_1)|^2}{|x_1 - y_1|^2} dx_1 dy_1.$$

From the trace theorem, we easily have

$$\int_0^H \int_0^H \frac{|\partial_\tau Z(x_1) - \partial_\tau Z(y_1)|^2}{|x_1 - y_1|^2} dx_1 dy_1 \leq C|\bar{E}_h \bar{v}|_{2,e}^2 \leq C|\bar{v}|_{2,h,e}^2. \tag{3.17}$$

On the other hand, $Z \in H_0^2(\gamma_1\gamma_2)$, applying the standard technique as in [1], [9], we see

$$\int_0^H \frac{|\partial_\tau Z|^2}{x_1} dx_1 \leq C(1+\log\frac{H}{h})|\partial_\tau Z|_{0,\infty,\gamma_1\gamma_2}^2 \leq C(1+\log\frac{H}{h})^2|\bar{v}|_{2,h,e}^2. \tag{3.18}$$

Now we proceed to attack the estimate of the second term in (3.16). Note that $\partial_n Z \in H_0^1(\gamma_1\gamma_2)$, utilizing the similar deduction as above, we know

$$\int_0^H \frac{|\partial_n Z|^2}{x_1} dx_1 \leq C(1+\log\frac{H}{h})|\partial_n Z|_{0,\infty,\gamma_1\gamma_2}^2 \leq C(1+\log\frac{H}{h})^2|\bar{v}|_{2,h,e}^2, \tag{3.19}$$

and

$$\int_0^H \int_0^H \frac{|\partial_n Z(x_1) - \partial_n Z(y_1)|^2}{|x_1 - y_1|^2} dx_1 dy_1 \le C|\bar{v}|^2_{2,h,e}. \tag{3.20}$$

The estimate of $a_{h,e}(v_{e\cup e_2}, v_{e\cup e_2})$ can be achieved in the same manners. Consequently, from (3.12)-(3.20) and the quasi-uniformity and the regularity of the triangulations T_h and T_H, we have

$$a_H(v_0, v_0) + \sum_{(i,j)\in S} a_h(v_{ij}, v_{ij}) \le C(1 + \log\frac{H}{h})^2 |v|^2_{2,h,\Omega} \le C(1 + \log\frac{H}{h})^2 a_h(v, v).$$

Theorem 3.6 then follows in the last.

4. From Weak Overlap to Nonoverlap

We shall first give the matrix description of the preconditioned problem (2.6) in order to understand it more intuitively and make out the concrete structure of the induced preconditioner. Let $\hat{\Omega}$, with element number $|\hat{\Omega}|$, be the interior nodal parameter set of the Morley element, on which we define a proper order; the point set $\hat{\Omega}_{ij} \subset \Omega_{ij}$, with element number $|\hat{\Omega}_{ij}|$, is defined as above similarly, on which there exists the same order as that on $\hat{\Omega}$. Here, for $(i,j) \in S$, Ω_{ij} is the quadrilateral formed by Ω_i and Ω_j. We denote by $\{\phi_i\}$ the shape basis functions of V_0^h according to the above order, and $\bar{A}_h$ and $\bar{A}_{ij}$ are the related stiffness matrices for the original and the induced local problems, i.e., $\bar{A}_h \equiv (a_h(\phi_i, \phi_j))_{i,j\in\hat{\Omega}}$, $\bar{A}_{ij} \equiv (a_h(\phi_k, \phi_l))_{k,l\in\hat{\Omega}_{ij}}$. We next introduce the following extension operator E_{ij} for $(i,j) \in S$:

$$\begin{cases} E_{ij} : R^{|\hat{\Omega}_{ij}|} \to R^{|\hat{\Omega}|}, & \text{for arbitrary } v \in R^{|\hat{\Omega}_{ij}|}, \\ E_{ij}v(k) = v(k), \quad k \in \hat{\Omega}_{ij};\ 0,\ \text{otherwise}. \end{cases} \tag{4.1}$$

That is to say E_{ij} is an extension operator which, keeping its original order, transfers a local expression to the related global one. For the coarse triangulation T_H, $\hat{\Omega}_H$ and etc. are defined in the same way. Here and henceforth, all definitions for T_h can also be converted for T_H. Let D_h denote the matrix description of $(\cdot,\cdot)_h$, i.e., for v, $w \in V_0^h$, which have the nodal parameters x, $y \in R^{|\hat{\Omega}|}$ respectively, the following equality holds:

$$(v, w)_h = [D_h x, y],$$

where $[\cdot,\cdot]$ is the euclidean natural inner product. Obviously, D_h is a diagonal matrix. We also denote by $\bar{I}_H^h$ the matrix description of I_H^h , i.e., for $v \in V_0^H$ with the nodal parameters $x \in R^{|\hat{\Omega}_H|}$, the nodal parameters of $I_H^h v$ is just $\bar{I}_H^h\ x \in R^{|\hat{\Omega}|}$; $\bar{I}_h^H$ is defined similarly. Then, from their definitions in (2.5), we get $\bar{I}\ {}_h^H = D_H^{-1}(\bar{I}_H^h\ )^t\ D_h$, here t means the transpose operation. In the same way, we can also see that the matrix description of A_h is $D_h^{-1}\bar{A}_h$ and the related description of $\bar{A}_{ij}^{-1}Q_{ij}A_h = P_{ij}$ is $E_{ij}\bar{A}_{ij}^{-1}$ E_{ij}^t . Hence, we know the matrix description of (2.6) is

$$\begin{cases} \bar{B}_1 \equiv \bar{I}_H^h \bar{A}_H^{-1}(\bar{I}_H^h)^t + \sum\limits_{(i,j)\in S} E_{ij}\bar{A}_{ij}^{-1}E_{ij}^t, \\ \bar{B}_1\bar{A}_h\bar{u} = \bar{B}_1\bar{f}, \end{cases} \tag{4.2}$$

here $\bar{u}$ is the nodal parameters of u_h, and $\bar{f}$ denotes the vector with the components (f, ϕ_i), $i \in \hat{\Omega}$. Thus, the preconditioner is $\bar{B}_1$. After that, we proceed to simplify this preconditioner by substituting some spectrally equivalent matrices for $\bar{A}_H^{-1}$ and $\bar{A}_{ij}^{-1}$. Suppose $v \in V_0^H$, with nodal parameters $y \in R^{|\hat{\Omega}_H|}$. Then from the proof of the Theorem3.6, we see

$$[\bar{A}_H y, y] = a_H(v, v) \approx \sum_{e\in T_H}\sum_{i=1}^{3} |\partial_n(v - I_1 v)(m_i)|^2 \equiv [\bar{\tilde{A}}_H y, y].$$

Hence, $\bar{A}_H$ is spectrally equivalent to $\bar{\tilde{A}}_H$. Here and from now on, $f \approx g$ means that there exist two constants C_1 and C_2 independent of H and h such that $C_1 g \le f \le C_2 g$. In practice, we do not change $\bar{A}_H$ for $\bar{\tilde{A}}_H$ so as to keep the convenience of program compiling. In order to discuss the other terms, we first give several definitions. Let $\Gamma_{ij} = (\partial\Omega_i \cap \partial\Omega_j)^0$(open set) denote the common edge of Ω_i and Ω_j. Then $\hat{\Omega}_i$, $\hat{\Gamma}_{ij}$ are defined as before, E_i is the extension operator from $\hat{\Omega}_i$ to $\hat{\Omega}$, and $E_{\Gamma_{ij}}$ is the extension operator from $\hat{\Gamma}_{ij}$ to $\hat{\Omega}$, they both are defined by (4.1) similarly. Moreover, $\bar{A}_i$ denotes the stiffness matrix on Ω_i, i.e., $\bar{A}_i \equiv (a_{h,\Omega_i}(\phi_k, \phi_l))_{k,l\in\hat{\Omega}_i}$, $\bar{A}_{i,ij} \equiv (a_{h,\Omega_i}(\phi_k, \phi_l))_{k\in\hat{\Omega}_i, l\in\hat{\Gamma}_{ij}}$. Then, noting that $\bar{A}_{ij}$ is the local stiffness matrix in Ω_{ij}, by the Lemma3.4 and the standard deduction for two-subregions domain decomposition method, we get to know $\bar{A}_{ij}$ is spectrally equivalent to $\bar{\tilde{A}}_{ij} = \begin{bmatrix} \bar{A}_i & 0 & \bar{A}_{i,ij} \\ 0 & \bar{A}_j & \bar{A}_{j,ij} \\ \bar{A}_{i,ij}^t & \bar{A}_{j,ij}^t & W_{ij} \end{bmatrix}$, where

$$W_{ij} \equiv \bar{B}_{ij} + \bar{A}_{i,ij}^t \bar{A}_i^{-1}\bar{A}_{i,ij} + \bar{A}_{j,ij}^t \bar{A}_j^{-1}\bar{A}_{j,ij},$$

and $\bar{B}_{ij}$ is defined by

$$[\bar{B}_{ij}y, y] \approx \|\partial_\tau(\bar{E}_h v)\|^2_{H_{00}^{1/2}(\Gamma_{ij})} + \|\partial_n(\bar{E}_h v)\|^2_{H_{00}^{1/2}(\Gamma_{ij})},$$

for any $v \in V_{ij}^h$, which is discrete biharmonic in Ω_i and Ω_j, and has nodal parameters y in Γ_{ij}. The construction of $\bar{B}_{ij}$ is similar to that in [1].

Therefore, we achieve the following spectrally equivalent preconditioner for $\bar{B}_1$:

$$\bar{B}_2 \equiv \bar{I}_H^h \bar{A}_H^{-1}\bar{I}_h^H + \sum_{(i,j)\in S} E_{ij}\bar{\tilde{A}}_{ij}^{-1}E_{ij}^t.$$

By a direct computation, we easily have, for $x \in R^{|\hat{\Omega}|}$, $[\bar{B}_2 x$,x] is equivalent to

$$[\bar{I}_H^h \bar{A}_H^{-1} \bar{I}_h^H x, x] + \sum_{i=1}^{M} [E_i \bar{A}_i^{-1} E_i^t x, x] + \sum_{(i,j)\in S} [\bar{B}_{ij}^{-1} y_{ij}, y_{ij}],$$

where $y_{ij} \equiv (E_{\Gamma_{ij}}^t - \bar{A}_{i,ij}^t \bar{A}_i^{-1} E_i^t - \bar{A}_{j,ij}^t \bar{A}_j^{-1} E_j^t)x \equiv Z_{ij} x$. Thus, in other words, $\bar{B}_1$ is spectrally equivalent to

$$\bar{B} \equiv \bar{I}_H^h \bar{A}_H^{-1} \bar{I}_h^H + \sum_{i=1}^{M} E_i \bar{A}_i^{-1} E_i^t + \sum_{(i,j)\in S} Z_{ij}^t B_{ij}^{-1} Z_{ij}. \tag{4.3}$$

Consequently, the following theorem holds clearly:

Theorem 4.1. *For the nonoverlapping preconditioner $\bar{B}$ defined by (4.3), the following estimate holds:*

$$Cond_2(\bar{B} A_h) \le C(1 + \log \frac{H}{h})^2.$$

Finally, we give a remark on the computation of $\bar{B}x$, which is the main work for each iteration of the PCG algorithm, i.e., we can first compute $\bar{A}_i^{-1} E_i^t x$, $i = 1, \cdots, M$, and then use these information for the computation of $Z_{ij} x$.

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