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GENERALIZED DIFFERENCE METHODS ON ARBITRARY QUADRILATERAL NETWORKS*

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Abstract

This paper considers the generalized difference methods on arbitrary networks for Poisson equations. Convergence order estimates are proved based on some a priori estimates. A supporting numerical example is provided.

Key words: Quadrilateral elements, Dual grids, Bilinear functions, Generalized difference methods, Priori estimates, Error estimates.

1. Introduction

Consider the boundary value problem of the Poisson equation

$$\begin{cases} -\Delta u = f(x,y), & (x,y) \in \Omega & (1.1) \\ u = 0, & (x,y) \in \Gamma = \partial\Omega & (1.2) \end{cases}$$

where Ω is a convex polygon regon; $\Gamma = \partial\Omega$ the boundary of Ω and $f(x,y)$ a known function on Ω.

The generalized difference methods on quadrilateral networks for elliptic equations are proposed in [11], where the convergence order estimates are given for rectangular networks. Quadrilateral networks are structured networks, the so called "finite volume method on structured networks" (cf. [7] - [9]), a popular method in computational fluid, is identical to the generalized difference method in [3](cf.[4] and [11]). The generalized difference methods have the same convergence orders as the corresponding finite element methods, but they require less computational expenses, and keep the mass conservation (cf. [5]). The aim of this paper is to provide a theory for the generalized difference method on arbitrary quadrilateral networks, and to obtain the optimal convergence order estimates. A generalized difference method with bilinear element is constructed in §2. Some a priori estimates are deduced in §3. §4 is devoted to the error order estimates. Finally, a numerical example is given in §5 to show the effectiveness of the method.

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2. Generalized Difference Methods

Let Ω be a convex polygonal region. Decompose Ω into the union of finite number of strictly convex and nonoverlapping quadrilateral elements.Two nodes are called adjacent if they are the endpoints of the same side of an element. The set of all the quadrilateral elements is denoted by T_h, where h is the maximum length of all the sides.

Connect the midpoints of the opposite side of a quadrilateral element, and call the joint of the two connecting lines the averaging center. Now we construct the dual subdivision of T_h. Let P be an inner node as in Fig.1; $\square PP_1P_2P_3$, $\square PP_3P_4P_5$, $\square PP_5P_6P_7$, $\square PP_7P_8P_1$ are the quadrilaterals with a common node P; and Q_1, Q_2, Q_3, Q_4 respectively are their averaging center. Let M_1, M_2, M_3, M_4 be the midpoints of $\overline{PP_1}, \overline{PP_3}, \overline{PP_5}, \overline{PP_7}$. Connect M_1, Q_1, M_2, Q_2, M_3, Q_3, M_4, Q_4, M_1, successively to obtain a polygonal region K_P^* surrounding P, called a dual element. The set of all the dual elements is denoted by T_h^*, and called the dual subdivision (cf. [11] or [5]).

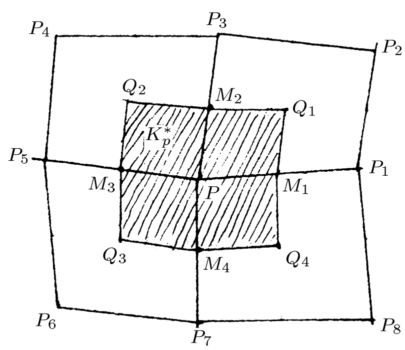

Fig. 1

Let $\bar{\Omega}_h$ be the set of nodes of T_h; $\mathring{\Omega}_h = \bar{\Omega}_h - \partial\Omega$ the set of the inner nodes; and Ω_h^* the set of nodes of the dual grid. Denote by K_Q the quadrilateral element with averaging center $Q \in \Omega_h^*$, and by S_Q, S_P^* the areas of the element K_Q and the dual element K_P^* respectively.

Suppose T_h and T_h^* are quasi-uniformly, that is, there exist constants $C_1, C_2 > 0$ independent of h, such that

$$C_1 h^2 \le S_Q \le h^2, \quad Q \in \Omega_h^* \tag{2.1$_1$}$$

$$C_1 h^2 \le S_P^* \le C_2 h^2, \quad P \in \bar{\Omega}_h \tag{2.1$_2$}$$

Remark 1. $(2.1)_2$ can be deduced from $(2.1)_1$ under the above assumptions on the dual grid.

In order to define the trial function space U_h, we take a unite square $\hat{K} = \hat{E}=$ [0,1]×[0,1] on (ξ, η) plane as the reference element. For any convex quadrilateral

$K_Q(Q \in \Omega_h^*)$, there is a unique invertible bilinear transformation (cf. [6])

$$F_{K_Q}: \begin{cases} x = x_1 + a_1\xi + a_2\eta + a_3\xi\eta \\ y = y_1 + b_1\xi + b_2\eta + b_3\xi\eta \end{cases} \tag{2.2$_1$}$$

$$\begin{array}{lll} a_1 = x_2 - x_1, & a_2 = x_3 - x_1, & a_3 = x_4 - x_3 - x_2 + x_1 \\ b_1 = y_2 - y_1, & b_2 = y_3 - y_1, & b_3 = y_4 - y_3 - y_2 + y_1 \end{array} \tag{2.2$_2$}$$

which maps $\hat{E}$ onto K_Q, where (x_i, y_i) is the coordinate of the node P_i of the element K_Q, see Fig. 2.

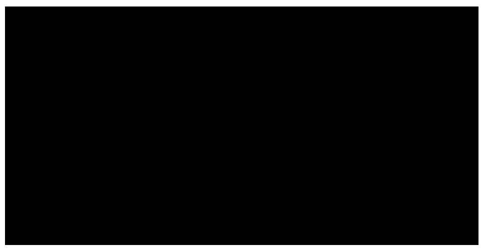

Fig. 2

Remark 2. When K_Q is a parallelogram (including rectangular) we have $a_3 = b_3 = 0$, so the transformation $(2.2)_1$ becomes linear (affine).

Denote by $P_{1,1}$ the space of all the bilinear function $P_{\hat{E}}(\xi,\eta) = c_0 + c_1\xi + c_2\eta + c_3\xi\eta$ defined on $\hat{E}$, Define the trial function space as:

$$U_h = \{u_h \in C^0(\bar{\Omega}), u_h|_K = P_{\hat{E}} \circ F_K^{-1}, u_h|_\Gamma = 0, P_{\hat{E}} \in P_{1,1}\} \tag{2.3}$$

where $K = K_Q$ is any quadrilateral element and $P_{\hat{E}} \circ F_K^{-1}$ denotes the compound function of $P_{\hat{E}}(\xi,\eta)$ and the inverse function F_K^{-1}. For $u_h \in U_h$, set $u_P = u_h(P)$, then the restriction of u_h on K_Q is

$$u_h|_{K_Q} = P_{\hat{E}} \circ F_{K_Q}^{-1} \tag{2.4}$$

where F_{K_Q} is defined in (2.2), and

$$P_{\hat{E}}(\xi,\eta) = c_0 + c_1\xi + c_2\eta + c_3\xi\eta$$

$$c_0 = u_{P_1}, c_1 = u_{P_2} - u_{P_1}, c_2 = u_{P_3} - u_{P_1}, c_3 = u_{P_4} - u_{P_3} - u_{P_2} + u_{P_1} \tag{2.5}$$

The test function space is

$$V_h = \{v_h \in L^2(\Omega), v_h|_{K_P^*} = constant, P \in \overset{\circ}{\Omega}_h\} \tag{2.6}$$

For any $P \in \overset{\circ}{\Omega}_h$, denote by ψ_P the characteristic function of K_P^*, then any $v_h \in V_h$ can be written:

$$v_h = \sum_{P \in \overset{\circ}{\Omega}_h} v_h(P)\psi_P \tag{2.7}$$

Let $\Pi_h u$ be the interpolation projection of $u \in U = H_0^1(\Omega)\cap H_2(\Omega)$ onto the trial function space U_h, then we have (see [1])

$$|u-\Pi_h u|_m \leq Ch^{2-m}|u|_2, m=0,1 \tag{2.8}$$

Let $\Pi_h^* u_h$ be the interpolation projection of $u_h \in U_h$ onto the test function space V_h.

As in [11](cf.[5]), the generalized difference method for (1.1),(1.2) is: Find $u_h \in U_h$, such that

$$a(u_h, \psi_{P_0}) = (f, \psi_{P_0}), \forall P_0 \in \overset{\circ}{\Omega}_h \tag{2.9}$$

where

$$a(u_h, \psi_{P_0}) = \int_{\partial K_{P_0}^*} \left(-\frac{\partial u_h}{\partial x}dy + \frac{\partial u_h}{\partial y}dx\right) \tag{2.10}$$

$$(f, \psi_{P_0}) = \int_{K_{P_0}^*} f dx dy \tag{2.11}$$

(2.10) is obtained by multiplying (1.1) by ψ_{P_0}, integrating it on $K_{P_0}^*$ and applying the Green formula. Let $u_{P_i} = u_h(P_i)$,and let φ_{P_i} be the basis function of the node P_i, namely,$\varphi_{P_i}(P_j) = \delta_{ij}, \varphi_{P_i} \in U_h$. So $u_h = \sum_{P_i \in \overset{\circ}{\Omega}_h} u_{P_i}\varphi_{P_i}$, and (2.9) can be rewriten as:

$$\sum_{P_i \in \overset{\circ}{\Omega}_h} u_{P_i} a(\varphi_{P_i}, \psi_{P_0}) = (f, \psi_{P_0}) \tag{2.12}$$

where

$$a(\varphi_{P_i}, \psi_{P_0}) = \int_{\partial K_{P_0}^*} \left(-\frac{\partial \varphi_{P_i}}{\partial x}dy + \frac{\partial \varphi_{P_i}}{\partial y}dx\right) \tag{2.13}$$

(2.9) is a linear system for $\{u_{P_i}\}$. Its formation involves a great number of integrates like (2.13). To simplify the computation, one should decomposite the integral (2.13) into an integration sum over $\partial K_{P_0}^* \cap K_{Q_l} (l = 1, 2, 3, 4)$, and then transform it by the bilinear transformation into a definite integral of ξ and η on the reference element $\hat{K}$.

3. A Priori Estimates

In this section we shall prove the positive definiteness of $a(u_h, \Pi_h^* u_h)$, which is the key point to the error estimates.

As in Fig.3, let the four nodes of the quadrilateral element K_Q be $P_i = (x_i, y_i) (i = 1, 2, 3, 4)$, and the midpoints of the four sides by $M_i = (x_{M_i}, y_{M_i}) (i = 1, 2, 3, 4)$. Q denotes the joint of $\overline{M_1M_3}$ and $\overline{M_2M_4}$, that is, the averaging center of K_Q, then Q becomes the midpoint of both $\overline{M_1M_3}$ and $\overline{M_2M_4}$. $\overline{P_1P_4}$ and $\overline{P_2P_3}$ are the two diagonals intersecting at R.

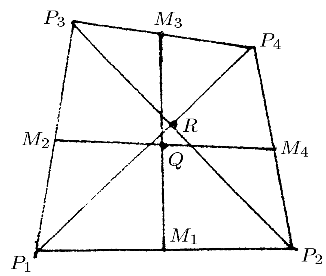

Fig. 3

Let us write

$$\begin{array}{llll} f_1 = x_4 - x_3, & f_2 = x_4 - x_2, & d_1 = x_{M_3} - x_{M_1}, & d_2 = x_{M_4} - x_{M_2} \\ g_1 = y_4 - y_3, & g_2 = y_4 - y_2, & e_1 = y_{M_3} - y_{M_1}, & e_2 = y_{M_4} - y_{M_2} \\ l_{12} = |\overline{P_1P_2}|, & l_{13} = |\overline{P_1P_3}|, & l_{24} = |\overline{P_2P_4}|, & l_{34} = |\overline{P_3P_4}| \\ l_1 = |\overline{P_2P_3}|, & l_2 = |\overline{P_1P_4}|, & m_1 = |\overline{M_1M_3}|, & m_2 = |\overline{M_2M_4}| \\ \theta = \angle M_3QM_4, & \phi = \angle P_3RP_4 & & \end{array}$$

Let $|\overline{RP_2}| = k_1l_1$, $|\overline{RP_1}| = k_2l_2$, then

$$|\overline{RP_3}| = (1-k_1)l_1, |\overline{RP_4}| = (1-k_2)l_2$$

Denote the areas of the quadrilateral K_Q,$\triangle P_1P_2P_3$, $\triangle P_1P_2P_4$, $\triangle P_1P_4P_3$ and $\triangle$ $P_2P_4P_3$, respectively, by S, S_{123}, S_{124}, S_{143}, S_{243}, then

$$S_{124} = k_1S, S_{123} = k_2S, S_{143} = (1-k_1)S, S_{243} = (1-k_2) \tag{3.1}$$

Introduce over U_h the discrete semi-norm

$$|u_h|_{1,h} = (\sum_{Q\in\Omega_h^*} |u_h|^2_{1,K_Q,h})^{\frac{1}{2}} \tag{3.2}$$

where

$$|u_h|^2_{1,K_Q,h} = (u_{P_2} - u_{P_1})^2 + (u_{P_4} - u_{P_2})^2 + (u_{P_4} - u_{P_3})^2 + (u_{P_3} - u_{P_1})^2 \tag{3.3}$$

$$u_{P_i} = u_h(P_i), i = 1,2,3,4.$$

Proposition 1. *The semi-norms $|u_h|_{1,h}$ and $|u_h|_1$ are equivalent over U_h, that is, there exist constants β_1 and β_2 independent of h such that*

$$\beta_1|u_h|_{1,h} \le |u_h|_1 \le \beta_2|u_h|_{1,h} \qquad \forall u_h \in U_h \tag{3.4}$$

Proof. We only have to show the equivalence of $|u_h|_{1,K_Q}$ and $|u_h|_{1,K_Q,h}$. By (2.4),(2.5), we have

$$\begin{cases} \frac{\partial u_h}{\partial x} = \frac{\partial u_h}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_h}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial u_h}{\partial y} = \frac{\partial u_h}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u_h}{\partial \eta}\frac{\partial \eta}{\partial y} \end{cases} \tag{3.5}$$

where

$$\begin{cases} \frac{\partial u_h}{\partial \xi} = c_1 + c_3\eta, \frac{\partial u_h}{\partial \eta} = c_2 + c_3\xi \\ c_1 = u_{P_2} - u_{P_1}, c_2 = u_{P_3} - u_{P_1}, c_3 = u_{P_4} - u_{P_3} - u_{P_2} + u_{P_1} \end{cases} \tag{3.6}$$

Then, transformation (2.2) results in

$$\begin{cases} \frac{\partial x}{\partial \xi} = a_1 + a_3\eta, \frac{\partial x}{\partial \eta} = a_2 + a_3\xi \\ \frac{\partial y}{\partial \xi} = b_1 + b_3\eta, \frac{\partial y}{\partial \eta} = b_2 + b_3\xi \end{cases}$$

Denote the Jacobian of (2.2) by $J(x, y)$, then

$$\det J(x,y) = (a_1b_2 - a_2b_1) + (a_1b_3 - a_3b_1)\xi + (a_3b_2 - a_2b_3)\eta \tag{3.7}$$

By the differentiation of inverse functions we have

$$\begin{cases} \frac{\partial \xi}{\partial x} = (b_2 + b_3\xi)/\det J(x,y), \frac{\partial \xi}{\partial y} = -(a_2 + a_3\xi)/\det J(x,y) \\ \frac{\partial \eta}{\partial x} = -(b_1 + b_3\eta)/\det J(x,y), \frac{\partial \eta}{\partial x} = (a_1 + a_3\eta)/\det J(x,y) \end{cases} \tag{3.8}$$

Combining (3.5), (3.6) and (3.8) leads to

$$\begin{cases} \frac{\partial u_h}{\partial x} = [(c_1 + c_3\eta)(b_2 + b_3\xi) - (c_2 + c_3\xi)(b_1 + b_3\eta)]/\det J(x,y) \\ \frac{\partial u_h}{\partial y} = [-(c_1 + c_3\eta)(a_2 + a_3\xi) + (c_2 + c_3\xi)(a_1 + a_3\eta)]/\det J(x,y) \end{cases} \tag{3.9}$$

Write $\bigtriangledown u_h = (\frac{\partial u_h}{\partial x}, \frac{\partial u_h}{\partial y})^T$, $\hat{\bigtriangledown} u_h = (\frac{\partial u_h}{\partial \xi}, \frac{\partial u_h}{\partial \eta})^T$. Then (3.5) gives

$$\begin{gathered} \bigtriangledown u_h = J(\xi,\eta)\hat{\bigtriangledown} u_h \\ \hat{\bigtriangledown} u_h = J^{-1}(\xi,\eta)\bigtriangledown u_h \end{gathered}$$

where

$$J(\xi,\eta) = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix}, J^{-1}(\xi,\eta) = (J(\xi,\eta))^{-1} = \begin{pmatrix} a_1 + a_3\eta & b_1 + b_3\eta \\ a_2 + a_3\xi & b_2 + b_3\xi \end{pmatrix}$$

Use $||\bigtriangledown u_h||_2$, $||J(\xi,\eta)||_2$ and $||J(\xi,\eta)||_F$ to denote the Euclidian norm of vectors, and the spectral norm of matrices and the Frobenius norm of matrices, respectively, then

$$\begin{gathered} ||\bigtriangledown u_h||_2^2 \le ||J(\xi,\eta)||_2^2||\hat{\bigtriangledown} u_h||_2^2 \le ||J(\xi,\eta)||_F^2||\hat{\bigtriangledown} u_h||_2^2 \\ ||\hat{\bigtriangledown} u_h||_2^2 \le ||J^{-1}(\xi,\eta)||_2^2||\bigtriangledown u_h||_2^2 \le ||J^{-1}(\xi,\eta)||_F^2||\bigtriangledown u_h||_2^2 \end{gathered}$$

So

$$|u_h|_{1,K_Q}^2 = \int_{K_Q} ||\bigtriangledown u_h||_2^2 dxdy \le \int_{\hat{E}} (||J(\xi,\eta)||_F^2 \det J(x,y))||\hat{\bigtriangledown} u_h||_2^2 d\xi d\eta \tag{3.10}$$

and

$$\begin{aligned} \int_{\hat{E}} ||\hat{\bigtriangledown} u_h||_2^2 d\xi d\eta &\le \int_{\hat{E}} ||J^{-1}(\xi,\eta)||_F^2 ||\bigtriangledown u_h||_2^2 d\xi d\eta \\ &\le \int_{K_Q} (||J^{-1}(\xi,\eta)||_F^2/\det J(x,y))||\bigtriangledown u_h||_2^2 dxdy \end{aligned} \tag{3.11}$$

A direct calculation gives

$$\begin{aligned}
&||J(\xi,\eta)||_F^2 \det J(x,y) \\
&= ||J^{-1}(\xi,\eta)||_F^2/\det J(x,y) \\
&= [(a_1+a_3\eta)^2+(b_1+b_3\eta)^2+(a_2+a_3\xi)^2+(b_2+b_3\xi)^2]/\det J(x,y)
\end{aligned} \tag{3.12}$$

By (3.1), (3.7) and the relations between the area and the node coordinates of the triangle, one get

$$\begin{aligned}
\det J(x,y) &= 2S_{123} + 2(S_{124}-S_{123})\xi + 2(S_{143}-S_{123})\eta \\
&= [2k_2 + 2(k_1-k_2)\xi + 2(1-k_1-k_2)\eta]S
\end{aligned} \tag{3.13}$$

where $(\xi,\eta) \in \hat{E} = [0,1]\times[0,1]$.

Suppose that k_1, k_2 satisfy

$$\sigma \le k_1, k_2 \le 1-\sigma, \qquad 0<\sigma\le \frac{1}{2}$$

and note that $\det J(x,y)$ is a linear function of ξ, η, then on the reference element $\hat{E}$ we have

$$\begin{aligned}
\det J(x,y) \ge \min\{&\det J(x(0,0),y(0,0)), \det J(x(0,1),y(0,1)), \\
&\det J(x(1,0),y(1,0)), \det J(x(1,1),y(1,1))\} \\
&= \min\{2k_2S, 2(1-k_1)S, 2k_1S, 2(1-k_2)S\} \\
&\ge 2\sigma S
\end{aligned}$$

On the other hand, we have for the numerator of the right-hand side of the second equality of (3.12)

$$\begin{aligned}
& (a_1+a_3\eta)^2+(b_1+b_3\eta)^2+(a_2+a_3\xi)^2+(b_2+b_3\xi)^2 \\
= \; & [a_1(1-\eta)+f_1\eta]^2 + [b_1(1-\eta)+g_1\eta]^2 \\
& +[a_2(1-\xi)+f_2\xi]^2+[b_2(1-\xi)+g_2\xi]^2 \\
\le \; & 2[(a_1^2+b_1^2)(1-\eta)^2+(f_1^2+g_1^2)\eta^2+(a_2^2+b_2^2)(1-\xi)^2+(f_2^2+g_2^2)\xi^2] \\
= \; & 2[l_{12}^2(1-\eta)^2+l_{34}^2\eta^2+l_{13}^2(1-\xi)^2+l_{24}^2\xi^2] \\
\le \; & 2[\max\{l_{12}^2,l_{34}^2\}+\max\{l_{13}^2,l_{24}^2\}] \\
\le \; & 4h^2
\end{aligned}$$

Hence

$$||J^{-1}(\xi,\eta)||_F^2/\det J(x,y) = ||J(\xi,\eta)||_F^2 \det J(x,y) \le \tfrac{4h^2}{2\sigma S} \le \tfrac{2}{C_1\sigma}$$

This together with (3.10) and (3.11) leads to

$$\tfrac{C_1\sigma}{2}|u_h|_{1,K_Q}^2 \le \int_{\hat{E}} ||\hat{\nabla} u_h||_2^2 d\xi d\eta \le \tfrac{2}{C_1\sigma}|u_h|_{1,K_Q}^2 \tag{3.14}$$

set

$$z_1 = u_{P_2} - u_{P_1}, z_2 = u_{P_4} - u_{P_3}, z_3 = u_{P_3} - u_{P_1}, z_4 = u_{P_4} - u_{P_2} \tag{3.15}$$

Then

$$\int_{\hat{E}} ||\hat{\bigtriangledown} u_h||_2^2 d\xi d\eta = \frac{1}{3}(z_1^2 + z_2^2 + z_1 z_2 + z_3^2 + z_4^2 + z_3 z_4)$$

It is easy to show that the right-hand side of the above equality is a positive definite quadratic form of z_1, z_2, z_3 and z_4, and hence is equivalent to $z_1^2+z_2^2+z_3^2+z_4^2$. Therefore $|u_h|_{1,K_Q,h}$ and $|u_h|_{1,K_Q}$ are equivalent by (3.3) and (3.14). This completes the proof.

Next we turn to show the positive definiteness of $a(u_h, \Pi_h^* u_h)$. As in [11], we rearrange the line integrals of the rigth-hand side of (2.10) to get

$$a(u_h, \Pi_h^* \bar{u}_h) = \sum_{Q \in \Omega_h^*} I_Q(u_h, \Pi_h^* \bar{u}_h) \tag{3.16}$$

where

$$I_Q(u_h, \Pi_h^* \bar{u}_h) = \sum_{P \in \overset{\circ}{K}} \int_{\partial K_P^* \cap K_Q} \left(-\frac{\partial u_h}{\partial x} dy + \frac{\partial u_h}{\partial y} dx\right) \bar{u}_h(P) \tag{3.17}$$

Denote by $\overset{\circ}{K}$ the set of the four nodes of $K_Q = \square P_1 P_2 P_3 P_4$(cf. Fig.3), and merge the two integrals with opposite directions on the same segment of the right-hand side of (3.17), then we have

$$\begin{aligned} I_Q(u_h, \Pi_h^* \bar{u}_h) = & \int_{\overline{M_1 Q}} \left(-\frac{\partial u_h}{\partial x} dy + \frac{\partial u_h}{\partial y} dx\right)(\bar{u}_h(P_1) - \bar{u}_h(P_2)) \\ & + \int_{\overline{Q M_3}} \left(-\frac{\partial u_h}{\partial x} dy + \frac{\partial u_h}{\partial y} dx\right)(\bar{u}_h(P_3) - \bar{u}_h(P_4)) \\ & + \int_{\overline{M_2 Q}} \left(-\frac{\partial u_h}{\partial x} dy + \frac{\partial u_h}{\partial y} dx\right)(\bar{u}_h(P_3) - \bar{u}_h(P_1)) \\ & + \int_{\overline{Q M_4}} \left(-\frac{\partial u_h}{\partial x} dy + \frac{\partial u_h}{\partial y} dx\right)(\bar{u}_h(P_4) - \bar{u}_h(P_2)) \end{aligned} \tag{3.18}$$

It is easy to see by Fig. 3, transformations (2.2), (3.9) and (3.15), that on $\overline{M_1 Q M_3}$ $(\xi = \frac{1}{2})$

$$\left\{ \begin{array}{l} x = (x_1 + \frac{1}{2} a_1) + (a_2 + \frac{1}{2} a_3)\eta = x_{M_1} + d_1 \eta \\ y = (y_1 + \frac{1}{2} b_1) + (b_2 + \frac{1}{2} b_3)\eta = y_{M_1} + e_1 \eta \end{array} \right. \tag{3.19$_1$}$$

$$\left\{ \begin{array}{l} \frac{\partial u_h}{\partial x} = [(z_1(1-\eta) + z_2 \eta) e_1 + \frac{1}{2}(z_3 + z_4)(-b_1 - b_3 \eta)] \\ \qquad /\det J(x(\frac{1}{2}, \eta), y(\frac{1}{2}, \eta)) \\ \frac{\partial u_h}{\partial y} = [(z_1(1-\eta) + z_2 \eta)(-d_1) + \frac{1}{2}(z_3 + z_4)(a_1 + a_3 \eta)] \\ \qquad /\det J(x(\frac{1}{2}, \eta), y(\frac{1}{2}, \eta)) \end{array} \right. \tag{3.19$_2$}$$

and on $\overline{M_2 Q M_4}(\eta = \frac{1}{2})$

$$\left\{ \begin{array}{l} x = (x_1 + \frac{1}{2} a_2) + (a_1 + \frac{1}{2} a_3)\xi = x_{M_2} + d_2 \xi \\ y = (y_1 + \frac{1}{2} b_2) + (b_1 + \frac{1}{2} b_3)\xi = y_{M_2} + e_2 \xi \end{array} \right. \tag{3.20$_1$}$$

$$\left\{\begin{array}{l}\frac{\partial u_h}{\partial x}=[\frac{1}{2}(z_1+z_2)(b_2+b_3\xi)+(z_3(1-\xi)+z_4\xi)(-e_2)] \\ \qquad /\mathrm{det}J(x(\xi,\frac{1}{2}),y(\xi,\frac{1}{2})) \\ \frac{\partial u_h}{\partial y}=[\frac{1}{2}(z_1+z_2)(-a_2-a_3\xi)+(z_3(1-\xi)+z_4\xi)d_2] \\ \qquad /\mathrm{det}J(x(\xi,\frac{1}{2}),y(\xi,\frac{1}{2}))\end{array}\right. \tag{3.20$_2$}$$

Insert $(3.19)_1$ and $(3.20)_1$ into (3.18), to obtain

$$\begin{aligned}I_Q(u_h,\Pi_h^*u_h)=&\int_0^{\frac{1}{2}}(-\frac{\partial u_h}{\partial x}e_1+\frac{\partial u_h}{\partial y}d_1)d\eta(u_{P_1}-u_{P_2})\\&+\int_{\frac{1}{2}}^1(-\frac{\partial u_h}{\partial x}e_1+\frac{\partial u_h}{\partial y}d_1)d\eta(u_{P_3}-u_{P_4})\\&+\int_0^{\frac{1}{2}}(-\frac{\partial u_h}{\partial x}e_2+\frac{\partial u_h}{\partial y}d_2)d\xi(u_{P_3}-u_{P_1})\\&+\int_{\frac{1}{2}}^1(-\frac{\partial u_h}{\partial x}e_2+\frac{\partial u_h}{\partial y}d_2)d\xi(u_{P_4}-u_{P_2})\end{aligned} \tag{3.21}$$

Write

$$\begin{aligned}&A_1=2S\int_0^{\frac{1}{2}}\frac{1-\eta}{\mathrm{det}J(x(\frac{1}{2},\eta),y(\frac{1}{2},\eta))}d\eta,A_2=2S\int_0^{\frac{1}{2}}\frac{\eta}{\mathrm{det}J(x(\frac{1}{2},\eta),y(\frac{1}{2},\eta))}d\eta\\&A_3=2S\int_{\frac{1}{2}}^1\frac{1-\eta}{\mathrm{det}J(x(\frac{1}{2},\eta),y(\frac{1}{2},\eta))}d\eta,A_4=2S\int_{\frac{1}{2}}^1\frac{\eta}{\mathrm{det}J(x(\frac{1}{2},\eta),y(\frac{1}{2},\eta))}d\eta\\&B_1=2S\int_0^{\frac{1}{2}}\frac{1-\xi}{\mathrm{det}J(x(\xi,\frac{1}{2}),y(\xi,\frac{1}{2}))}d\xi,B_2=2S\int_0^{\frac{1}{2}}\frac{\xi}{\mathrm{det}J(x(\xi,\frac{1}{2}),y(\xi,\frac{1}{2}))}d\xi\\&B_3=2S\int_{\frac{1}{2}}^1\frac{1-\xi}{\mathrm{det}J(x(\xi,\frac{1}{2}),y(\xi,\frac{1}{2}))}d\xi,B_4=2S\int_{\frac{1}{2}}^1\frac{\xi}{\mathrm{det}J(x(\xi,\frac{1}{2}),y(\xi,\frac{1}{2}))}d\xi\end{aligned} \tag{3.22}$$

$$\begin{aligned}&\alpha_1=a_1d_1+b_1e_1=\overrightarrow{P_1P_2}\cdot\overrightarrow{M_1M_3},\alpha_2=f_1d_1+g_1e_1=\overrightarrow{P_3P_4}\cdot\overrightarrow{M_1M_3}\\&\alpha_3=a_2d_2+b_2e_2=\overrightarrow{P_1P_3}\cdot\overrightarrow{M_2M_4},\alpha_4=f_2d_2+g_2e_2=\overrightarrow{P_2P_4}\cdot\overrightarrow{M_2M_4}\end{aligned} \tag{3.23}$$

By (3.21), $(3.19)_2$, $(3.20)_2$, (3.22), (3.23) and (3.15), we have

$$\begin{aligned}&I_Q(u_h,\Pi_h^*u_h)\\=\;&[(z_1A_1+z_2A_2)m_1^2-\frac{1}{2}(z_3+z_4)(A_1\alpha_1+A_2\alpha_2)]\frac{z_1}{2S}\\&+[(z_1A_3+z_2A_4)m_1^2-\frac{1}{2}(z_3+z_4)(A_3\alpha_1+A_4\alpha_2)]\frac{z_2}{2S}\\&+[-\frac{1}{2}(z_1+z_2)(B_1\alpha_3+B_2\alpha_4)+(z_3B_1+z_4B_2)m_2^2]\frac{z_3}{2S}\\&+[-\frac{1}{2}(z_1+z_2)(B_3\alpha_3+B_4\alpha_4)+(z_3B_3+z_4B_4)m_2^2]\frac{z_4}{2S}\\=\;&z^TWz\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}&z=(z_1,z_2,z_3,z_4)^T\\&W=\frac{-1}{4S}\times\\&\begin{pmatrix}-2A_1m_1^2 & -2A_2m_1^2 & A_1\alpha_1+A_2\alpha_2 & A_1\alpha_1+A_2\alpha_2\\-2A_3m_1^2 & -2A_4m_1^2 & A_3\alpha_1+A_4\alpha_2 & A_3\alpha_1+A_4\alpha_2\\B_1\alpha_3+B_2\alpha_4 & B_1\alpha_3+B_2\alpha_4 & -2B_1m_2^2 & -2B_2m_2^2\\B_3\alpha_3+B_4\alpha_4 & B_3\alpha_3+B_4\alpha_4 & -2B_3m_2^2 & -2B_4m_2^2\end{pmatrix}\end{aligned} \tag{3.25}$$

Using (3.13) we compute the integrals in (3.22):

$$\begin{array}{ll} A_1 = \frac{1}{1-k}[-\frac{1}{2} - \frac{(2-k)lnk}{2(1-k)}], & A_2 = \frac{1}{1-k}[\frac{1}{2} + \frac{klnk}{2(1-k)}] \\ A_3 = \frac{1}{1-k}[-\frac{1}{2} + \frac{(2-k)ln(2-k)}{2(1-k)}], & A_4 = \frac{1}{1-k}[\frac{1}{2} - \frac{kln(2-k)}{2(1-k)}] \\ B_1 = \frac{1}{1-\hat{k}}[-\frac{1}{2} - \frac{(2-\hat{k})ln\hat{k}}{2(1-\hat{k})}], & B_2 = \frac{1}{1-\hat{k}}[\frac{1}{2} + \frac{\hat{k}ln\hat{k}}{2(1-\hat{k})}] \\ B_3 = \frac{1}{1-\hat{k}}[-\frac{1}{2} + \frac{(2-\hat{k})ln(2-\hat{k})}{2(1-\hat{k})}], & B_4 = \frac{1}{1-\hat{k}}[\frac{1}{2} - \frac{\hat{k}ln(2-\hat{k})}{2(1-\hat{k})}] \end{array} \tag{3.26}$$

where $k = k_1 + k_2$, $\hat{k} = 1 - k_1 + k_2$, k_1 and k_2 are defined as before, $0 < k_1, k_2 < 1$.

Remark 1. If $k, \hat{k} \to 1$, in (3.26), then by Taylor Formula we have

$$\begin{array}{ll} A_1 = \frac{3}{4} + O(k-1), & A_2 = \frac{1}{4} + O(k-1) \\ A_3 = \frac{1}{4} + O(k-1), & A_4 = \frac{3}{4} + O(k-1) \\ B_1 = \frac{3}{4} + O(\hat{k}-1), & B_2 = \frac{1}{4} + O(\hat{k}-1) \\ B_3 = \frac{1}{4} + O(\hat{k}-1), & B_4 = \frac{3}{4} + O(\hat{k}-1) \end{array} \tag{3.27}$$

Note $k \to 1, \hat{k} \to 1$ are equivalent to $k_1 \to \frac{1}{2}, k_2 \to \frac{1}{2}$.

Since M_1, M_2, M_3 and M_4 are the midpoints of the corresponding sides (sse Fig. 3), we have

$$\overrightarrow{M_1M_3} = \frac{1}{2}(\overrightarrow{P_1P_4} + \overrightarrow{P_2P_3}), \qquad \overrightarrow{M_2M_4} = \frac{1}{2}(\overrightarrow{P_1P_4} + \overrightarrow{P_3P_2})$$

So

$$\overrightarrow{P_1P_4} = \overrightarrow{M_1M_3} + \overrightarrow{M_2M_4}, \qquad \overrightarrow{P_2P_3} = \overrightarrow{M_1M_3} + \overrightarrow{M_4M_2}$$

By the definition of k_1 and k_2, we have

$$\begin{array}{ll} \overrightarrow{P_1P_2} = k_1\ \overrightarrow{P_3P_2} + k_2\ \overrightarrow{P_1P_4}, & \overrightarrow{P_3P_4} = (1-k_1)\ \overrightarrow{P_3P_2} + (1-k_2)\ \overrightarrow{P_1P_4} \\ \overrightarrow{P_1P_3} = (1-k_1)\ \overrightarrow{P_2P_3} + k_2\ \overrightarrow{P_1P_4}, & \overrightarrow{P_2P_4} = k_1\ \overrightarrow{P_2P_3} + (1-k_2)\ \overrightarrow{P_1P_4} \end{array}$$

So

$$\begin{array}{l} \alpha_1 = (\hat{k}-1)m_1^2 + km_1m_2cos\theta \\ \alpha_2 = (1-\hat{k})m_1^2 + (2-k)m_1m_2cos\theta \\ \alpha_3 = \hat{k}m_1m_2cos\theta + (k-1)m_2^2 \\ \alpha_4 = (2-\hat{k})m_1m_2cos\theta + (1-k)m_2^2 \end{array} \tag{3.28}$$

where $k = k_1 + k_2, \hat{k} = 1 - k_1 + k_2, k_1, k_2$ as defined before.

We symmetrize the quadratic form (3.24) to obtain

$$I_Q(u_h, \Pi_h^* u_h) = z^T W_0 z \tag{3.29}$$

where

$$z = (z_1, z_2, z_3, z_4)^T$$

$$W_0 = \frac{1}{4S}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{22} & a_{34} & a_{44} \end{pmatrix} \tag{3.30}$$

$$\begin{aligned} &a_{11} = 2A_1 m_1^2, a_{12} = (A_2 + A_3)m_1^2, a_{22} = 2A_4 m_1^2 \\ &a_{33} = 2B_1 m_2^2, a_{34} = (B_2 + B_3)m_2^2, a_{22} = 2B_4 m_2^2 \\ &a_{13} = -\tfrac{1}{2}(A_1\alpha_1 + A_2\alpha_2 + B_1\alpha_3 + B_2\alpha_4) \\ &a_{14} = -\tfrac{1}{2}(A_1\alpha_1 + A_2\alpha_2 + B_3\alpha_3 + B_4\alpha_4) \\ &a_{23} = -\tfrac{1}{2}(A_3\alpha_1 + A_4\alpha_2 + B_1\alpha_3 + B_2\alpha_4) \\ &a_{24} = -\tfrac{1}{2}(A_3\alpha_1 + A_4\alpha_2 + B_3\alpha_3 + B_4\alpha_4) \end{aligned} \tag{3.31}$$

Set

$$\begin{aligned} &G_1 = \tfrac{1-\hat{k}}{1-k}[1 + \tfrac{lnk}{1-k}],\ G_2 = \tfrac{1-\hat{k}}{1-k}[1 - \tfrac{ln(2-k)}{1-k}] \\ &G_3 = \tfrac{1-k}{1-\hat{k}}[1 + \tfrac{ln\hat{k}}{1-\hat{k}}],\ G_4 = \tfrac{1-k}{1-\hat{k}}[1 - \tfrac{ln(2-\hat{k})}{1-\hat{k}}] \end{aligned} \tag{3.32}$$

Then by (3.26), (3.28), (3.31) and (3.32) we have

$$\begin{aligned} &a_{13} = -\tfrac{1}{2}(G_1 m_1^2 + G_3 m_2^2) - m_1 m_2 cos\theta \\ &a_{14} = -\tfrac{1}{2}(G_1 m_1^2 + G_4 m_2^2) - m_1 m_2 cos\theta \\ &a_{23} = -\tfrac{1}{2}(G_2 m_1^2 + G_3 m_2^2) - m_1 m_2 cos\theta \\ &a_{24} = -\tfrac{1}{2}(G_2 m_1^2 + G_4 m_2^2) - m_1 m_2 cos\theta \end{aligned} \tag{3.33}$$

To estimate the minimum eigenvalue λ_{W_0} of W_0, we decomposite W_0 into a sum of three matrices:

$$W_0 = \frac{1}{4S}(W_1 + W_2 + W_3)$$

where

$$\begin{aligned} &W_1 = \begin{pmatrix} \frac{3m_1^2}{2} & \frac{m_1^2}{2} & c & c \\ \frac{m_1^2}{2} & \frac{3m_1^2}{2} & c & c \\ c & c & \frac{3m_2^2}{2} & \frac{m_2^2}{2} \\ c & c & \frac{m_2^2}{2} & \frac{3m_2^2}{2} \end{pmatrix}, \\ &W_2 = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{12} & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & b_{34} & b_{44} \end{pmatrix}, W_3 = \begin{pmatrix} 0 & 0 & b_{13} & b_{14} \\ 0 & 0 & b_{23} & b_{24} \\ b_{13} & b_{23} & 0 & 0 \\ b_{14} & b_{24} & 0 & 0 \end{pmatrix} \end{aligned} \tag{3.34}$$

$$\begin{aligned} &c = -m_1 m_2 cos\theta \\ &b_{11} = (2A_1 - \tfrac{3}{2})m_1^2, b_{12} = (A_2 + A_3 - \tfrac{1}{2})m_1^2, b_{22} = (2A_4 - \tfrac{3}{2})m_1^2 \\ &b_{33} = (2B_1 - \tfrac{3}{2})m_2^2, b_{34} = (B_2 + B_3 - \tfrac{1}{2})m_2^2, b_{44} = (2B_4 - \tfrac{3}{2})m_2^2 \\ &b_{13} = -\tfrac{1}{2}(G_1 m_1^2 + G_3 m_2^2), b_{14} = -\tfrac{1}{2}(G_1 m_1^2 + G_4 m_2^2) \\ &b_{23} = -\tfrac{1}{2}(G_2 m_1^2 + G_3 m_2^2), b_{24} = -\tfrac{1}{2}(G_2 m_1^2 + G_4 m_2^2) \end{aligned} \tag{3.35}$$

Next we estimate successively the minimum eigenvalue λ_{W_1} of W_1 and the spectral radii $\rho(W_2), \rho(W_3)$.

Assume $m_2 \geq m_1$, and let $m_2 = \tau m_1$, then $\tau \geq 1$.

It is easy to show that the four eigenvalues of W_1 are

$$\begin{aligned}
&\lambda_1 = m_1^2, \lambda_2 = m_2^2 \\
&\lambda_3 = m_1^2 + m_2^2 + \sqrt{(m_1^2 + m_2^2)^2 - 4m_1^2 m_2^2 sin^2\theta} \\
&\lambda_4 = m_1^2 + m_2^2 - \sqrt{(m_1^2 + m_2^2)^2 - 4m_1^2 m_2^2 sin^2\theta}
\end{aligned}$$

Obviously, $\lambda_3 > \lambda_2 \geq \lambda_1$.

Write $\rho = sin\theta$, we have

$$\lambda_4 = m_1^2[1 + \tau^2 - \sqrt{(1+\tau^2)^2 - 4\tau^2\rho^2}]$$

Let $\lambda_{\min} = \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \min\{\lambda_1, \lambda_4\}$, then

$$\lambda_{\min} = \begin{cases} m_1^2, & \rho \in [\sqrt{1+2\tau^2}/2\tau, 1] \\ \lambda_4, & \rho \in (0, \sqrt{1+2\tau^2}/2\tau] \end{cases} \tag{3.36}$$

For any $\varepsilon_0 \in (0,1)$ in order to have

$$\lambda_4 \geq \varepsilon_0 m_1^2 \tag{3.37}$$

we need only

$$\rho \geq \rho_0 = \sqrt{2\varepsilon_0(1+\tau^2)}/2\tau = [\tfrac{1}{2}(1 + \tfrac{1}{\tau^2})]^{\frac{1}{2}}\sqrt{\varepsilon_0}$$

that is

$$\theta \in (\theta_0, \pi - \theta_0) \tag{3.38}$$

where $\theta_0 > 0$ is small enough.

Thus, if (3.38) holds, we have

$$\lambda_{\min} \geq \varepsilon_0 m_1^2 \tag{3.39}$$

We denote by D_1 and D_2 the up-left and down-right two-by-two matrices of W_2. Then the eigenvalues of W_1 are the eigenvalues of D_1 and D_2. The bigger eigenvalues in norm of D_1 and D_2, respectively, are

$$\lambda_{D_1} = U(k)m_1^2, \qquad \lambda_{D_2} = U(\hat{k})m_2^2$$

where

$$\begin{aligned}
U(k) &= (A_1 + A_4 - \tfrac{3}{2}) + \sqrt{(A_1 - A_4)^2 + (A_2 + A_3 - \tfrac{1}{2})^2} \\
U(\hat{k}) &= (B_1 + B_4 - \tfrac{3}{2}) + \sqrt{(B_1 - B_4)^2 + (B_2 + B_3 - \tfrac{1}{2})^2}
\end{aligned} \tag{3.40}$$

Noting (3.26), we know that $U(k)$ and $U(\hat{k})$ are the values of the same function $U(t)$ at $t = k$ and $\hat{k}$ respectively.

By (3.27) and (3.40), we obtain

$$\begin{aligned} U(k) &= O(k-1), k\to 1, \text{or} \quad k_1,k_2\to \tfrac{1}{2} \\ U(\hat{k}) &= O(\hat{k}-1), \hat{k}\to 1, \text{or} \quad k_1,k_2\to \tfrac{1}{2} \end{aligned} \tag{3.41}$$

In addition, the function $U(k)$ is symmetric on (0, 2) with respect to $k=1$, and for taking $\sigma\in(0,0.5]$, we have

$$\max_{k\in[2\sigma,1]} U(k) = U(2\sigma)$$

Write $\mathcal{D}_\sigma=\{(k_1,k_2),\sigma\le k_1,k_2\le 1-\sigma, 0<\sigma\le 0.5\}$.

Then

$$\begin{aligned} \max_{(k_1,k_2)\in\mathcal{D}_\sigma}\rho(W_2) &= max\{U(2\sigma)m_1^2, U(2\sigma)m_2^2\} \\ &= U(2\sigma)\tau^2m_1^2 \end{aligned} \tag{3.42}$$

Next, we consider the spectral radius of W_3. By solving the square of the eigenvalues of W_3, we obtain

$$\begin{aligned} [\rho(W_3)]^2 = \; & \tfrac{1}{8}[(G_1m_1^2+G_3m_2^2)^2+(G_1m_1^2+G_4m_2^2)^2 \\ & +(G_2m_1^2+G_3m_2^2)^2+(G_2m_1^2+G_4m_2^2)^2] \\ & +\tfrac{1}{4}\times\sqrt{(G_1m_1^2+G_4m_2^2)^2+(G_2m_1^2+G_3m_2^2)^2}\times \\ & \sqrt{(G_1m_1^2+G_3m_2^2)^2+(G_2m_1^2+G_4m_2^2)^2} \end{aligned} \tag{3.43}$$

From (3.32) we know that the right-hand side of (3.43) is a function of $k(k=k_1+k_2)$ and $\hat{k}(\hat{k}=1-k_1+k_2)$ for fixed m_1 and m_2, then we have

$$\begin{aligned} \max_{(k_1,k_2)\in\mathcal{D}_\sigma}[\rho(W_3)]^2 &= [\rho(W_3)]^2|_{(k_1,k_2)=(\sigma,\sigma)} \\ &= \tfrac{(1-2\sigma)^2\tau^4m_1^4}{4} \end{aligned}$$

So

$$\rho(W_3)\le(\tfrac{1}{2}-\sigma)\tau^2m_1^2, \quad (k_1,k_2)\in\mathcal{D}_\sigma \tag{3.44}$$

Finally, by (3.39), (3.42), (3.44) and

$$S=\frac{1}{2}l_1l_2sin\phi=m_1m_2sin\theta$$

we obtain

$$\begin{aligned} \lambda_{W_0} &\ge \tfrac{1}{4S}[\varepsilon_0m_1^2-U(2\sigma)\tau^2m_1^2-(\tfrac{1}{2}-\sigma)\tau^2m_1^2] \\ &= \tfrac{m_1^2}{4m_1m_2sin\theta}[\varepsilon_0-U(2\sigma)\tau^2-(\tfrac{1}{2}-\sigma)\tau^2] \\ &\ge \tfrac{1}{4\tau}[\varepsilon_0-U(2\sigma)\tau^2-(\tfrac{1}{2}-\sigma)\tau^2] \end{aligned} \tag{3.45}$$

Thus,we have

Proposition 2. *Suppose*

$$(1) \quad \tau_1\le\tau=\tfrac{m_2}{m_1}\le\tau_2 \tag*{$(3.46)_1$}$$

(2) *There exists* $\theta_0 > 0$, *such that*

$$\theta_0 \le \theta \le \pi - \theta_0 \tag{3.46$_2$}$$

(3) k_1, k_2 *approximate* $\frac{1}{2}$ *such that*

$$U(2\sigma)\tau^2 + (\tfrac{1}{2} - \sigma)\tau^2 \le \varepsilon_1 < \varepsilon_0$$

Or replace (3) by

$$(3)' \qquad k_1, k_2 \to \tfrac{1}{2}, \quad \text{as} \quad h \to 0 \tag{3.46$_3$}$$

Then for sufficiently small h, there exists constant $C_0 > 0$ such that

$$\lambda_{W_0} \ge C_0 \tag{3.47}$$

Hence

$$I_Q(u_h, \Pi_h^* u_h) \ge C_0 ||z||_2^2 = C_0 |u_h|_{1,K_Q,h}^2 \tag{3.48}$$

By (3.48) and Proposition 1 we have

$$I_Q(u_h, \Pi_h^* u_h) \ge \tilde{C} |u_h|_{1,K_Q}^2$$

So (3.16) implies

$$a(u_h, \Pi_h^* u_h) \ge \tilde{C} |u_h|_1^2 \ge \gamma ||u_h||_1^2 \tag{3.49}$$

Where $\tilde{C}, \gamma$ are positive constants.

This is the positiveness of $a(u_h, \Pi_h^* u_h)$. From this one can deduce the existence and uniqueness of the solutions of the generalized difference methods.

Remark 2. Denote the midpoints of $\overline{P_1P_4}$ and $\overline{P_2P_3}$ by M_{14} and M_{23} respectively, then

$$|\ \overrightarrow{P_1P_3} + \overrightarrow{P_4P_2}\ | = |\ \overrightarrow{P_1P_2} + \overrightarrow{P_4P_3}\ | = 2|\overline{M_{14}M_{23}}| \tag{3.50}$$

Also note

$$|\overline{M_{14}M_{23}}|^2 = (\hat{k}-1)^2 m_1^2 + (k-1)^2 m_2^2 + 2(\hat{k}-1)(k-1) m_1 m_2 cos\theta \tag{3.51}$$

Suppose $\square P_1P_2P_3P_4$ is a quasi-paralell quadrilateral element, namely (cf.[5])

$$|\ \overrightarrow{P_1P_3} + \overrightarrow{P_4P_2}\ | \le Ch^2 \tag{3.52}$$

(3.50) shows that the condition (cf. [10])

$$|\overline{M_{14}M_{23}}| \le Ch^2 \tag{3.53}$$

is equivalent to (3.52).

By (3.50)-(3.53), we know the condition $(3.46)_3$ of the Proposition 2 is similar to (3.52) and (3.53), moreover, we can deduce that the condition $(3.46)_3$ is weaker than (3.52) and (3.53).

4. Error Estimates

Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution to the problem (1.1), (1.2) and $u_h \in U_h$ to the generalized difference scheme (2.9). Then

$$a(u, \Pi_h^* v_h) = (f, \Pi_h^* v_h), \qquad \forall v_h \in U_h \tag{4.1}$$

$$a(u_h, \Pi_h^* v_h) = (f, \Pi_h^* v_h), \qquad \forall v_h \in U_h \tag{4.2}$$

So

$$a(u - u_h, \Pi_h^* v_h) = 0, \qquad \forall v_h \in U_h$$

By the positive definiteness (3.49), we have

$$\begin{aligned} ||\Pi_h u - u_h||_1^2 &\le \tfrac{1}{\gamma} a(\Pi_h u - u_h, \Pi_h^*(\Pi_h u - u_h)) \\ &= \tfrac{1}{\gamma} a(\Pi_h u - u, \Pi_h^*(\Pi_h u - u_h)) \end{aligned}$$

Hence

$$||\Pi_h u - u_h||_1 \le \tfrac{1}{\gamma} \sup_{w_h \in U_h} \frac{|a(\Pi_h u - u, \Pi_h^* w_h)|}{||w_h||_1} \tag{4.3}$$

where

$$a(\Pi_h u - u, \Pi_h^* w_h) = \sum_{Q \in \Omega_h^*} I_Q(\Pi_h u - u, \Pi_h^* w_h) \tag{4.4}$$

Note

$$\begin{aligned} I_Q(\Pi_h u - u, \Pi_h^* w_h) = &\int_{\overline{M_1 Q}} [\tfrac{\partial(\Pi_h u - u)}{\partial x} dy - \tfrac{\partial(\Pi_h u - u)}{\partial y} dx](w_{P_2} - w_{P_1}) \\ &+ \int_{\overline{M_3 Q}} [\tfrac{\partial(\Pi_h u - u)}{\partial x} dy - \tfrac{\partial(\Pi_h u - u)}{\partial y} dx](w_{P_3} - w_{P_4}) \\ &+ \int_{\overline{M_2 Q}} [\tfrac{\partial(\Pi_h u - u)}{\partial x} dy - \tfrac{\partial(\Pi_h u - u)}{\partial y} dx](w_{P_1} - w_{P_3}) \\ &+ \int_{\overline{M_4 Q}} [\tfrac{\partial(\Pi_h u - u)}{\partial x} dy - \tfrac{\partial(\Pi_h u - u)}{\partial y} dx](w_{P_4} - w_{P_2}) \end{aligned} \tag{4.5}$$

It follows from the definition (3.3) of the discrete semi-norm that

$$\begin{aligned} &|w_{P_2} - w_{P_1}| \le |w_h|_{1,K_Q,h}, \qquad |w_{P_3} - w_{P_4}| \le |w_h|_{1,K_Q,h} \\ &|w_{P_1} - w_{P_3}| \le |w_h|_{1,K_Q,h}, \qquad |w_{P_4} - w_{P_2}| \le |w_h|_{1,K_Q,h} \end{aligned} \tag{4.6}$$

Write $\varphi_1 = \frac{\partial(\Pi_h u - u)}{\partial x}$, $\varphi_2 = \frac{\partial(\Pi_h u - u)}{\partial y}$, then

$$\begin{aligned} &|\int_{\overline{M_k Q}} [\tfrac{\partial(\Pi_h u - u)}{\partial x} dy - \tfrac{\partial(\Pi_h u - u)}{\partial y} dx]| \qquad (k = 1, 2, 3, 4) \\ &\le \int_{\overline{M_k Q}} (|\varphi_1| + |\varphi_2|) ds \\ &\le C h^{\frac{1}{2}} [\int_{\overline{M_k Q}} (\varphi_1^2 + \varphi_2^2) ds]^{\frac{1}{2}} \end{aligned} \tag{4.7}$$

The inverse transformation $F_{K_Q}^{-1}$ of (2.2) transforms the element K_Q into the reference element $\hat{K}_Q$, the functions $\varphi_i(x,y)$ on K_Q into functions $\hat{\varphi}_i(\xi,\eta) = \varphi_(x,y)(i = 1,2)$, and Q, M_k, P_k into $\hat{Q}$, $\hat{M}_k$, $\hat{P}_k(k = 1,2,3,4)$ respectively. Then

$$\int_{\overline{M_k Q}} |\varphi_i|^2 ds \leq h \int_{\overline{\hat{M}_k \hat{Q}}} |\hat{\varphi}_i|^2 ds, \quad i = 1,2 \tag{4.8}$$

The trace formula implies that there exists a constant $C > 0$ independent of K_Q, such that

$$\int_{\overline{\hat{M}_k \hat{Q}}} |\hat{\varphi}_i|^2 ds \leq ||\hat{\varphi}_i||^2_{1,\hat{K}_Q}, \quad i = 1,2 \tag{4.9}$$

Also note

$$\begin{cases} |\hat{\varphi}_i|_{0,\hat{K}_Q} \leq Ch^{-1}|\varphi_i|_{0,K_Q}, & i = 1,2 \\ |\hat{\varphi}_i|_{1,\hat{K}_Q} \leq C|\varphi_i|_{1,K_Q}, & i = 1,2 \end{cases} \tag{4.10}$$

Combining (4.8)-(4.10) and using (2.8), we have

$$\begin{aligned} &\int_{\overline{M_k Q}} |\varphi_i|^2 ds \leq Ch||\hat{\varphi}_i||^2_{1,\hat{K}_Q} \\ &= Ch(|\hat{\varphi}_i|^2_{0,\hat{K}_Q} + |\hat{\varphi}_i|^2_{1,\hat{K}_Q}) \\ &\leq Ch(|\hat{\varphi}_i|_{0,\hat{K}_Q} + |\hat{\varphi}_i|_{1,\hat{K}_Q})^2 \\ &\leq Ch(h^{-1}|\varphi_i|_{0,K_Q}| + |\varphi_i|_{1,K_Q})^2 \\ &\leq Ch(h^{-1}|u - \Pi_h u|_{1,K_Q} + |u - \Pi_h u|_{2,K_Q})^2 \\ &\leq Ch|u|^2_{2,K_Q} \end{aligned} \tag{4.11}$$

Insert this into (4.7) and employ (4.6), then we have from (4.5) that

$$I_Q(\Pi_h u - u, \Pi_h^* w_h) \leq Ch|u|_{2,K_Q}|w_h|_{1,K_Q,h} \tag{4.12}$$

Finally, the continuity estimate results from (4.12), (4.4) and the equivalence (3.4) of the norms:

$$|a(\Pi_h u - u, \Pi_h^* w_h)| \leq Ch|u|_2|w_h|_1 \tag{4.13}$$

Substitute (4.13) into the right-hand side of (4.3) to obtain

$$||\Pi_h u - u_h||_1 \leq Ch|u|_2 \tag{4.14}$$

Then we have the error estimate for the generalized difference method:

$$||u - u_h||_1 \leq ||\Pi_h u - u_h||_1 + ||\Pi_h u - u||_1 \leq Ch|u|_2 \tag{4.15}$$

Thus we have proved the following theorem.

Theorem 1. *Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution to (1.1),(1.2), and $u_h \in U_h$ to (2.9). Suppose the quasi-uniformly subdivision condition (2.1) and the condition (3.46) of the proposition 2 are valid. Then the following error estimate holds*

$$||u - u_h||_1 \leq Ch|u|_2 \tag{4.16}$$

Remark 1. Obviously, the result (4.16) holds for rectangular grids. And in [11], the superconvergence estimate whose order is higher than (4.16) is obtained for rectangular grids.

Remark 2. As known, an error estimate like (4.16) can be found in [2]. But there are two important differences from those in [2]. Firstly, the method in our paper is based on the Petrov-Galerkin formulations, so we can construct the higher accuracy methods by taking tried and test spaces in higher order. On the contrary, the accuracy of the method in [2] can not be increased, one can not obtain the scheme with higher accuracy by methods in [2]. Secondly, the estimate in [2] is obtained on a strict restriction, i. e., the networks must be locally irregular networks, in other words, the elements of the subdivision consist essentially of equilateral triangles and rectangles.

5. Numerical Example

As a numerical example we use the generalized difference method to solve the following problems:

$$\textbf{Example1}. \begin{cases} -\Delta u = f(x,y), & (x,y) \in \Omega = (0,1) \times (0,1) \\ u = 0, & (x,y) \in \partial\Omega \end{cases} \tag{5.1}$$

where $f(x,y) = 2\pi^2 sin\pi x sin\pi y$, and the true solution is $u = sin\pi x sin\pi y$.

$$\textbf{Example2}. \begin{cases} -\Delta u = f(x,y), & (x,y) \in \Omega = (0,1) \times (0,1) \\ u(0,y) = u(1,y) = 0, & 0 \le y \le 1 \\ u(x,0) = sin\pi x, & 0 \le x \le 1 \\ u(x,1) = e(sin\pi x), & 0 \le x \le 1 \end{cases} \tag{5.2}$$

where $f(x,y) = (\pi^2 - 1)e^y sin\pi x$, and the true solution is $u = e^y sin\pi x$.

To solve these two problems, we decomposite the region $\bar{\Omega} = [0,1] \times [0,1]$ into $10 \times 10 = 100$ small squares, ending up with a square mesh as in Fig.4; Then we obtain a triangular mesh by drawing the diagonal of each small square as in Fig. 5.

Two generalized difference methods are used to solve (5.1) and (5.2).

(1)The linear element generalized difference method on triangular meshes (see [3]), denoted by $TGDM$.

(2)The generalized method on quadrilateral networks (see [11] and this paper), denoted by $QGDM$.

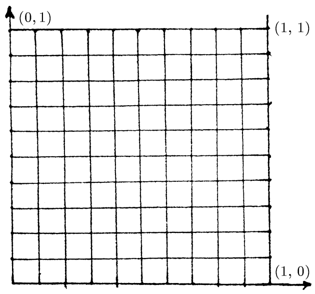

Fig. 4

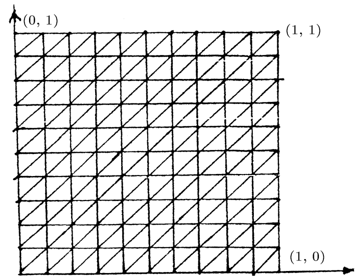

Fig. 5

Gaussian elimination method is used to solve the generalized difference equations of the above two methods. The numerical results of (5.1) (respectively (5.2)) and the corresponding true solutions (TS) are given in Table 1(respectively Table 2). We see that for Example 1, the generalized difference method on the triangular mesh behaves better than the one on the quadrilateral mesh, while for Example 2, the quadrilateral network is better. In practice, it depends on the geometry of the region Ω to determine which kind of mesh to use.

Table 1

(x_i, y_j)	$TGDM$ u_h	$QGDM$ u_h	TS u
(0.1,0.9)	0.096281	0.097473	0.095491
(0.2,0.9)	0.183137	0.185406	0.181636
(0.3,0.9)	0.252066	0.255189	0.250000
(0.4,0.9)	0.296322	0.299993	0.293893
(0.5,0.9)	0.311571	0.315431	0.309017
(0.2,0.8)	0.348347	0.352662	0.345491
(0.3,0.8)	0.479458	0.485398	0.475528
(0.4,0.8)	0.563637	0.570620	0.559017
(0.5,0.8)	0.592643	0.599985	0.587785
(0.3,0.7)	0.659918	0.668093	0.654508
(0.4,0.7)	0.775780	0.785391	0.769421
(0.5,0.7)	0.815703	0.825809	0.809017
(0.4,0.6)	0.911984	0.923282	0.904508
(0.5,0.6)	0.958917	0.970796	0.951056
(0.5,0.5)	1.008265	1.020755	1.000000

Table 2

(x_i, y_j)	$TGDM$	$QGDM$	TS
	u_h	u_h	u
(0.1,0.9)	0.761336	0.760967	0.760059
(0.2,0.9)	1.448147	1.447446	1.445719
(0.3,0.9)	1.993203	1.992238	1.989861
(0.4,0.9)	2.343151	2.342016	2.339222
(0.5,0.9)	2.463735	2.462542	2.459603
(0.1,0.8)	0.689988	0.689192	0.687730
(0.2,0.8)	1.312054	1.310921	1.308140
(0.3,0.8)	1.805887	1.804328	1.800501
(0.4,0.8)	2.122948	2.121114	2.116615
(0.5,0.8)	2.232199	2.230272	2.225541
(0.1,0.7)	0.624762	0.624042	0.622284
(0.2,0.7)	1.188367	1.186999	1.183654
(0.3,0.7)	1.635647	1.633764	1.629160
(0.4,0.7)	1.922819	1.920605	1.915193
(0.5,0.7)	2.021771	2.019444	2.013753
(0.1,0.6)	0.565698	0.564932	0.563066
(0.2,0.6)	1.076021	1.074565	1.071015
(0.3,0.6)	1.481016	1.479012	1.474125
(0.4,0.6)	1.741039	1.738682	1.732938
(0.5,0.6)	1.830637	1.828159	1.822119

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