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A SIMPLE WAY CONSTRUCTING SYMPLECTIC RUNGE-KUTTA METHODS*

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Abstract

With the help of symplecticity conditions of Partitioned Runge-Kutta methods, a simple way constructing symplectic methods is derived. Examples including several classes of high order symplectic Runge-Kutta methods are given, and showed up the relationship between existing high order Runge-Kutta methods.

Key words: Symplecticity condition, Partitioned Runge-Kutta method.

1. Introduction and Preliminaries

Let Ω be a domain in the oriented Euclidean space $\mathbb{R}^{2d}$ of point $(p,q)=((p_1,\cdots,p_d)^T,(q_1,\cdots,q_d)^T)$. If $H(p,q)$ is a sufficiently smooth real function defined in Ω, then the Hamiltonian system of differential equations with Hamiltonian $H(p,q)$ is given by

$$\frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}=:f_i(p,q),\quad \frac{dq_i}{dt}=\frac{\partial H}{\partial p_i}=:g_i(p,q), 1\le i\le d. \tag{1.1}$$

The integer d is called the number of degrees of freedom and Ω is the phase space. Here we assume that all Hamiltonians considered are autonomous, i.e., time- independent.

Definition 1.1. *A one-step method is called symplectic if, as applied to the Hamiltonian system* $(1,1)$, *the underlying formula generating numerical solutions* (p^{n+1},q^{n+1}) *is a symplectic transformation, that is,*

$$\frac{\partial(p^{n+1},q^{n+1})^T}{\partial(p^n,q^n)}J\frac{\partial(p^{n+1},q^{n+1})}{\partial(p^n,q^n)}=J,\quad \forall(p^n,q^n)\in\Omega, \tag{1.2}$$

Where $J=\begin{pmatrix}0 & I_d\\ -I_d & 0\end{pmatrix}$ *is the standard symplectic matrix.*

Definition 1.2. *One step of an s-stage Partitioned Runge-Kutta (PRK) method with stepsize h and initial values* (p^n,q^n) *applied to (1.1) reads*

$$P_i=p^n+h\sum_{j=1}^{s}a_{ij}F_j(P_j,Q_j),\quad Q_i=q^n+h\sum_{j=1}^{s}\overline{a}_{ij}G_j(P_j,Q_j), \tag{1.3a}$$

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$$p^{n+1} = p^n + h\sum_{i=1}^{s} b_i F_i(P_i, Q_i), \quad q^{n+1} = q^n + h\sum_{i=1}^{s} \overline{b}_i G_i(P_i, Q_i), \tag{1.3b}$$

Where a_{ij}, b_i *and* $\overline{a}_{ij}, \overline{b}_i$ *represent two different Runge-Kutta schemes,* $F = (f_1, f_2, \cdots, f_d)^T$, $G = (g_1, g_2, \cdots, g_d)^T$.

Definition 1.3. *The local error of a PRK method (1.3) is defined by*

$$\delta_{p_h}(t_n) = p^{n+1} - p(t_n + h), \quad \delta_{q_h}(t_n) = q^{n+1} - q(t_n + h)$$

Where $(p(t), q(t))$ *is the exact solutions of (1.1) possing through* (p^n, q^n) *at* t_n.

By definition 1.1, an s-stage symplectic PRK method can be characterized as follows:[7],[9],[12]

Theorem 1.4. *If the coefficients of an* s*-stage PRK method (1.3) satisfy the relation*

$$b_i = \overline{b}_i \quad for \quad i = 1, \cdots, s \tag{1.4a}$$

$$b_i \overline{a}_{ij} + \overline{b}_j a_{ji} - b_i \overline{b}_j = 0 \quad for \quad i, j = 1, \cdots, s, \tag{1.4b}$$

then the PRK method is symplectic.

Remark 1. Symplectic Runge-Kutta methods are a special case of symplectic PRK methods with coefficients $\overline{a}_{ij} = a_{ij}, i, j = 1, \cdots, s$.

Starting from a known s-stage RK method with $b_i \neq 0 (i = 1, \cdots, s)$, an s-stage symplectic PRK method can be defined uniquely as follows:[9]

Theorem 1.5. *Suppose that an* s*-stage RK method with coefficients* $a_{ij}, b_i \neq 0$ *and distinct* c_i*, satisfies the following simplifying assumptions*

$$B(p): \sum_{i=1}^{s} b_i c_i^{k-1} = \frac{1}{k} \quad for \quad k = 1, 2, \cdots, p,$$
$$C(\eta): \sum_{j=1}^{s} a_{ij} c_j^{k-1} = \frac{c_i^k}{k} \quad for \quad i = 1, \cdots, s, k = 1, \cdots, \eta,$$
$$D(\zeta): \sum_{i=1}^{s} b_i c_i^{k-1} a_{ij} = \frac{b_j}{k}(1 - c_j^k) \quad for \quad j = 1, \cdots, s, k = 1, \cdots, \zeta,$$

then the s*-stage PRK method with coefficients* $a_{ij}, \overline{b}_i = b_i, \overline{c}_i = c_i$ *and* $\overline{a}_{ij} = b_j(1 - a_{ji}/b_i)$ *is symplectic and satisfies*

$$\delta_{p_h}(t_n) = O(h^{r+1}), \quad \delta_{q_h}(t_n) = O(h^{r+1}),$$

i.e., at least, order $r = \min(p, 2\eta + 2, 2\zeta + 2, \eta + \zeta + 1)$.

Remark 2. By using the W-transformation of Hairer and Wanner[5] it can be shown that the RK method with coefficients $\overline{a}_{ij} = b_j(1 - a_{ji}/b_i), b_i$ and c_i satisfies $B(p), C(\zeta)$ and $D(\eta)$.

Remark 3. Moreover, suppose that the stability functions of RK methods with coefficients $a_{ij}, \overline{b}_i = b_i, \overline{c}_i = c_i$ and $\overline{a}_{ij} = b_j(1 - a_{ji}/b_i)$ are respectively $R(Z)$ and $\overline{R}(Z)$, if, the corresponding symplectic PRK method applied to a specially linear Hamiltonian system (referred to as a test equations of linear Hamiltonian systems)

$$\begin{cases} \dfrac{dp}{dt} = -\lambda p \\ \dfrac{dq}{dt} = \lambda q, \end{cases}$$

then symplecticity conditions (1.2) become

$$R(-Z)\overline{R}(Z) = 1. \tag{1.5}$$

In particular with a symmetric RK method, then $R(Z) = \overline{R}(Z)$, with an L-stable RK method, then the other is not A-stable, and with $\overline{a}_{ij} = a_{ij}(i, j = 1, \cdots, s)$ then (1.5)[4] is symplecticity condition for linear Hamiltonian systems.

Go a step further, let $a^*_{ij} = \frac{1}{2}(a_{ij} + \overline{a}_{ij}) = \frac{1}{2}(a_{ij} + b_j - b_j a_{ji}/b_i)$, by Remarks 2 and Butcher's order theorem[2], we immediately obtain:

Theorem 1.6. *The s-stage RK method with coefficients $a^*_{ij} = \frac{1}{2}(a_{ij} + \overline{a}_{ij}), b^*_i = b_i$ and $c^*_j = c_i$ is symplectic, and at least satisfies $B(p), C(\xi)$ and $D(\xi)$, i.e., order $r = \min(p, 2\xi + 1)$, where $\xi = \min(\eta, \zeta)$.*

Proof. Since

$$\begin{aligned} & b_i a^*_{ij} + b_j a^*_{ji} - b_i b_j \\ & = \frac{1}{2} b_i(a_{ij} + \overline{a}_{ij}) + \frac{1}{2} b_j(a_{ji} + \overline{a}_{ji}) - b_i b_j \\ & = \frac{1}{2}\{(b_i \overline{a}_{ij} + b_j a_{ji} - b_i b_j) + (b_i a_{ij} + b_j \overline{a}_{ji} - b_i b_j)\} \\ & = 0 \quad \text{for} \quad i, j = 1, \cdots, s, \end{aligned}$$

then the s-stage RK method is symplectic, the rest of this Theorem is a direct consequence of remarks 2 and Butcher's order Theorem[2].

Remark 4. Suppose that an s-stage RK method with coefficients (a_{ij}, b_i, c_i) satisfies $b_i > 0 (i = 1, \cdots, s)$, then the s-stage symplectic RK method generated by coefficients $\left(a^*_{ij} = \frac{1}{2}\left(a_{ij} + b_j - \frac{b_j a_{ji}}{b_i}\right), b_i, c_i\right)$ is algebraic stable.

It can be seen from Theorem 1.6. that starting from a known high order RK method with $b_i \neq 0$, in order to obtain the high order symplectic RK method, values η and ζ in $C(\eta)$ and $D(\zeta)$ must be very close, that is $\eta \sim \zeta$.

In next section, by Theorem 1.5. and Theorem 1.6., high order symplectic Runge-Kutta methods can be derived very easily starting from known high order RK methods, and Examples given will show up the relationship between existing high order Runge-Kutta methods.

2. Construction of Symplectic RK Methods

In this Section examples of constructing several classes of high order symplectic Runge-Kutta methods are given, and will show up the relationship between existing high order RK methods.

Starting from an s-stage known high order symplectic PRK method (or an s-stage known high order RK method with coefficients $(a_{ij}, b_i \neq 0, c_i)$, satisfying $B(p), C(\eta)$ and $D(\zeta)$, by Theorem 1.5. an s-stage symplectic PRK method is uniquely defined, here the RK method with coefficients $(\overline{a}_{ij} = b_j(1 - a_{ji}/b_i), \overline{b}_i = b_i, \overline{c}_i = c_i)$, satisfies $(B(p), C(\zeta), D(\eta)))$, and then let $a^*_{ij} = \dfrac{1}{2}(a_{ij} + \overline{a}_{ij}), b^*_i = b$ and $c^*_i = c_i$, by Theorem 1.6., then the s-stage RK method with coefficients (a^*_{ij}, b^*_i, c^*_i) is symplectic and, at least, satisfies $B(p), C(\xi)$ and $D(\xi)$, i.e., order $r = \min(p, 2\xi + 1), \xi = \min(\zeta, \eta)$.

Example 1. For the s-stage symplectic PRK method Lobatto IIIA-IIIB with coefficients $(A, b, c) - (\overline{A}, b, c)$, satisfying $(B(2s-2), c(s), D(s-2)) - (B(2s-2), C(s-2), D(s))$, (see [9]) let $a^*_{ij} = \dfrac{1}{2}(a_{ij} + \overline{a}_{ij})$ (i.e., $A^* = \dfrac{1}{2}(A + \overline{A})$), by Theorem 1.6., then the s-stage RK method with coefficients (A^*, b, c) is symplectic and satisfies $(B(2s-2), C(s-2), D(s-2))$. In fact, that is Lobatto IIIS method with the special case $\sigma = \dfrac{1}{2}$ (see [3]). For example, its members with 2,3 and 4-stage are generated as follows:

$$\begin{array}{c|cc} & 0 & 0 \\ & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \quad \text{and} \quad \begin{array}{c|cc} & \frac{1}{2} & 0 \\ & \frac{1}{2} & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \quad \Longrightarrow \quad \begin{array}{c|cc} & \frac{1}{4} & 0 \\ & \frac{1}{2} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array},$$

$$\begin{array}{c|ccc} & 0 & 0 & 0 \\ & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array} \quad \text{and} \quad \begin{array}{c|ccc} & \frac{1}{6} & -\frac{1}{6} & 0 \\ & \frac{1}{6} & \frac{1}{3} & 0 \\ & \frac{1}{6} & \frac{5}{6} & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array} \quad \Longrightarrow \quad \begin{array}{c|ccc} & \frac{1}{12} & -\frac{1}{12} & 0 \\ & \frac{3}{16} & \frac{1}{3} & -\frac{1}{48} \\ & \frac{1}{6} & \frac{9}{12} & \frac{1}{12} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

$$\text{and} \quad \begin{array}{c|cccc} & 0 & 0 & 0 & 0 \\ & \frac{11+\sqrt{5}}{120} & \frac{25-\sqrt{5}}{120} & \frac{25-13\sqrt{5}}{120} & \frac{-1+\sqrt{5}}{120} \\ & \frac{11-\sqrt{5}}{120} & \frac{25+13\sqrt{5}}{120} & \frac{25+\sqrt{5}}{120} & \frac{-1-\sqrt{5}}{120} \\ & \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \\ \hline & \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \end{array} \quad \text{and} \quad \begin{array}{c|cccc} & \frac{1}{12} & \frac{-1-\sqrt{5}}{24} & \frac{-1+\sqrt{5}}{24} & 0 \\ & \frac{1}{12} & \frac{25+\sqrt{5}}{120} & \frac{25-13\sqrt{5}}{120} & 0 \\ & \frac{1}{12} & \frac{25+13\sqrt{5}}{120} & \frac{25-\sqrt{5}}{120} & 0 \\ & \frac{1}{12} & \frac{11-\sqrt{5}}{24} & \frac{11+\sqrt{5}}{24} & 0 \\ \hline & \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \end{array}$$

$$\Longrightarrow \quad \begin{array}{c|cccc} & \frac{1}{24} & \frac{-1-\sqrt{5}}{48} & \frac{-1+\sqrt{5}}{48} & 0 \\ & \frac{21+\sqrt{5}}{240} & \frac{5}{24} & \frac{25-13\sqrt{5}}{120} & \frac{-1+\sqrt{5}}{240} \\ & \frac{21-\sqrt{5}}{240} & \frac{25+13\sqrt{5}}{120} & \frac{5}{24} & \frac{-1-\sqrt{5}}{240} \\ & \frac{1}{12} & \frac{21-\sqrt{5}}{48} & \frac{21+\sqrt{5}}{48} & \frac{1}{24} \\ \hline & \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \end{array}$$

Since s-stage Lobatto IIIA (or IIIB) method is symmetric, by Remarks 3, then their stability function is all the same.

Example 2. For s-stage symplectic PRK method Lobatto IIIC-III$\overline{C}$ (referred to as III-process by Butcher [1]) with coefficients $(A,b,c)-(\overline{A},b,c)$, all satisfying $(B(2s-2),C(s-1),D(s-1))$, (see [11]) let $a_{ij}^{*}=\frac{1}{2}(a_{ij}+\overline{a}_{ij})$, by Theorem 1.6. then the s-stage RK method with coefficients (A^{*},b,c) is symplectic and, satisfies $(B(2s-2),C(s-1),D(s-1))$. In fact, that is Lobatto III E method (see [6], [3]). For example, its members with 2,3 and 4-stage are generated as follows:

$$\begin{array}{c|cc} & \frac{1}{2} & -\frac{1}{2} \\ & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \quad \text{and} \quad \begin{array}{c|cc} & 0 & 0 \\ & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \quad \Longrightarrow \quad \begin{array}{c|cc} & \frac{1}{4} & -\frac{1}{4} \\ & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array},$$

$$\begin{array}{c|ccc} & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ & \frac{1}{6} & \frac{5}{12} & -\frac{1}{12} \\ & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array} \quad \text{and} \quad \begin{array}{c|ccc} & 0 & 0 & 0 \\ & \frac{1}{4} & \frac{1}{4} & 0 \\ & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array} \quad \Longrightarrow \quad \begin{array}{c|ccc} & \frac{1}{12} & -\frac{1}{6} & \frac{1}{12} \\ & \frac{5}{24} & \frac{4}{12} & -\frac{1}{24} \\ & \frac{1}{12} & \frac{5}{6} & \frac{1}{12} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

$$\text{and} \quad \begin{array}{c|cccc} & \frac{1}{12} & -\frac{\sqrt{5}}{12} & \frac{\sqrt{5}}{12} & -\frac{1}{12} \\ & \frac{1}{12} & \frac{1}{4} & \frac{10-7\sqrt{5}}{60} & \frac{\sqrt{5}}{60} \\ & \frac{1}{12} & \frac{10+7\sqrt{5}}{60} & \frac{1}{4} & \frac{-\sqrt{5}}{60} \\ & \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \\ \hline & \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \end{array} \quad \text{and} \quad \begin{array}{c|cccc} & 0 & 0 & 0 & 0 \\ & \frac{5+\sqrt{5}}{60} & \frac{1}{6} & \frac{15-7\sqrt{5}}{60} & 0 \\ & \frac{5-\sqrt{5}}{60} & \frac{15+7\sqrt{5}}{60} & \frac{1}{6} & 0 \\ & \frac{1}{6} & \frac{5-\sqrt{5}}{12} & \frac{5+\sqrt{5}}{12} & 0 \\ \hline & \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \end{array}$$

$$\Longrightarrow \quad \begin{array}{c|cccc} & \frac{1}{24} & -\frac{\sqrt{5}}{24} & \frac{\sqrt{5}}{24} & -\frac{1}{24} \\ & \frac{10+\sqrt{5}}{120} & \frac{5}{24} & \frac{25-14\sqrt{5}}{120} & \frac{\sqrt{5}}{120} \\ & \frac{10-\sqrt{5}}{120} & \frac{25+14\sqrt{5}}{120} & \frac{5}{24} & -\frac{\sqrt{5}}{120} \\ & \frac{1}{8} & \frac{10-\sqrt{5}}{24} & \frac{10+\sqrt{5}}{24} & \frac{1}{24} \\ \hline & \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \end{array}.$$

Since s-stage Lobatto IIIC method is L-stable, by Remarks 3, the s-stage Lobatto III $\overline{C}$ method (III-process) is not A-stable.

Example 3. Starting from s-stage Lobatto X method with transformation matrix

$$X = \begin{pmatrix} \frac{1}{2} & -\xi_1 & & & & \\ \xi_1 & 0 & & & & \\ & & \ddots & & & \\ & & & & -\xi_{s-2} & \\ & & & \xi_{s-2} & 0 & -\xi_{s-1}\sigma_1 \\ & & & & \xi_{s-1}\sigma_2 & 0 \end{pmatrix}$$

where $\sigma_1 \neq \sigma_2 \neq \dfrac{1}{b^T P_{s-1}^2(C)}$ (see [10]), by Theorem 1.5. and Theorem 1.6., such method leads to symmetric and symplectic s-stage Lobatto IIIS method satisfying $(B(2s-2), C(s-2), D(s-2))$ $(see[3])$.

Remark 5. It can be seen from Example 3 that the starting scheme of generating s-stage Lobatto IIIS method is not unique. For example, all taking $\sigma_1 = 0, \sigma_2 =$ arbitary real parameter and $\sigma_2 \neq \sigma_1 \neq 0$ lead to the s-stage Lobatto IIIS method.

Example 4. For s-stage symplectic PRK method Radau $IA - I\overline{A}$ (referred to as I-process by Bucther [1]) with coefficients $(A, b, c) - (\overline{A}, b, c)$, satisfying $(B(2s-1), C(s-1), D(s)) - (B(2s-1), C(s), D(s-1))$ (see [11]), let $a^*_{ij} = \frac{1}{2}(a_{ij} + \overline{a}_{ij})$, by Theorem 1.6., then the s-stage RK method with coefficients (A^*, b, c) is symplectic and satisfies $(B(2s-1), C(s-1), D(s-1))$, i.e., what is called s-stage Radau IB method (see [8]). For example, its members with 1,2 and 3-stage are generated as follows:

$$\begin{array}{c|c} & 1 \\ \hline & 1 \end{array} \quad \text{and} \quad \begin{array}{c|c} & 0 \\ \hline & 1 \end{array} \quad \Longrightarrow \quad \begin{array}{c|c} & \frac{1}{2} \\ \hline & 1 \end{array}$$

$$\begin{array}{c|cc} & \frac{1}{4} & -\frac{1}{4} \\ & \frac{1}{4} & \frac{5}{12} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array} \quad \text{and} \quad \begin{array}{c|cc} & 0 & 0 \\ & \frac{1}{3} & \frac{1}{3} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array} \quad \Longrightarrow \quad \begin{array}{c|cc} & \frac{1}{8} & -\frac{1}{8} \\ & \frac{7}{24} & \frac{3}{8} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

$$\text{and} \quad \begin{array}{c|ccc} & \frac{1}{9} & \frac{-(1+\sqrt{6})}{18} & \frac{-1+\sqrt{6}}{18} \\ & \frac{1}{9} & \frac{88+7\sqrt{6}}{360} & \frac{88-43\sqrt{6}}{360} \\ & \frac{1}{9} & \frac{88+43\sqrt{6}}{360} & \frac{88-7\sqrt{6}}{360} \\ \hline & \frac{1}{9} & \frac{16+\sqrt{6}}{36} & \frac{16-\sqrt{6}}{36} \end{array} \quad \text{and} \quad \begin{array}{c|ccc} & 0 & 0 & 0 \\ & \frac{9+\sqrt{6}}{75} & \frac{24+\sqrt{6}}{120} & \frac{168-73\sqrt{6}}{600} \\ & \frac{9-\sqrt{6}}{75} & \frac{168+73\sqrt{6}}{600} & \frac{24-\sqrt{6}}{120} \\ \hline & \frac{1}{9} & \frac{16+\sqrt{6}}{36} & \frac{16-\sqrt{6}}{36} \end{array}$$

$$\Longrightarrow \quad \begin{array}{c|ccc} & \frac{1}{18} & \frac{-1-\sqrt{6}}{36} & \frac{-1+\sqrt{6}}{36} \\ & \frac{52+3\sqrt{6}}{450} & \frac{16+\sqrt{6}}{72} & \frac{472-217\sqrt{6}}{1800} \\ & \frac{52-3\sqrt{6}}{450} & \frac{472+217\sqrt{6}}{1800} & \frac{16-\sqrt{6}}{72} \\ \hline & \frac{1}{9} & \frac{16+\sqrt{6}}{36} & \frac{16-\sqrt{6}}{36} \end{array}$$

Since s-stage Radau IA method is L-stable, by Remarks 3, the s-stage Radau $I\overline{A}$ method (I-process) is not A-stable.

Example 5. For s-stage symplectic PRK method Radau $IIA - II\overline{A}$ (referred to as II-process by Butcher [1]) with coefficients $(A,b,c)-(\overline{A},b,c)$, satisfying $(B(2s-1),C(s),D(s-1))-(B(2s-1),C(s-1),D(s))$ (see [11]), let $a^*_{ij}=\frac{1}{2}(a_{ij}+\overline{a}_{ij})$, by Theorem1.6., then the s-stage RK method with coefficients (A^*,b,c) is symplectic and satisfies $(B(2s-1),C(s-1),D(s-1))$, i.e., what is called the s-stage Radau IIB method (see [8]). For example, its members with 1,2 and 3-stage are generated as follows:

$$\begin{array}{c|c} & 1 \\ \hline & 1 \end{array} \quad \text{and} \quad \begin{array}{c|c} & 0 \\ \hline & 1 \end{array} \quad \Longrightarrow \quad \begin{array}{c|c} & \frac{1}{2} \\ \hline & 1 \end{array}$$

$$\begin{array}{c|cc} & \frac{5}{12} & -\frac{1}{12} \\ & \frac{3}{4} & \frac{1}{4} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array} \quad \text{and} \quad \begin{array}{c|cc} & \frac{1}{3} & 0 \\ & 1 & 0 \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array} \quad \Longrightarrow \quad \begin{array}{c|cc} & \frac{3}{8} & -\frac{1}{24} \\ & \frac{7}{8} & \frac{1}{8} \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

$$\text{and} \quad \begin{array}{c|ccc} & \frac{88-7\sqrt{6}}{360} & \frac{296-169\sqrt{6}}{1800} & \frac{-2+3\sqrt{6}}{225} \\ & \frac{296+169\sqrt{6}}{1800} & \frac{88+7\sqrt{6}}{360} & \frac{-(2+3\sqrt{6})}{225} \\ & \frac{16-\sqrt{6}}{36} & \frac{16+\sqrt{6}}{36} & \frac{1}{9} \\ \hline & \frac{16-\sqrt{6}}{36} & \frac{16+\sqrt{6}}{36} & \frac{1}{9} \end{array} \quad \text{and} \quad \begin{array}{c|ccc} & \frac{24-\sqrt{6}}{120} & \frac{14-11\sqrt{6}}{120} & 0 \\ & \frac{24+11\sqrt{6}}{120} & \frac{24+\sqrt{6}}{120} & 0 \\ & \frac{6-\sqrt{6}}{12} & \frac{6+\sqrt{6}}{12} & 0 \\ \hline & \frac{16-\sqrt{6}}{36} & \frac{16+\sqrt{6}}{36} & \frac{1}{9} \end{array}$$

$$\Longrightarrow \quad \begin{array}{c|ccc} & \frac{16-\sqrt{6}}{72} & \frac{328-167\sqrt{6}}{1800} & \frac{-2+3\sqrt{6}}{450} \\ & \frac{328+167\sqrt{6}}{1800} & \frac{16+\sqrt{6}}{72} & \frac{-2-3\sqrt{6}}{450} \\ & \frac{85-10\sqrt{6}}{180} & \frac{85+10\sqrt{6}}{180} & \frac{1}{18} \\ \hline & \frac{16-\sqrt{6}}{36} & \frac{16+\sqrt{6}}{36} & \frac{1}{9} \end{array}.$$

In the same reason as example 4, the s-stage Radau $II\overline{A}$ method (II-process) is not A-stable.

Example 6. For the s-stage PRK method Gauss $IA - I\overline{A}$ with coefficients $(A,b,c)-(\overline{A},b,c)$, all satisfying $(B(2s),C(s-1),D(s-1))$ (see [11]), let $a^*_{ij}=\frac{1}{2}(a_{ij}+\overline{a}_{ij})$, by Theorem 1.6., then the s-stage RK method with coefficients (A^*,b,c) is symplectic and, at least, satisfies $(B(2s),C(s-1),D(s-1))$. In fact, that is s-stage Gauss method with order $2s$. For example its members with 2 and 3-stage are generated as follows:

$$\begin{array}{c|cc} & \frac{1\pm 2\sigma}{4} & \frac{1\mp 2\sigma}{4}-\frac{\sqrt{3}}{6} \\ & \frac{1\mp 2\sigma}{4}+\frac{\sqrt{3}}{6} & \frac{1\pm 2\sigma}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} = \begin{array}{c} a_{ij} \\ \overline{a}_{ij} \\ \sigma > 0 \end{array} \Longrightarrow \begin{array}{c} (a_{ij})(\text{or } (\overline{a}_{ij})) \text{ with } \sigma = 0, \\ \text{i.e., Gauss method with order 4.} \end{array}$$

and
$$\begin{array}{c|ccc}
 & \frac{5\pm 8\sigma}{36} & \frac{2\mp 4\sigma}{9} - \frac{\sqrt{15}}{15} & \frac{5\pm 8\sigma}{36} - \frac{\sqrt{15}}{30} \\
 & \frac{5\mp 10\sigma}{36} + \frac{\sqrt{15}}{24} & \frac{2\pm 5\sigma}{9} & \frac{5\mp 10\sigma}{36} - \frac{\sqrt{15}}{24} \\
 & \frac{5\pm 8\sigma}{36} + \frac{\sqrt{15}}{30} & \frac{2\mp 4\sigma}{9} + \frac{\sqrt{15}}{15} & \frac{5\pm 8\sigma}{36} \\
\hline
 & \frac{5}{18} & \frac{4}{9} & \frac{5}{18}
\end{array} = \frac{a_{ij}}{\overline{a}_{ij}} \quad \sigma > 0$$

$\Longrightarrow (a_{ij})$(or $(\overline{a}_{ij})$) with $\sigma = 0$, i.e., Gauss method with order 6.

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