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THE GPL-STABILITY OF RUNGE-KUTTA METHODS FOR DELAY DIFFERENTIAL SYSTEMS*[1)]

Biao Yang Lin Qiu Jiao-xun Kuang
(*Department of Mathematics, Shanghai Normal University, Shanghai 200234, China.*)

Abstract

This paper deals with the GPL-stability of the Implicit Runge-Kutta methods for the numerical solutions of the systems of delay differential equations. We focus on the stability behaviour of the Implicit Runge-Kutta(IRK) methods in the solutions of the following test systems with a delay term

$$\begin{aligned} y'(t) &= Ly(t) + My(t-\tau), \quad t \geq 0, \\ y(t) &= \Phi(t), \quad t \leq 0, \end{aligned}$$

where L, M are $N \times N$ complex matrices,$\tau > 0$,$\Phi(t)$ is a given vector function. We shall show that the IRK methods is GPL-stable if and only if it is L-stable,when we use the IRK methods to the test systems above.

Key words: Delay differential equation, Implicit Runge-Kutta methods, GPL-stability.

1. Introduction

Before dealing with the numerical stabililty analysis of the IRK methods for systems of DDEs,we consider the following initial value problem

$$y'(t) = f(t, y(t)), \quad t > t_0, \tag{1}$$

$$y(t_0) = y_0, \tag{2}$$

where f is a given function and $y(t)$ is unknown for $t > t_0$.

For the initial problem (1)-(2),consider an Implicit Runge-Kutta method,

$$K_{n,i} = hf(t_n + c_i h, y_n + \sum_{j=1}^{v} a_{ij} K_{n,j}), \quad i = 1, 2, \ldots, v, \tag{3}$$

$$y_{n+1} = y_n + \sum_{i=1}^{v} b_i K_{n,i}, \quad n = 0, 1, 2, \ldots, \tag{4}$$

where $\sum_{i=1}^{v} b_i = 1, c_i = \sum_{j=1}^{v} a_{ij}, 1 \leq i \leq v, y_n \sim y(t_n), t_n = t_0 + nh$ and $h > 0$ is a stepsize.

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When we want to analyze the numerical stability of the IRK methods,we focus on the stability behaviour of the IRK methods with respect to the following linear test equations

$$y'(t) = \lambda y(t), \quad Re(\lambda) < 0, \tag{5}$$

$$y(t_0) = y_0. \tag{6}$$

We get the numerical recurrence formula,(see [7])

$$y_{n+1} = r(\bar{h}) y_n, \quad n \geq 0, \tag{7}$$

$$\begin{aligned} r(\bar{h}) &= 1 + \bar{h}\mathrm{b}^T (I - \bar{h}A)^{-1} e \\ &= \frac{det[I - \bar{h}(A - e\mathrm{b}^T)]}{det[I - \bar{h}A]}, \text{ if } det[I - \bar{h}A] \neq 0. \end{aligned} \tag{8}$$

Definition 1.1. (see [7]) *Let $R(q)$ be a function of q.*

(a) *If $Re(q) < 0 \Longrightarrow |R(q)| < 1$,then we say $R(q)$ is A-acceptable;*

(b) *If $q < 0 \Longrightarrow |R(q)| < 1$,then we say $R(q)$ is A_0-acceptable;*

(c) *If $R(q)$ is A-acceptable and $\lim_{Re(q) \to -\infty} |R(q)| = 0$, then we say $R(q)$ is L-acceptable.*

From Definition 1.1, we have the following statements. For the Implicit Runge-Kutta methods (3)-(4),
(1) it is A-stable if and only if $r(\bar{h})$ is A-acceptable;
(2) it is L-stable if and only if $r(\bar{h})$ is L-acceptable.

2. The GPL-Stability of the IRK Methods

For the following systems of delay differential equations

$$y'(t) = Ly(t) + My(t - \tau), \quad t \geq 0, \tag{9}$$

$$y(t) = \Phi(t), \quad t \leq 0, \tag{10}$$

where $y(t) = (y_1(t), y_2(t), \ldots, y_N(t))^T$,$L$ and M are constant complex $N \times N$ matrices,$\tau > 0$,$\Phi(t)$ denotes a given vector value function and $y(t)$ is unknown for $t > 0$.

We consider the exponential solutions of (9)-(10) in the form

$$y(t) = \xi \cdot e^{\zeta t}, \quad \xi \in C^N. \tag{11}$$

We have

Lemma 2.1. (see [5]) *The systems (9) has nonzero exponential solutions if and only if*

$$det[\zeta I - L - M e^{-\zeta \tau}] = 0. \tag{12}$$

The equation (12) is called the characteristic equation of (9),and (9) is asymptotically stable if and only if every root ζ of (12) satisfies $Re(\zeta) < 0$.

Lemma 2.2. (see [5]) *Assume that the coefficients of (9) satisfy*

$$\eta(L) = \frac{1}{2}\lambda_{\max}(L + L^*) < 0, \tag{13}$$

$$||M|| < -\eta(L), \tag{14}$$

then all roots of the equation (12) have negative real parts and the systems of (9) is asymptotically stable,i.e.,$\lim_{t \to -\infty} y(t) = 0$.

Definition 2.1. *If L and M satisfy (13)-(14),then a numerical method is called P-stable if and only if the numerical solutions y_n of (9)-(10) satisfy*

$$\lim_{n\to\infty} y_n = 0, \tag{15}$$

where $h = m^{-1}\tau$,$m \geq 1$ be a positive integer.

Definition 2.2. *A numerical method is called GP-stable if and only if $y_n \to 0$, as $n \to \infty$ and for every $h > 0$.*

Many results on the P-stability and GP-stability of the IRK methods have been given in [1,2,6].

For the exponential solution of (9),we have $y(t+h) = \xi \cdot e^{\zeta(t+h)} = \xi \cdot e^{\zeta t} e^{\zeta h} = y(t) \cdot e^{\zeta h}$,then $||y(t+h)||/||y(t)|| = |e^{\zeta h}|$.

If $Re(\zeta) < 0$,then

$$\lim_{Re(\zeta)\to-\infty} \frac{||y(t+h)||}{||y(t)||} = \lim_{Re(\zeta)\to-\infty} e^{\zeta h} = 0. \tag{16}$$

The following theorem plays a key role in this paper.

Theorem 2.1. *For the characteristic equation (12),if L and M satisfy (13)-(14) and the following (P),*

$$(P) \qquad \lim_{\eta(L)\to-\infty} \frac{||M||}{\eta(L)} = 0,$$

then $\lim_{\eta(L)\to-\infty} Re(\zeta) = -\infty$,for every root ζ of (12).

Proof. Let $\zeta = x + iy$ be a root of (12). If $\eta(L) \to -\infty$, but x doesn't tend to $-\infty$,for the characteristic equation (12),there must be a nonzero vector root $\xi_0 \neq 0$,$<\xi_0, \xi_0>= 1$,satisfies $(\zeta I - L - Me^{-\zeta\tau})\xi_0 = 0$. To make the inner product with ξ_0,we get

$$\zeta - < L\xi_0, \xi_0 > - < M\xi_0, \xi_0 > e^{-\zeta\tau} = 0,$$

Let $L = H_1 + iH_2$, $< M\xi_0, \xi_0 >= be^{i\theta}$, where $H_1 = \frac{1}{2}(L + L^*)$ and $H_2 = \frac{1}{2i}(L - L^*)$,

then

$$x - < H_1\xi_0, \xi_0 > - be^{-x\tau}\cos(\theta + y\tau) = 0,$$

it implies

$$\left|\frac{x}{< H_1\xi_0, \xi_0 >} - 1\right| = \left|\frac{be^{-x\tau}\cos(y\tau + \theta)}{< H_1\xi_0, \xi_0 >}\right| \leq \frac{||M||e^{-x\tau}}{|< H_1\xi_0, \xi_0 >|}.$$

Since $\min \lambda_{H_1} \leq < H_1\xi, \xi > \leq \eta(L)$ and $\eta(L) \to -\infty$, we get

$$\frac{||M||}{|\eta(L)|} \geq \frac{||M||}{|< H_1\xi_0, \xi_0 >|} \geq \left|\frac{x}{< H_1\xi_0, \xi_0 >} - 1\right| e^{x\tau}.$$

If $< H_1\xi_0, \xi_0 > \to -\infty$,but x does not tend to $-\infty$,then $|x/ < H_1\xi_0, \xi_0 > -1|e^{x\tau}$ does not tend to zero,it means $\frac{||M||}{|\eta(L)|}$ does not tend to zero, this contradicts condition (P) and complete the proof of Theorem 2.1.

By using Theorem 2.1 to (16),we arrive at

$$\lim_{\eta(L)\to-\infty} \frac{||y(t+h)||}{||y(t)||} = \lim_{\eta(L)\to-\infty} e^{\zeta h} = 0. \tag{17}$$

Also we expect that the numerical solutions $\{y_n\}$ of the IRK methods satisfy

$$\lim_{\eta(L)\to-\infty} \frac{||y_{n+1}||}{||y_n||} = 0, \text{ for any } n \geq 1.$$

Definition 2.2. *If L and M satisfy (13)-(14) and (P),then a numerical method is called PL-stable if and only if it is P-stable and the numerical solutions $\{y_n\}$ of (9)-(10) satisfy*

$$\lim_{\eta(L)\to-\infty} \frac{||y_{n+1}||}{||y_n||} = 0, \tag{18}$$

$$\textit{for } n \geq 0, h = m^{-1}\tau, m \geq 1.$$

Definition 2.3. *A numerical method for DDEs is called GPL-stable if and only if it is GP-stable and the (18) is fulfilled for any $h > 0$.*

Some results on the P_mL-stability of the IRK methods have been given in [8], and the PL-stability of the block θ- methods of delay differential equations in [4]. Now we use the Implicit Runge-Kutta methods to (9)-(10),

$$\begin{aligned}(I_{v\times N} - A\otimes\overline{L})\mathrm{K}_n &= (e\otimes\overline{L})y_n + e\otimes\overline{M}(\sum_{p=-r}^{s} L_p(\delta)y_{n-m+p}) + \\ & \quad A\otimes\overline{M}\sum_{p=-r}^{s} L_p(\delta)\mathrm{K}_{n-m+p} \qquad (19)\\ y_{n+1} &= y_n + \mathrm{b}^T\otimes I_N\mathrm{K}_n. \qquad (20)\end{aligned}$$

where the symbol $\otimes$ denotes the Kronecker product,

$$\begin{aligned}\mathrm{K}_n &= (K_{n,1}, K_{n,2}, \ldots, K_{n,v})^T,\\ \mathrm{b} &= (b_1, b_2, \ldots, b_v)^T,\\ e &= (1, 1, \ldots, 1)^T,\\ L_p(\delta) &= \prod_{k=-r,k\neq p}^{s} [(\delta-k)/(p-k)],\\ m &\geq s+1.\end{aligned}$$

we can write (20) in matrices form

$$\begin{pmatrix} I_{v\times N} - (A\otimes\overline{L}) & 0 \\ -\mathrm{b}^T\otimes I_N & I_N \end{pmatrix}\begin{pmatrix} \mathrm{K}_n \\ y_{n+1}\end{pmatrix} = \begin{pmatrix} 0 & e\otimes\overline{L} \\ 0 & I_N\end{pmatrix}\begin{pmatrix}\mathrm{K}_{n-1} \\ y_n\end{pmatrix} + \begin{pmatrix} A\otimes\overline{M} & 0 \\ 0 & 0\end{pmatrix} \cdot \begin{pmatrix}\sum_{p=-r}^{s} L_p(\delta)\mathrm{K}_{n-m+p} \\ \sum_{p=-r}^{s} L_p(\delta)y_{n-m+p+1}\end{pmatrix} + \begin{pmatrix} 0 & e\otimes\overline{M} \\ 0 & 0\end{pmatrix} \cdot \begin{pmatrix}\sum_{p=-r}^{s} L_p(\delta)\mathrm{K}_{n-m+p-1} \\ \sum_{p=-r}^{s} L_p(\delta)y_{n-m+p}\end{pmatrix}. \tag{21}$$

Then the characteristic equation of the above difference equation becomes

$$\det[\begin{pmatrix} I_{v\times N} - (A\otimes \overline{L}) & 0 \\ -\mathrm{b}^T \otimes I_N & I_N \end{pmatrix} z^n - \begin{pmatrix} 0 & e\otimes \overline{L} \\ 0 & I_N \end{pmatrix} z^{n-1} -$$
$$\begin{pmatrix} A\otimes \overline{M} & 0 \\ 0 & 0 \end{pmatrix} \textstyle\sum_{p=-r}^{s} L_p(\delta) \quad z^{n-m+p} -$$
$$\begin{pmatrix} 0 & e\otimes \overline{M} \\ 0 & 0 \end{pmatrix} \textstyle\sum_{p=-r}^{s} L_p(\delta) \quad z^{n-m+p+1}] = 0. \qquad (22)$$

Let

$$\begin{aligned} T_1(z) &= z^{m+1}[I_{v\times N} - A\otimes(\overline{L} + \overline{M}\sum_{p=-r}^{s} L_p(\delta)z^{p-m})], \\ T_2(z) &= -z^m[e\otimes(\overline{L} + \overline{M}\sum_{p=-r}^{s} L_p(\delta)z^{p-m})], \\ T_3(z) &= -(\mathrm{b}^T\otimes I_N)\cdot z^{m+1}, \\ T_4(z) &= I_N(z^{m+1} - z^m), \\ \overline{L} &= hL, \\ \overline{M} &= hM. \end{aligned}$$

Then (22) can be written in the form

$$det \begin{bmatrix} T_1(z) & T_2(z) \\ T_3(z) & T_4(z) \end{bmatrix} = 0, \qquad (23)$$

Lemma 2.3. (see Matrix theory: Gaotmaher) *If*

$$det \begin{bmatrix} T_1(z) & T_2(z) \\ T_3(z) & T_4(z) \end{bmatrix} = 0 \quad \Longrightarrow det T_1(z) \neq 0,$$

then (23) is equivalent to $det[T_4 - T_3T_1^{-1}T_2] = 0$.

Now we focus on the polynomial

$$\gamma(z,\delta) = \sum_{p=-r}^{s} L_p(\delta)z^{p+r}.$$

Lemma 2.4. (see [1]) *The condition*

$$|\gamma(z,\delta)| \leq 1, (whenever\ |z| = 1, 0\leq \delta < 1),$$

is equivalent to condition $r \leq s \leq r+2$*.Moreover, if* $r+s > 0, r \leq s \leq r+2$*,* $|z| = 1, 0 < \delta < 1$*,then* $|\gamma(z,\delta)| = 1$ *if and only if* $z = 1$.

Let

$$R(z,\delta) = \sum_{p=-r}^{s} L_p(\delta)\cdot z^{p-m}.$$

When $|z| = 1$,by using Lemma 2.4 we get $|R(z,\delta)| \leq 1$, for $\delta \in [0,1)$. When $z = \infty$,we have $|R(\infty,\delta)| = 0$,since $m \geq s+1$. Then we use the maximum modulus principle for analytic function to obtain

$$|R(z,\delta)| \leq 1 \text{ for } z \geq 1, \delta \in [0,1). \qquad (24)$$

From condition (14),namely,

$$||M|| < -\frac{1}{2}\lambda_{\max}(L+L^*),$$

we can show that

$$Re(\lambda_i(L+M\omega)) < 0, \text{ for every } |\omega| \le 1, i = 1, 2, \cdots, N, \tag{25}$$

where $\lambda_i(L+M\omega)$ denotes the eigenvalue of $L+M\omega$.

From (24) and (25),we get

$$\begin{aligned} det[T_1(z)] &= \prod_{j=1}^{N} det[I_v - \lambda_j(\overline{L} + \overline{M}\sum_{p=-r}^{s} L_p(\delta)\cdot z^{p-m})A] \\ &\neq 0, \quad \text{for} |z| \ge 1, \end{aligned}$$

where $\lambda_j(\overline{L}+\overline{M}\sum_{p=-r}^{s} L_p(\delta)\cdot z^{p-m})$,$j = 1, 2, \cdots, N$, denote the eigenvalues of the matrix $(\overline{L}+\overline{M}\sum_{p=-r}^{s} L_p(\delta)z^{p-m})$. The above inequality is obtained from A-stability of the IRK methods (see [7]), so the condition of Lemma 2.3 holds for $|z| \ge 1$.

Using Lemma 2.3 to (23),we arrive at

$$\begin{aligned} det[\quad & I_N(z^{m+1}-z^m) - (\mathrm{b}^T\otimes I_N)z^{m+1}(z^{m+1} \quad [I_{vN} - A\otimes(\overline{L}+\overline{M}\sum_{p=-r}^{s} L_p(\delta)z^{p-m})]^{-1}\cdot \\ & \cdot(e\otimes(\overline{L}+\overline{M}\textstyle\sum_{p=-r}^{s} L_p(\delta)z^{p-m})z^m) \quad] = 0, \text{ for } |z| \ge 1, \end{aligned} \tag{26}$$

i.e.,

$$\begin{aligned} det[\quad & z^m((z-1)I_N - (\mathrm{b}^T\otimes I_N) \quad [I_{vN} - A\otimes(\overline{L}+\overline{M}\sum_{p=-r}^{s} L_p(\delta)z^{p-m})]^{-1}\cdot \\ & \cdot(e\otimes(\overline{L}+\overline{M}\textstyle\sum_{p=-r}^{s} L_p(\delta)z^{p-m})) \quad] = 0, \text{ for } |z| \ge 1. \end{aligned} \tag{27}$$

we have (see [3])

$$det[zI_N - r(Q(z,\delta))] = 0, \text{ for } |z| \ge 1, \tag{28}$$

where $r(Q(z,\delta)) = I_N + (\mathrm{b}^T\otimes I_N)(I_{vN} - A\otimes Q(z,\delta))^{-1}(e\otimes Q(z,\delta))$, $Q(z,\delta) = \overline{L} + \overline{M}\sum_{p=-r}^{s} L_p(\delta)\cdot z^{p-m}$.

Now we give the main theorem of this paper.

Theorem 2.2. *If L and M satisfy (13)-(14) and the condition (P),then the IRK methods is GPL-stable if and only if (3)-(4) is L-stable.*

Proof. If (3)-(4) is L-stable.We first show that the IRK methods for (9)-(10) is GP-stable. We only need to prove that all roots of (23) satisfy $|z| < 1$.

Assume some root $\tilde{z}$ of (23) such that $|\tilde{z}| \ge 1$. From (28),we have

$$\det[\tilde{z}I_N - r(Q(\tilde{z},\delta))] = 0, \tag{29}$$

where $Q(\tilde{z},\delta) = \overline{L}+\overline{M}\sum_{p=-r}^{s} L_P(\delta)\tilde{z}^{p-m}$.

By the Spectral Mapping Theorem (see [3] or [9]), we have

$$\lambda_{r(Q(\tilde{z},\delta))} = r(\lambda_{Q(\tilde{z},\delta)}). \tag{30}$$

Then from (29),(30),$Re(\lambda(Q(\tilde{z},\delta))) < 0$ and the A-stability of (3)-(4),we obtain

$$|\tilde{z}| = |r(\lambda_{Q(\tilde{z},\delta)})| < 1.$$

It is impossible by the assumptation $|\tilde{z}| \ge 1$.So it is GP-stable.

Next we want to prove the IRK methods is GPL-stable. We only need to prove any root z of (23) satisfies $\lim_{\eta(L)\to-\infty} z = 0$.

Contradictly,let $\hat{z}$ be a root of (23) such that $\hat{z}$ can not tend to zero as $\eta(L)$ tends to $-\infty$,then we have sequences $\{\hat{z}_n\}$, $\{L_n\}$,$\{M_n\}$,which satisfy (13)–(14) and condition (P) for every n,then there exists a positive number σ,$0<\sigma<1$,such that

$$det\begin{bmatrix} T_1(\hat{z}_n) & T_2(\hat{z}_n) \\ T_3(\hat{z}_n) & T_4(\hat{z}_n) \end{bmatrix} = 0, |\hat{z}_n| \geq \sigma, 0<\sigma<1, n \geq 0.$$

Let

$$\lambda(\overline{L}_n + \overline{M}_n \sum_{p=-r}^{s} L_p(\delta)\hat{z}_n^{p-m}) = x_n + iy_n, \tag{31}$$

where $\lambda(A)$ denotes the eigenvalue of the matrix A.

Since $det[(x_n+iy_n)I - (\overline{L}_n + \overline{M}_n \sum_{p=-r}^{s} L_p(\delta)\hat{z}_n^{p-m})] = 0$,then there exists a vector nonezero β_n,$<\beta_n,\beta_n>=1$,such that

$$[(x_n+iy_n)I - (\overline{L}_n + \overline{M}_n \sum_{p=-r}^{s} L_p(\delta)\hat{z}_n^{p-m})]\beta_n = 0.$$

Then

$$((x_n+iy_n) - <\overline{L}_n\beta_n,\beta_n> - <\overline{M}_n \sum_{p=-r}^{s} L_p(\delta)\hat{z}_n^{p-m}\beta_n,\beta_n> = 0, \tag{32}$$

where $\overline{L}_n = \overline{H}_{1n} + i\overline{H}_{2n}$,$\overline{H}_{1n} = \frac{1}{2}(\overline{L}_n + \overline{L}_n^*)$, $\overline{H}_{2n} = \frac{1}{2i}(\overline{L}_n - \overline{L}_n^*)$.

Let $<\overline{M}_n \sum_{p=-r}^{s} L_p(\delta)\hat{z}_n^{p-m}\beta_n,\beta_n> = r_n e^{i\psi_n}$,then (32) implies

$$x_n - <\overline{H}_{1n}\beta_n,\beta_n> - r_n\cos\psi_n = 0. \tag{33}$$

Since the sum $\sum_{p=-r}^{s} L_p(\delta), (0 \leq \delta < 1)$ is independent on n, and $|\hat{z}_n| \geq \sigma, (0 \leq \sigma < 1)$, then there exists a positive number $\alpha > 0$ such that

$$|\sum_{p=-r}^{s} L_p(\delta)\hat{z}_n^{p-m}| \leq \alpha.$$

Then $|r_n| \leq \alpha||\overline{M}_n||$ by Schwartz' inequality,and we apply $\frac{1}{2}\lambda_{\min}(\overline{L}_n+\overline{L}_n^*) \leq <\overline{H}_{1n}\beta_n, \beta_n> \leq \frac{1}{2}\lambda_{\max}(\overline{L}_n + \overline{L}_n^*) < 0$ to get $|<\overline{H}_{1n}\beta_n,\beta_n>| \geq -\lambda_{\max}(\overline{L}_n + \overline{L}_n^*) = -\eta(\overline{L}_n)$, and (33) can be

$$\begin{aligned} \left|\frac{x_n}{<\overline{H}_{1n}\beta_n,\beta_n>} - 1\right| &= \frac{|r_n\cos\psi_n|}{|<\overline{H}_{1n}\beta_n,\beta_n>|} \leq \frac{|r_n|}{|<\overline{H}_{1n}\beta_n,\beta_n>|} \\ &\leq \frac{\alpha||\overline{M}_n||}{|<\overline{H}_{1n}\beta_n,\beta_n>|} \leq \frac{\alpha||\overline{M}_n||}{-\eta(\overline{L}_n)}, \end{aligned}$$

Then

$$\lim_{\eta(L_n)\to-\infty} \left|\frac{x_n}{<\overline{H}_{1n}\beta_n,\beta_n>} - 1\right| = 0.$$

Because $<\overline{H}_{1n}\beta_n,\beta_n>$ tends to $-\infty$ as n tends to ∞, then x_n must tend to $-\infty$. It means there exists some positive integer N_0,such that $x_n < 0$ for $n \geq N_0$,from (31),we have $Re\lambda[\overline{L}_n+\overline{M}_n\sum_{p=-r}^{s}L_p(\delta)\hat{z}_n^{p-m}]<0$, for $n\geq N_0$,then

$$\begin{aligned} det[T_1(\hat{z}_n)] &= det[I_{v\times N}-A\otimes(\overline{L}_n+\overline{M}_n\sum_{p=-r}^{s}L_p(\delta)\hat{z}_n^{p-m}] \\ &= \prod_{i=1}^{N}det[I_v-A\cdot\lambda_i(\overline{L}_n+\overline{M}_n\sum_{p=-r}^{s}L_p(\delta)\hat{z}_n^{p-m})] \\ &\neq 0, \text{ for } n\geq N_0, \end{aligned}$$

since the IRK methods is A-stable and $Re\lambda_i(\overline{L}_n+\overline{M}_n\sum_{p=-r}^{s}L_p(\delta)\hat{z}_n^{p-m})<0, \quad i=1,2,\ldots,N$. From Lemma 2.3,

$$det[I\hat{z}_n-r(Q(\hat{z}_n,\delta))]=0,$$

where $Q(\hat{z}_n,\delta)=\overline{L}_n+\overline{M}_n\sum_{p=-r}^{s}L_p(\delta)\cdot\hat{z}_n^{p-m}$, by the Spectral Mapping Theorem,we arrive at

$$|\hat{z}_n|=|\lambda_{r(Q(\hat{z}_n,\delta))}|=|r(\lambda_{Q(\hat{z}_n,\delta)})|,$$

then

$$\lim_{Re(\lambda_{Q(\hat{z}_n,\delta)})\to-\infty}|r(\lambda_{Q(\hat{z}_n,\delta)})|=0, \text{ by L-stability of the IRK methods,}$$

and moreover we have

$$\lim_{\eta(L_n)\to-\infty}|\lambda_{Q(\hat{z}_n,\delta)}|=-\infty,$$

so $\lim_{\eta(L_n)\to-\infty}\hat{z}_n=0$,this is a contradict with the assumptation $|\tilde{z}_n|\geq\sigma>0$. And we complete the proof of Theorem 2.2.

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