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THE OVERLAPPING DOMAIN DECOMPOSITION METHOD FOR HARMONIC EQUATION OVER EXTERIOR THREE-DIMENSIONAL DOMAIN*[1)]

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Abstract

In this paper, the overlapping domain decomposition method, which is based on the natural boundary reduction[1] and first suggested in [2], is applied to slove the exterior boundary value problem of harmonic equation over three-dimensional domain. The convergence and error estimates both for the continuous case and the discrete case are given. The contraction factor for the exterior spherical domain is also discussed. Moreover, numerical results are given which show that the accuracy and the convergence are in accord with the theoretical analyses.

Key words: DDM, Exterior harmonic problem, Natural boundary reduction.

1. Introduction

Many scientific and engineering problems can be reduced to exterior boundary value problems of partial differential equations. Although the numerical methods to solve boundary value problems, such as the finite element method and the finite difference method, are very effective on bounded domains, yet we often find it difficult to use them to deal with unbounded problems. The boundary reduction is a forceful means to solve some problems over unbounded domains. Among many boundary reductions, the natural boundary reduction founded by K. Feng and D.H. Yu has some distinctive advantages[1]. However, it also has its own limitation. How to Combine the natural boundary element method with the traditional finite element method effectively to solve problems over unbounded domains is an interesting and worthy work. Up to now, there have been some advances in this field in the two-dimensional cases[1−4]. Our goal here will be to extend this work to three-dimensional field.

Consider the following Dirichlet exterior boundary value problem

$$\begin{cases} -\triangle u = 0, & \text{in } \Omega^c, \\ u = g_0, & \text{on } \Sigma_0, \end{cases} \tag{1}$$

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where Ω is a bounded domain in R^3 with regular boundary $\partial\Omega = \Sigma_0$ and Ω^c denotes $R^3\backslash\overline{\Omega}$. In order to assure the existence and uniqueness of the solution of (1), the assumption that u vanishes at infinity is needed. The corresponding variational form of (1) is

$$\begin{cases} \text{Find } w = u - \widetilde{u} \in \overset{\circ}{W}{}_0^1(\Omega^c), \text{ such that} \\ D_{\Omega^c}(w,\ v) = -D_{\Omega^c}(\widetilde{u},\ v),\ \forall\ v \in \overset{\circ}{W}{}_0^1(\Omega^c), \end{cases} \tag{2}$$

where

$$D_{\Omega^c}(u,\ v) = \int_{\Omega^c} \nabla u \bullet \nabla v dxdydz,$$

$$\overset{\circ}{W}{}_0^1(\Omega^c) = \{v \in W_0^1(\Omega^c) \ \mid \ v|_{\Sigma_0} = 0\},$$

$$W_0^1(\Omega^c) = \{v \mid \frac{v}{\sqrt{1+x^2+y^2+z^2}},\ \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial v}{\partial z} \in L^2(\Omega^c)\},$$

$\widetilde{u} \in W_0^1(\Omega^c)$ has bounded support and $\widetilde{u}\ |_{\Sigma_0} = g_0$. $|u|_1 = \sqrt{D_{\Omega^c}(u,\ u)}$ is an equivalent norm of $\overset{\circ}{W}{}_0^1(\Omega^c)$. If $g_0 \in H^{\frac{1}{2}}(\Sigma_0)$, then there exists $\widetilde{u}$ such that the solution of (2) exists and is uniquely determined[5]. Particularly, if Ω is a spherical domain with radius R whose centre is the origin of coordinates and g_0 is continuous on Σ_0, the solution of (2) is given by the following Poisson integral formula

$$u(r,\ \theta,\ \varphi) = \frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{(r^2-R^2)\sin\theta' g_0(\theta',\ \varphi')}{(R^2+r^2-2Rr\cos\gamma)^{3/2}} d\theta' d\varphi',\ r > R, \tag{3}$$

where $(r,\ \theta,\ \varphi)$ denote the spherical coordinates and

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi').$$

By using (3), we will develop an overlapping domain decomposition algorithm for general unbounded domain Ω^c.

2. Schwarz Alternating Algorithm and Its Convergence

Make a sphere $\Sigma_2 = \{(r,\ \theta,\ \varphi)|r = R\}$ for an appropriate R such that Σ_0 is surrounded by Σ_2. Then make a regular closed surface Σ_1 which surrounds Σ_2. Let Ω_1 denote the annular domain between Σ_0 and Σ_1. Define Ω_2 as the exterior unbounded domain with boundary Σ_2. Then the original problem is turned into two subproblems over subdomains Ω_1 and Ω_2. Construct the following Schwarz alternating algorithm

$$\begin{cases} -\Delta u_1^{(2k)} = 0, & \text{in } \Omega_1, \\ u_1^{(2k)} = g_0, & \text{on } \Sigma_0, \\ u_1^{(2k)} = u_2^{(2k-1)}, & \text{on } \Sigma_1, \end{cases} \tag{4}$$

and

$$\begin{cases} -\Delta u_2^{(2k+1)} = 0, & \text{in } \Omega_2, \\ u_2^{(2k+1)} = u_1^{(2k)}, & \text{on } \Sigma_2, \end{cases} \tag{5}$$

where $k = 0,\ 1,\ 2,\ \cdots$ (We will employ this assumption throughout this paper without any further mention) and $u_2^{(-1)} = \tilde{u}$.

For simplicity, set

$$V = \overset{\circ}{W}{}_0^1\,(\Omega^c),\quad V_1 = H_0^1(\Omega_1),\quad V_2 = \overset{\circ}{W}{}_0^1\,(\Omega_2).$$

V_1 and V_2 can be regarded as the subspaces of V if their elements are extended by zero. (4) and (5) are respectively equivalent to the following variational forms:

$$\begin{cases} \text{Find } w_1^{(2k)} = u_1^{(2k)} - u^{(2k-1)}|_{\Omega_1} \in V_1, \text{ such that} \\ D_{\Omega_1}(w_1^{(2k)},\ v) = -D_{\Omega_1}(u^{(2k-1)},\ v),\ \forall v \in V_1, \end{cases} \tag{6}$$

and

$$\begin{cases} \text{Find } w_2^{(2k+1)} = u_2^{(2k+1)} - u^{(2k)}|_{\Omega_2} \in V_2, \text{ such that} \\ D_{\Omega_2}(w_2^{(2k+1)},\ v) = -D_{\Omega_2}(u^{(2k)},\ v),\ \forall v \in V_2, \end{cases} \tag{7}$$

where

$$u^{(2k)} = \begin{cases} u_1^{(2k)}, & \text{in } \Omega_1, \\ u_2^{(2k-1)}, & \text{on } \Omega^c \backslash \Omega_1, \end{cases} \tag{8}$$

$$u^{(2k+1)} = \begin{cases} u_1^{(2k)}, & \text{on } \Omega^c \backslash \Omega_2, \\ u_2^{(2k+1)}, & \text{in } \Omega_2, \end{cases} \tag{9}$$

$$u^{(-1)} = \tilde{u}\ . \tag{10}$$

From (2), (6) and (7), we have

$$\begin{cases} D_{\Omega^c}(u - u^{(2k)},\ v) = 0, & \forall\ v \in V_1, \\ D_{\Omega^c}(u - u^{(2k+1)},\ v) = 0, & \forall\ v \in V_2. \end{cases} \tag{11}$$

Noticing

$$u^{(2k)} - u^{(2k-1)} \in V_1,\quad u^{(2k+1)} - u^{(2k)} \in V_2$$

and

$$u - u^{(2k-1)} \in V,\quad u - u^{(2k)} \in V,$$

we obtain

$$\begin{cases} u^{(2k)} - u^{(2k-1)} = P_{V_1}(u - u^{(2k-1)}), \\ u^{(2k+1)} - u^{(2k)} = P_{V_2}(u - u^{(2k)}). \end{cases}$$

This is equivalent to

$$\begin{cases} u - u^{(2k)} = P_{V_1^\perp}(u - u^{(2k-1)}), \\ u - u^{(2k+1)} = P_{V_2^\perp}(u - u^{(2k)}). \end{cases} \tag{12}$$

Here $P_W : V \longrightarrow W$ denotes the projection operator under the inner product $D_{\Omega^c}(\bullet,\ \bullet)$. Let

$$e^{(k-1)} = u - u^{(k-1)}$$

denote the errors. Then, (12) can be rewritten as

$$\begin{cases} e^{(2k)} = P_{V_1^\perp} e^{(2k-1)}, \\ e^{(2k+1)} = P_{V_2^\perp} e^{(2k)}. \end{cases} \tag{13}$$

This yields to

$$\begin{cases} e^{(2k+2)} = P_{V_1^\perp} P_{V_2^\perp} e^{(2k)}, \\ e^{(2k+1)} = P_{V_2^\perp} P_{V_1^\perp} e^{(2k-1)}. \end{cases} \tag{14}$$

Lemma 1. *$V = V_1 + V_2$ and for any $v \in V$ there exists a positive constant C_0 such that*

$$(|v_1|_1^2 + |v_2|_1^2)^{1/2} \leq C_0 |v|_1, \quad v = v_1 + v_2, \quad v_1 \in V_1, \quad v_2 \in V_2, \tag{15}$$

$$|v|_1 \leq C_0 (|P_{V_1} v|_1^2 + |P_{V_2} v|_1^2)^{1/2}. \tag{16}$$

Proof. The proof of $V = V_1 + V_2$ is similar to that of Lemma 1 in [2] while the proofs of (15) and (16) are given in [6].

Theorem 1. *There exists a constant $\alpha \in [0,\ 1)$, such that*

$$|e^{(2k+1)}|_1 \leq \alpha^{2k} |e^{(1)}|_1 \quad \text{and } |e^{(2k+2)}|_1 \leq \alpha^{2k+2} |e^{(0)}|_1 \tag{17}$$

hold true.

Proof. Substituting $P_{V_1^\perp} v$ for v in (16), we have

$$|P_{V_1^\perp} v|_1^2 \leq C_0^2 |P_{V_2} P_{V_1^\perp} v|_1^2.$$

Here we can choose $C_0 \geq 1$ without any harm to our argument. Thus

$$|P_{V_1^\perp} v|_1^2 = |P_{V_2^\perp} P_{V_1^\perp} v|_1^2 + |P_{V_2} P_{V_1^\perp} v|_1^2 \geq |P_{V_2^\perp} P_{V_1^\perp} v|_1^2 + \frac{1}{C_0^2} |P_{V_1^\perp} v|_1^2,$$

It follows that

$$|P_{V_2^\perp} P_{V_1^\perp} v|_1 \leq \alpha |P_{V_1^\perp} v|_1, \quad \forall v \in V,$$

where

$$\alpha = \sqrt{1 - \frac{1}{C_0^2}} \in [0,\ 1).$$

Similarly, we have

$$|P_{V_1^\perp} P_{V_2^\perp} v|_1 \leq \alpha |P_{V_2^\perp} v|_1, \quad \forall v \in V.$$

From (13) and (14), it comes that

$$\begin{aligned} |e^{(2k+1)}|_1 &\leq \alpha |P_{V_1^\perp} e^{(2k-1)}|_1 = \alpha |P_{V_1^\perp} P_{V_2^\perp} e^{(2k-2)}|_1 \\ &\leq \alpha^2 |P_{V_2^\perp} e^{(2k-2)}|_1 = \alpha^2 |e^{(2k-1)}|_1. \end{aligned}$$

By recursion we obtain the first part of (17). Similarly, we can obtain the second part of (17).

Theorem 1 shows the above Schwarz alternating algorithm is geometrically convergent and the contraction factor is α.

3. Analyses of the Convergence Rate

The convergence rate of the above Schwarz alternating algorithm is closely related to the overlapping extent of Ω_1 and Ω_2. Although it can be deduced intuitively that the larger the overlapping part is, the faster the convergence rate will be, yet we find it difficult to analyse the convergence rate for general unbounded domain Ω^c. However, under certain assumptions, we can find out the relationship between contraction factor α and overlapping extent of Ω_1 and Ω_2. Set

$$\Sigma_i = \{(r,\ \theta,\ \varphi)|\ r = R_i\},\ i = 0,\ 1,\ 2,\ R_0 < R_2 < R_1. \tag{18}$$

Consider the following boundary value problem over domain Ω_1

$$\begin{cases} -\Delta u &= 0, \quad \text{in } \Omega_1, \\ u &= g_0, \quad \text{on } \Sigma_0, \\ u &= g_1, \quad \text{on } \Sigma_1. \end{cases} \tag{19}$$

If $g_i (i = 0,\ 1)$ are continuously bidifferentiable on Σ_i, then they can be expanded into the following absolutely and uniformly convergent series in terms of spherical harmonics[7]

$$g_i(\theta,\ \varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(A_{nm}^{(i)}\cos m\varphi + B_{nm}^{(i)}\sin m\varphi)P_n^m(\cos\theta),\ i = 0,\ 1, \tag{20}$$

with

$$A_{nm}^{(i)} = \frac{1}{N_{nm}}\int_0^{2\pi}\int_0^{\pi} g_i(\theta,\ \varphi)P_n^m(\cos\theta)\cos m\varphi\sin\theta d\theta d\varphi,\ i = 0,\ 1,$$

$$B_{nm}^{(i)} = \frac{1}{N_{nm}}\int_0^{2\pi}\int_0^{\pi} g_i(\theta,\ \varphi)P_n^m(\cos\theta)\sin m\varphi\sin\theta d\theta d\varphi,\ i = 0,\ 1,$$

$$N_{nm} = \frac{2\pi\varepsilon_m(n+m)!}{(2n+1)(n-m)!}, \qquad \varepsilon_m = \begin{cases} 2, & \text{if } m = 0, \\ 1, & \text{if } m > 0. \end{cases}$$

$P_n^m(x)$ denotes the adjoint function of Legendre polynomial $P_n(x)$, namely

$$\begin{aligned} P_n^m(x) &= (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x) \\ &= \frac{1}{2^n n!}(1-x^2)^{\frac{m}{2}}\frac{d^{n+m}}{dx^{n+m}}[(x^2-1)^n]. \end{aligned}$$

By separation of variables, we obtain the solution of (19) as follows

$$\begin{aligned} u(r,\ \theta,\ \varphi) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}[\frac{A_{nm}^{(0)}\cos m\varphi + B_{nm}^{(0)}\sin m\varphi}{R_1^{2n+1} - R_0^{2n+1}}R_0^{n+1}(\frac{R_1^{2n+1}}{r^{n+1}} - r^n) \\ &+ \frac{A_{nm}^{(1)}\cos m\varphi + B_{nm}^{(1)}\sin m\varphi}{R_1^{2n+1} - R_0^{2n+1}}R_1^{n+1}(r^n - \frac{R_0^{2n+1}}{r^{n+1}})]P_n^m(\cos\theta), \end{aligned} \tag{21}$$

where $R_0 \le r \le R_1$.

Theorem 2. *Suppose that g_0 is continuously bidifferentiable on Σ_0 and (18) holds. If we apply the Schwarz alternating algorithm given in Section 2 to problem (1), then*

$$\sup_{\overline{\Omega}_1} |u - u^{(2k)}| \le C_1 \alpha^k, \tag{22}$$

and

$$\sup_{\overline{\Omega}_2} |u - u^{(2k+1)}| \le C_2 \alpha^{k+1} \tag{23}$$

hold ture, here constant $C_i (i = 1,\ 2)$ depend only on g_0 and $\frac{R_0}{R_i}$ while

$$\alpha = \frac{R_2 - R_0}{R_1 - R_0}. \tag{24}$$

Proof. From the assumptions we know that (20) holds ture for i=0, so the solution of (1) can be expressed as

$$u(r,\ \theta,\ \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (A_{nm}^{(0)} \mathrm{cos} m\varphi + B_{nm}^{(0)} \mathrm{sin} m\varphi) \frac{R_0^{n+1}}{r^{n+1}} P_n^m(\mathrm{cos}\theta),\ r \ge R_0.$$

Choose

$$\widetilde{u}\,(r,\ \theta,\ \varphi) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{A_{nm}^{(0)} \mathrm{cos} m\varphi + B_{nm}^{(0)} \mathrm{sin} m\varphi}{R_1^{2n+1} - R_0^{2n+1}} R_0^{n+1} (\frac{R_1^{2n+1}}{r^{n+1}} - r^n) P_n^m(\mathrm{cos}\theta),\ R_0 \le r \le R_1, \\ 0, \quad r > R_1. \end{cases}$$

Using (21) and mathematical induction we obtain

$$u(r,\ \theta,\ \varphi) - u^{(2k)}(r,\ \theta,\ \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{A_{nm}^{(0)} \mathrm{cos} m\varphi + B_{nm}^{(0)} \mathrm{sin} m\varphi}{R_1^{2n+1} - R_0^{2n+1}} R_0^{n+1} (\frac{R_2^{2n+1} - R_0^{2n+1}}{R_1^{2n+1} - R_0^{2n+1}})^k (r^n - \frac{R_0^{2n+1}}{r^{n+1}}) P_n^m(\mathrm{cos}\theta),$$
$$R_0 \le r \le R_1.$$

Noticing that

$$0 < \frac{R_2^{2n+1} - R_0^{2n+1}}{R_1^{2n+1} - R_0^{2n+1}} \le \frac{R_2 - R_0}{R_1 - R_0}, \quad n = 0,\ 1,\ 2,\ \cdots$$

and the series in (20) is uniformly convergent, we have (22). Similarly we can obtain (23).

It can be seen from Theorem 2 that, the larger the overlapping part of Ω_1 and Ω_2 is, the smaller the contraction factor α will be, consequently, the faster the Schwarz alternating algorithm converges.

4. The Convergence and the Error Estimates of the Discrete Schwarz Alternating Algorithm

Subdivide Ω_1 into hexahedrons(or tetrahedrons). Let $S_h(\Omega_1)$ denote the linear finite element space over Ω_1. Putting

$$\overset{\circ}{S}_h\ (\Omega_1) = \{v_h \in S_h(\Omega_1) \mid \ v_h|_{\Sigma_0\cup\Sigma_1} = 0\}.$$

$\overset{\circ}{S}_h\ (\Omega_1)$ can be regarded as the subspace of V if its elements are extended by zero. We first establish the following discrete Schwarz alternating algorithm:

$$\begin{cases} \text{Find } \ w_{1h}^{(2k)} = u_{1h}^{(2k)} - u_h^{(2k-1)}|_{\Omega_1} \in \overset{\circ}{S}_h\ (\Omega_1), \text{ such that} \\ D_{\Omega_1}(w_{1h}^{(2k)},\ v_h) = -D_{\Omega_1}(u_h^{(2k-1)},\ v_h), \quad \forall\, v_h \in \overset{\circ}{S}_h\ (\Omega_1), \end{cases} \tag{25}$$

and

$$\begin{cases} \text{Find } \ w_{2h}^{(2k+1)} = u_{2h}^{(2k+1)} - u_h^{(2k)}|_{\Omega_2} \in V_2, \text{ such that} \\ D_{\Omega_2}(w_{2h}^{(2k+1)},\ v) = -D_{\Omega_2}(u_h^{(2k)},\ v), \quad \forall\, v \in V_2, \end{cases} \tag{26}$$

with

$$u_h^{(2k)} = \begin{cases} u_{1h}^{(2k)}, & \text{in } \Omega_1, \\ u_h^{(2k-1)}, & \text{on } \Omega^c\backslash\Omega_1, \end{cases}$$

$$u_h^{(2k+1)} = \begin{cases} u_h^{(2k)}, & \text{on } \Omega^c\backslash\Omega_2, \\ u_{2h}^{(2k+1)}, & \text{in } \Omega_2, \end{cases}$$

$$u_h^{(-1)} = \tilde{u}\ .$$

The solution of (26) can be given by Poisson integral formula, namely

$$u_{2h}^{(2k+1)} = P\ \gamma u_h^{(2k)}, \tag{27}$$

where $P:\ H^{\frac{1}{2}}(\Sigma_2) \longrightarrow W_0^1(\Omega_2)$ denotes Poisson integral operator and $\gamma:\ H^1(\Omega_1) \longrightarrow$

$H^{\frac{1}{2}}(\Sigma_2)$ denotes Dirichlet trace opertor. It is easy to verify that

$$u_h^{(2k)} = \tilde{u} + \begin{cases} \sum_{i=0}^{k} w_{1h}^{(2i)}, & \text{on } \overline{\Omega^c}\backslash\Omega_2, \\ \sum_{i=0}^{k} w_{1h}^{(2i)} + \sum_{j=0}^{k-1}[P\ \gamma w_{1h}^{(2j)} - w_{1h}^{(2j)}] + \delta_k(P\ \gamma\ \tilde{u} - \tilde{u}), & \text{in } \Omega_1\backslash(\overline{\Omega^c}\backslash\Omega_2), \\ \sum_{j=0}^{k-1} P\ \gamma w_{1h}^{(2j)} + \delta_k(P\ \gamma\ \tilde{u} - \tilde{u}), & \text{on } \Omega^c\backslash\Omega_1, \end{cases}$$

$$u_h^{(2k+1)} = \tilde{u} + \begin{cases} \sum_{i=0}^{k} w_{1h}^{(2i)}, & \text{on } \overline{\Omega^c}\backslash\Omega_2, \\ \sum_{i=0}^{k} w_{1h}^{(2i)} + \sum_{j=0}^{k}[P\ \gamma w_{1h}^{(2j)} - w_{1h}^{(2j)}] + (P\ \gamma\ \tilde{u} - \tilde{u}), & \text{in } \Omega_1\backslash(\overline{\Omega^c}\backslash\Omega_2), \\ \sum_{j=0}^{k} P\ \gamma w_{1h}^{(2j)} + (P\ \gamma\ \tilde{u} - \tilde{u}), & \text{on } \Omega^c\backslash\Omega_1, \end{cases}$$

$$\delta_k = \begin{cases} 0, & \text{if } k = 0, \\ 1, & \text{if } k > 0. \end{cases}$$

For $k = 0$, the terms corresponding to $\sum_{j=0}^{k-1}$ vanish. Define

$$A_h(\Omega_2) = \{P\ \gamma(v_h + \alpha\ \tilde{u} + \beta w) - (v_h + \alpha\ \tilde{u} + \beta w)|_{\overline{\Omega}_2} \mid v_h \in \overset{\circ}{S}_h\ (\Omega_1), \alpha, \beta \in R, w = u - \tilde{u}\}.$$

Then, extending the elements of $A_h(\Omega_2)$ by zero, we have

$$A_h(\Omega_2) \subset V_2 \subset V.$$

In order to acquire the convergence of the discrete Schwarz alternating algorithm, we first claim

Lemma 2. *For the constant C_0 given in Lemma 1 and any $v_h \in V_h = \overset{\circ}{S}_h\ (\Omega_1) + A_h(\Omega_2) \subset V$, there holds that*

$$|v_h|_1 \leq C_0(|P_{\overset{\circ}{S}_h(\Omega_1)}\ v_h|_1^2 + |P_{A_h(\Omega_2)}\ v_h|_1^2)^{1/2}. \tag{28}$$

Proof. Suppose $v_h = v_{1h} + v_{2h}$ with $v_{1h} \in \overset{\circ}{S}_h\ (\Omega_1) \subset V_1$ and $v_{2h} \in A_h(\Omega_2) \subset V_2$. It comes from (15) that

$$(|v_{1h}|_1^2 + |v_{2h}|_1^2)^{1/2} \leq C_0|v_h|_1.$$

Therefore

$$\begin{aligned} |v_h|_1^2 &= D_{\Omega^c}(v_h,\ v_{1h}) + D_{\Omega^c}(v_h,\ v_{2h}) \\ &= D_{\Omega^c}(P_{\overset{\circ}{S}_h(\Omega_1)}\ v_h,\ v_{1h}) + D_{\Omega^c}(P_{A_h(\Omega_2)}\ v_h,\ v_{2h}) \\ &\leq (|P_{\overset{\circ}{S}_h(\Omega_1)}\ v_h|_1^2 + |P_{A_h(\Omega_2)}\ v_h|_1^2)^{1/2}(|v_{1h}|_1^2 + |v_{2h}|_1^2)^{1/2} \\ &\leq C_0|v_h|_1(|P_{\overset{\circ}{S}_h(\Omega_1)}\ v_h|_1^2 + |P_{A_h(\Omega_2)}\ v_h|_1^2)^{1/2}. \end{aligned}$$

From this, (28) follows, completing the proof.

Now we introduce the following variational problem

$$\begin{cases} \text{Find } v_h^* \in \overset{\circ}{S}_h(\Omega_1) + A_h(\Omega_2), \text{ such that} \\ D_{\Omega^c}(v_h^*,\ v_h) = -D_{\Omega^c}(\widetilde{u},\ v_h),\ \forall\, v_h \in \overset{\circ}{S}_h(\Omega_1) + A_h(\Omega_2). \end{cases} \tag{29}$$

Obviously, there exists a unique solution for (29). Set

$$u_h^* = v_h^* + \widetilde{u}\ .$$

We have

Theorem 3. *For the series $\{u_h^{(k)}\}$ generated by the discrete Schwarz alternating algorithm and the constant α given in Theorem 1, there hold*

$$|u_h^* - u_h^{(2k+1)}|_1 \le \alpha^{2k} |u_h^* - u_h^{(1)}|_1, \tag{30}$$

$$|u_h^* - u_h^{(2k+2)}|_1 \le \alpha^{2k+2} |u_h^* - u_h^{(0)}|_1. \tag{31}$$

Proof. From (25), (26) and (29), we have

$$\begin{cases} D_{\Omega^c}(u_h^* - u_h^{(2k)},\ v_h) = 0,\ \forall\, v_h \in \overset{\circ}{S}(\Omega_1), \\ D_{\Omega^c}(u_h^* - u_h^{(2k+1)},\ w_h) = 0,\ \forall\, w_h \in A_h(\Omega_2). \end{cases} \tag{32}$$

Note that

$$u_h^{(2k)} - u_h^{(2k-1)} \in \overset{\circ}{S}_h(\Omega_1),\quad u_h^{(2k+1)} - u_h^{(2k)} \in A_h(\Omega_2).$$

Therefore, imitating the inference from (11) to (14), we obtain (30) and (31) by making use of Lemma 2 and duplicating the proof of Theorem 1 in $V_h = \overset{\circ}{S}_h(\Omega_1) + A_h(\Omega_2)$.

Theorem 3 implies that, the series $\{u_h^{(k)}\}$ generated by the discrete Schwarz alternating algorithm geometrically converges to u_h^* and the convergence rate is independent of mesh parameter h.

In order to acquire the error estimates for the above algorithm, the following preparatory work is needed.

Lemma 3. *There exists $v_{1h} \in \overset{\circ}{S}_h(\Omega_1)$, such that*

$$v_h^*|_{\overline{\Omega}_2} = P\ \gamma v_{1h} - (P\ \gamma w - w|_{\overline{\Omega}_2}). \tag{33}$$

Proof. Suppose $v_h^*|_{\overline{\Omega}_2} = v_{1h}|_{\overline{\Omega}_2} + v_{2h}$ with $v_{1h} \in \overset{\circ}{S}_h(\Omega_1)$ and $v_{2h} \in A_h(\Omega_2)$. From (2) and (29) it follows

$$D_{\Omega_2}(v_{1h} + v_{2h},\ w_h) = D_{\Omega_2}(w,\ w_h),\quad \forall\, w_h \in A_h(\Omega_2).$$

We note that

$$D_{\Omega_2}(P\ \gamma v_{1h},\ w_h) = D_{\Omega_2}(P\ \gamma w,\ w_h) = 0,\quad \forall\, w_h \in A_h(\Omega_2).$$

Hence

$$D_{\Omega_2}(v_{1h} - P\ \gamma v_{1h} + v_{2h} + P\ \gamma w - w,\ w_h) = 0,\ \ \forall\, w_h \in A_h(\Omega_2).$$

By using

$$v_{1h}|_{\overline{\Omega}_2} - P\ \gamma v_{1h} + v_{2h} + P\ \gamma w - w|_{\overline{\Omega}_2} \in A_h(\Omega_2),$$

(33) follows and the proof is completed.

Lemma 4. *Let $u = w + \widetilde{u}$ be the solution of variational problem (2) and suppose $w = w_1 + w_2$ with $w_1 \in V_1$ and $w_2 \in V_2$. If $w_1 \in H^2(\Omega_1)$, then*

$$|u - u_h^*|_1 \le C\ h \tag{34}$$

holds ture.

Proof. From (33), we have

$$D_{\Omega^c}(v_h^*,\ v) = D_{\Omega^c}(w,\ v),\ \ \forall\, v \in V_2.$$

Then it comes from (2) and (29) that

$$D_{\Omega^c}(w - v_h^*,\ v_h) = 0\ \ \forall\, v_h \in \overset{\circ}{S}_h\ (\Omega_1) + V_2.$$

Let w_1^I denote the interpolant of w_1 in $\overset{\circ}{S}_h\ (\Omega_1)$. We have

$$\begin{aligned} |u - u_h^*|_1^2 &= |w - v_h^*|_1^2 = D_{\Omega^c}(w - v_h^*,\ w - v_h^*) \\ &= D_{\Omega^c}(w - v_h^*,\ w_1 - w_1^I) \le C'|w - v_h^*|_1\ |w_1 - w_1^I|_1 \\ &\le Ch|w - v_h^*|_1. \end{aligned}$$

From this, (34) is obtained.

Theorem 3 and Lemma 4 directly yield to

Theorem 4. *For the discrete Schwarz alternating algorithm and the constant α in Theorem 1, there hold the following error estimates*

$$|u - u_h^{(2k+1)}|_1 \le C\ h + \alpha^{2k}|u_h^* - u_h^{(1)}|_1, \tag{35}$$

$$|u - u_h^{(2k+2)}|_1 \le C\ h + \alpha^{2k+2}|u_h^* - u_h^{(0)}|_1. \tag{36}$$

Remark. Since $u - u_h^{(k)} \in V$, (35) and (36) also hold true if the norm in their left hand sides are replaced by the norm of $W_0^1(\Omega^c)$.

5. Numerical Test

Example. Solve the boundary value problem (1) with the algorithm given in Section 4. Set $g_0 = \dfrac{x+z}{r^3}$. Let Ω be the cubic domain enclosed by planes $x = \pm a,\ y = \pm a,$ and $z = \pm a$ $(a > 0)$. The exact solution is $u = \dfrac{x+z}{r^3}$. Let Σ_1 be the boundary of the cubic domain enclosed by planes $x = \pm ma,\ y = \pm ma,$ and $z = \pm ma$ (integer m is no less than 2). Choose

$$\Sigma_2 = \{(r,\ \theta,\ \varphi)|r = R_2 \text{ and } \sqrt{3}a < R_2 < ma\}.$$

We subdivide the bounded domain Ω_1 using the following family of planes

$$x = \pm\frac{ia}{s},\ y = \pm\frac{ia}{s},\ z = \pm\frac{ia}{s},\quad i = 0,\ 1,\ \cdots,\ ms,$$

where s is a positive integer. Now we obtain a uniform cubic partition of Ω_1 with meshsize $h = \frac{a}{s}$. Let $S_h(\Omega_1)$ be the piecewise trilinear finite element space on Ω_1. Substitute

$$\tilde{u}^I = \sum_{P_i \in \Sigma_0} g_0(P_i) L_i$$

for $\tilde{u}$ (where L_i denotes the interpolation basis function correspoding to node P_i). Set

$$e(n) = \sup_{P_i \in \Omega_1} |u(P_i) - u_h^{(2n)}(P_i)|,$$

$$e_h(n) = \sup_{P_i \in \Omega_1} |u_h^{(2n+2)}(P_i) - u_h^{(2n)}(P_i)|,$$

$$q_h(n) = e_h(n-1)/e_h(n).$$

By computing, the resultes are as follows:

Table 1. $(a,\ R_2,\ m) = (1.0,\ 1.74,\ 2)$

s	n	0	1	2	3	4
2	e(n)	1.7194×10^{-1}	8.1408×10^{-2}	5.8035×10^{-2}	4.8811×10^{-2}	4.5187×10^{-2}
	$e_h(n)$		1.2145×10^{-1}	2.9833×10^{-2}	9.7995×10^{-3}	3.7307×10^{-3}
	$q_h(n)$			4.0710	3.0444	2.6267
4	e(n)	2.0578×10^{-1}	8.1404×10^{-2}	3.6367×10^{-2}	1.9919×10^{-2}	1.3758×10^{-2}
	$e_h(n)$		1.3983×10^{-1}	4.6883×10^{-2}	1.7871×10^{-2}	6.6741×10^{-3}
	$q_h(n)$			2.9824	2.6234	2.6776
8	e(n)	2.4242×10^{-1}	9.6055×10^{-2}	3.7921×10^{-2}	1.5406×10^{-2}	6.9625×10^{-3}
	$e_h(n)$		1.5833×10^{-1}	5.8765×10^{-2}	2.2742×10^{-2}	8.6643×10^{-3}
	$q_h(n)$			2.6943	2.5840	2.6248

Table 2. $(a,\ R_2,\ m) = (1.0,\ 1.74,\ 2)$

s	n	5	6	7	8	9
2	e(n)	4.3765×10^{-2}	4.3209×10^{-2}	4.2990×10^{-2}	4.2905×10^{-2}	4.2872×10^{-2}
	$e_h(n)$	1.4212×10^{-3}	5.5685×10^{-4}	2.1811×10^{-4}	8.5416×10^{-5}	3.3449×10^{-5}
	$q_h(n)$	2.6250	2.5522	2.5531	2.5535	2.5536
4	e(n)	1.1963×10^{-2}	1.1712×10^{-2}	1.1620×10^{-2}	1.1587×10^{-2}	1.1574×10^{-2}
	$e_h(n)$	2.4787×10^{-3}	9.1893×10^{-4}	3.4067×10^{-4}	1.2630×10^{-4}	4.6803×10^{-5}
	$q_h(n)$	2.6926	2.6974	2.6974	2.6973	2.6984
8	e(n)	3.7703×10^{-3}	2.8213×10^{-3}	2.7183×10^{-3}	2.6797×10^{-3}	2.6653×10^{-3}
	$e_h(n)$	3.2654×10^{-3}	1.2256×10^{-3}	4.5929×10^{-4}	1.7203×10^{-4}	6.4426×10^{-5}
	$q_h(n)$	2.6533	2.6644	2.6684	2.6699	2.6702

Table 3. $(a,\ R_2,\ m) = (1.0,\ 1.85,\ 2)$

s	n	0	1	2	3	4
2	e(n)	1.7194×10^{-1}	9.8964×10^{-2}	7.7826×10^{-2}	6.6901×10^{-2}	6.1263×10^{-2}
	$e_h(n)$		9.9900×10^{-2}	3.1721×10^{-2}	1.2250×10^{-2}	5.6377×10^{-3}
	$q_h(n)$			3.1493	2.5896	2.1728
4	e(n)	2.0578×10^{-1}	1.1266×10^{-1}	6.6819×10^{-2}	4.3495×10^{-2}	3.0916×10^{-2}
	$e_h(n)$		1.1963×10^{-1}	4.9227×10^{-2}	2.5192×10^{-2}	1.3291×10^{-2}
	$q_h(n)$			2.4303	1.9541	1.8955
8	e(n)	2.4242×10^{-1}	1.3406×10^{-1}	7.4710×10^{-2}	4.2618×10^{-2}	2.5054×10^{-2}
	$e_h(n)$		1.3179×10^{-1}	6.1868×10^{-2}	3.3781×10^{-2}	1.8384×10^{-2}
	$q_h(n)$			2.1302	1.8315	1.8375

Table 4. $(a,\ R_2,\ m) = (1.0,\ 1.85,\ 2)$

s	n	5	6	7	8	9
2	e(n)	5.8358×10^{-2}	5.6861×10^{-2}	5.6090×10^{-2}	5.5694×10^{-2}	5.5490×10^{-2}
	$e_h(n)$	2.9059×10^{-3}	1.4967×10^{-3}	7.7055×10^{-4}	3.9662×10^{-4}	2.0412×10^{-4}
	$q_h(n)$	1.9401	1.9415	1.9424	1.9428	1.9431
4	e(n)	2.4253×10^{-2}	2.0756×10^{-2}	1.8929×10^{-2}	1.7977×10^{-2}	1.7481×10^{-2}
	$e_h(n)$	7.0506×10^{-3}	3.7063×10^{-3}	1.9396×10^{-3}	1.0127×10^{-3}	5.2812×10^{-4}
	$q_h(n)$	1.8850	1.9023	1.9109	1.9153	1.9175
8	e(n)	1.5602×10^{-2}	1.0573×10^{-2}	7.9126×10^{-3}	6.5081×10^{-3}	5.7677×10^{-3}
	$e_h(n)$	9.8313×10^{-3}	5.2123×10^{-3}	2.7532×10^{-3}	1.4519×10^{-3}	7.6534×10^{-4}
	$q_h(n)$	1.8700	1.8862	1.8932	1.8963	1.8971

As can be seen from the above tables that, the discrete Schwarz alternating algorithm is geometrically convergent; the larger the overlapping part is, the faster the convergence rate will be; the convergence rate is nearly not affected by mesh parameter h. All these are in accord with the theoretical analyses.

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