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D-CONVERGENCE AND STABILITY OF A CLASS OF LINEAR MULTISTEP METHODS FOR NONLINEAR DDES*1)

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Abstract

This paper deals with the error behaviour and the stability analysis of a class of linear multistep methods with the Lagrangian interpolation (LMLMs) as applied to the nonlinear delay differential equations (DDEs). It is shown that a LMLM is generally stable with respect to the problem of class $D_{\sigma,\gamma}$, and a p-order linear multistep method together with a q-order Lagrangian interpolation leads to a D-convergent LMLM of order $\min\{p, q+1\}$.

Key words: D-Convergence, Stability, Multistep methods, Nonlinear DDEs.

1. Introduction

Consider the following nonlinear delay problem

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau)), & t \in [t_0, T], \quad (1.1a) \\ y(t) = \varphi(t), & t \in [t_0-\tau, t_0], \quad (1.1b) \end{cases}$$

where $y : R \to C^N, \tau > 0$ is a delay term, $f : [t_o, T] \times C^N \times C^N \to C^N$ and $\varphi(t) : [t_0-\tau, t_0] \to C^N$ denotes a given initial function. Thoroughout this paper , the problem (1.1) is supposed to have a unique solution y(t), which satisfies

$$\| y^{(i)}(t) \| \leq M_i, \quad t \in [t_0-\tau, T]$$

here norm $\| \bullet \|$ is defined by $\| x \|^2 = < x, x >$ $(\forall x \in C^N)$, and $M_i > 0$ are some constants.

Definition 1.1.[1] *The class of all delay problems of the form (1.1) with*

$$\begin{cases} Re < u-v, f(t,u,\tilde{u}) - f(t,v,\tilde{u}) > \leq \sigma \| u-v \|^2 & (1.2) \\ \| f(t,u,\tilde{u}) - f(t,u,\tilde{v} \| \leq \gamma \| \tilde{u} - \tilde{v} \|, & (1.3) \\ where\ t \in [t_o, T], u, \tilde{u}, v, \tilde{v} \in C^N,\ and\ constants\ \sigma,\ \gamma\ satisfy \\ 0 \leq \gamma \leq -\sigma \end{cases}$$

is denoted by $D_{\sigma,\gamma}$.

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The following proposition on stability of the problem (1.1) can be inferrded directly by a result of L. Torelli [1].

Proposition 1.1. *Suppose the problem* (1.1) *belongs to the class* $D_{\sigma,\gamma}$. *Then for any two solutins* $y(t)$ *and* $z(t)$ *of the equation* (1.1a) *we have*

$$\| y(t) - z(t) \| \leq \max_{x \in [t_0 - \tau, t_0]} \| \varphi(x) - \psi(x) \|,$$

where $\varphi(t)$ *and* $\psi(t)$ *are the two initial functions corresponding to the solutions* $y(t), z(t)$.

Moreover, it is remarkable that H.J.Tian and J.X.Kuang [2] gave a Theorem on asymptotic stability of (1.1) with an adaptation to the conditions (1.2)–(1.3).

So far, a lot of results on nonlinear stability and convergence of the numerical solutions of DDEs have been obtained ($cf.$[1 − 7]). However, these results were achived under the classical Lipschitz condition except those of the paper [1, 6, 7], which deal only with Runge-Kutta methods. In view of what aboves, we study convergence and stability of a class of variable-coefficient LMLMs for the problem of class $D_{\sigma,\gamma}$. and present some significant results in this paper.

2. The Methods and the Basic Lemmas

Consider variable-coefficient LMLMs ($cf.$[8]) for (1.1)

$$\sum_{i=0}^{k} \alpha_i [y_{n+i} - h\beta_i f(t_{n+i}, y_{n+i}, y^h(t_{n+i} - \tau))] = 0, \tag{2.1}$$

where k is a positive integer; $n = 0, 1, 2, \ldots, N$, and $(N + k)h \leq T - t_0, h > 0$ is a stepsize independent of n; the coefficients α_i, β_i are real-valued functions of h and there exists a constant $h_1 > 0$ such that for $h \in (0, h_1]$,

$$\alpha_k = 1, \quad \sum_{i=0}^{k} \alpha_i = 0, \quad \max_{i \in I_0} \alpha_i \leq 0, \quad \max_{i \in I_0} | \beta_i | \leq \beta_k < \beta, \tag{2.2}$$

where $I_0 = \{0, 1, 2, \ldots, k-1\}, \beta > 0$ is a constant; $y_{n+i}, y^h(t_{n+i} - \tau) \in C^N$ are approximations to $y(t_{n+i})$ and $y(t_{n+i} - \tau)$ respectively, and $y^h(\bullet)$ is determined by Lagrangian interpolation

$$y^h(t_m + \delta h) = \begin{cases} \sum_{j=-r}^{s} L_j(\delta) y_{m+j}, t_0 < t_m + \delta h \leq T, \\ \varphi(t_m + \delta h), t_0 - \tau \leq t_m + \delta h \leq t_0, \end{cases} \tag{2.3}$$

where $\delta \in [0, 1), r, s$ are positive integers, $t_m = t_0 + mh$ (m denotes a integer) and

$$L_j(\delta) = \prod_{\substack{l=-r \\ l \neq j}}^{s} (\frac{\delta - l}{j - l})$$

Refer to the paper [8 − 10], we introduce a nonnegative function, using the mapping f, for any $u, \tilde{u}, v, \tilde{v} \in C^N, t \in [t_0 - \tau, T]$ and $\lambda \in R$:

$$G_{u,v,\tilde{u},t,f}(\lambda) = \begin{cases} \| u - v - \lambda[f(t,u,\tilde{u}) - f(t,v,\tilde{u}) \|, & t \in [t_0, T], \\ 0, & t \in [t_0 - \tau, t_0] \end{cases} \tag{2.4}$$

For convenience, the $G_{u,v,\tilde{u},t,f}(\lambda)$ will be noted by $G(\lambda)$, and the following notations will be adopted:

$$\begin{cases} G_n(\lambda) = G_{y_n, y(t_n), y^h(t_n - \tau), t_n, f}(\lambda), \\ \tilde{G}_n(\lambda) = G_{y_n, \tilde{y}_n, y^h(t_n - \tau), t_n, f}(\lambda), \\ \hat{G}_n(\lambda) = G_{\tilde{y}_n, y(t_n), y^h(t_n - \tau), t_n, f}(\lambda), \end{cases} \tag{2.5}$$

where $\{\tilde{y}_n\}$ denotes the solution sequence of the following equation

$$\begin{aligned} &\tilde{y}_{n+k} - h\beta_k f(t_{n+k}, \tilde{y}_{n+k}, y^h(t_{n+k} - \tau)) \\ &= -\sum_{i=0}^{k-1} \alpha_i [y(t_{n+i}) - h\beta_i f(t_{n+i}, y(t_{n+i}), y(t_{n+i} - \tau))]. \end{aligned} \tag{2.6}$$

Lemma 2.1. *Suppose the mapping f satisfies* (1.2). *Then for any $a, b \in R$ with $| b | \le a$, it follows*

$$G(b) \le G(a)$$

Proof. In terms of the definition of function $G(\bullet)$, we need only to prove the case of $t \in [t_0, T]$. When $t \in [t_0, T]$ we have

$$\begin{aligned} \Delta G &:= G^2(a) - G^2(b) \\ &= 2(b-a) Re < u - v, f(t,u,\tilde{u}) - f(t,v,\tilde{u}) > \\ &\qquad + (a^2 - b^2) \| f(t,u,\tilde{u}) - f(t,v,\tilde{u}) \|^2 \\ &\ge 2(b-a)\sigma \| u - v \|^2 + (a^2 - b^2) \| f(t,u,\tilde{u}) - f(t,v,\tilde{u}) \|^2 \ge 0. \end{aligned}$$

Hence $G(b) \le G(a)$

Lemma 2.2. *For q-order ($q = r + s$) interpolation scheme* (2.3), *we have the estimation of global error*

$$\begin{aligned} &\max_{i=0\sim k} \| y^h(t_{n+i} - \tau) - y(t_{n+i} - \tau) \| \\ &\le \sup_{\delta \in [0,1)} \sum_{j=-r}^{s} | L_j(\delta) | \max_{i \in I_1} G_{n+i}(h\beta_k) + \hat{M}_1 h_{q+1}, \end{aligned}$$

where $\tau = (m - \delta)h, m(\ge s+1)$ is a positive integer, $\delta \in [0,1), I_1 = \{i \in Z \mid -(m+r) \le i \le k-1, Z$ denotes the set of integers $\}$ and $\hat{M}_1 = \dfrac{M_{q+1}}{(q+1)!} \sup_{\delta \in [0,1)} \prod_{j=-r}^{s} | \delta - j |$.

Proof. With the error formula of Lagrange interpolation, we have

$$\|y(t_{n+i} - \tau) - \hat{y}(t_{n+i} - \tau)\| \le \frac{M_{q+1}}{(q+1)!} h^{q+1} \prod_{j=-r}^{s} | \delta - j | .$$

where $\hat{y}(t_{n+i}-\tau)=\sum_{j=-r}^{s} L_j(\delta)y(t_{n+i-m+j})$.

Therefore

$$\begin{aligned}&\| y^h(t_{n+i}-\tau)-y(t_{n+i}-\tau)\|\\ \le&\| y^h(t_{n+i}-\tau)-\hat{y}(t_{n+i}-\tau)\| + \|\hat{y}(t_{n+i}-\tau)-y(t_{n+i}-\tau)\|\\ \le&\sum_{j=-r}^{s} | L_j(\delta) | G_{n+i-m+j}(0)+\hat{M}_1 h^{q+1}\\ \le&\sup_{\delta\in[0,1)}\sum_{j=-r}^{s} | L_j(\delta) | \max_{j=-r\sim s} G_{n+i-m+j}(h\beta_k)+\hat{M}_1 h^{q+1}.\end{aligned}$$

Further, we get

$$\max_{i=0\sim k}\| y^h(t_{n+i}-\tau)-y(t_{n+i}-\tau)\|\le\sup_{\delta\in[0,1)}\sum_{j=-r}^{s} | L_j(\delta) | \max_{i\in I_1} G_{n+i}(h\beta_k)+\hat{M}_1 h^{q+1}.$$

3. Analysis of Convergence and stability

In this section, we set to study the convergence and stability of the method (2.1)–(2.2) for the class $D_{\sigma,\gamma}$. At first, we introduce a new convergence concept.

Definition 3.1. *A LMLM* (2.1) – (2.3) *with* $y_i=y(t_i)(i=0\sim k-1)$ *is called D-convergent of order Q for the problem of class* $D_{\sigma,\gamma}$ *if this method produces an approximation sequence* $\{y_n\}$ *and the global error satisfies*

$$\| y(t_n)-y_n\|\le C(t_n)h^Q, h\in(0,h_0], n=0,1,2,\dots,$$

where the maximum stepsize depends only on the methods; the function $C(t)$ *depends only on the methods, delay* τ*, characteristic parameter* σ,γ *and bounds* M_i *of some derivatives* $y^{(i)}(t)$.

Theorem 3.1. *Suppose the method* (2.1) – (2.2) *has the classical cosistency order* p *and the interpolation scheme* (2.3) *is of order* q*. Then, when the method* (2.1) – (2.3) *applied to the problim* (1.1) *of class* $D_{\sigma,\gamma}$*, this method is D-convergent of order* $\min\{p,q+1\}$.

Proof. Since the method (2.1) – (2.3) has the classical consistency order p, there exists a constant $h_2>0$, which depends only on the method, such that

$$\sum_{i=0}^{k}\alpha_i[y_{n+i}-h\beta_i y'(t_{n+i})\|\le\hat{M}_2 h^{p+1},\quad h\in(0,h_2], \tag{3.1}$$

where $\hat{M}_2$ depends only on the method and bounds M_i of some derivatives $y^{(i)}(t)$.

In terms of Lemma 2.1, we know that

$$\| y_{n+k}-y(t_{n+k})\|=G_{n+k}(0)\le G_{n+k}(h\beta_k). \tag{3.2}$$

Whereas

$$G_{n+k}(h\beta_k) \le \tilde{G}_{n+k}(h\beta_k) + \hat{G}_{n+k}(h\beta_k). \tag{3.3}$$

Further, it follows from $(2.5), (2.6), (1.3), (2.2)$ and Lemma 2.1 that

$$\begin{aligned}
\tilde{G}_{n+k}(h\beta_k) = & \| y_{n+k} - \tilde{y}_{n+k} - h\beta_k[f(t_{n+k}, y_{n+k}, y^h(t_{n+k}-\tau)) \\
& -f(t_{n+k}, \tilde{y}_{n+k}, y^h(t_{n+k}-\tau))] \| \\
\le & \sum_{i=0}^{k-1} | \alpha_i | \| y_{n+i} - y(t_{n+i}) - h\beta_i[f(t_{n+i}, y_{n+i}, y^h(t_{n+i}-\tau) \\
& -f(t_{n+i}, y(t_{n+i}), y(t_{n+i}-\tau))] \| \\
\le & \sum_{i=0}^{k-1} | \alpha_i | G_{n+i}(h\beta_i) + h\sum_{i=0}^{k-1} | \alpha_i\beta_i | \| f(t_{n+i}, y(t_{n+i}-\tau), y^h(t_{n+i}-\tau)) \\
& -f(t_{n+i}, y(t_{n+i}), y(t_{n+i}-\tau)) \| \\
\le & \sum_{i=0}^{k-1} | \alpha_i | G_{n+i}(h\beta_k) + h\beta\gamma\sum_{i=0}^{k-1} | \alpha_i | \| y^h(t_{n+i}-\tau) - y(t_{n+i}-\tau) \| \\
\le & \max_{i\in I_0} G_{n+i}(h\beta_k) + h\beta\gamma \max_{i\in I_0} \| y^h(t_{n+i}-\tau) - y(t_{n+i}-\tau) \|, h \in (0, h_1].
\end{aligned} \tag{3.4}$$

On the other hand, putting $h_0 = min\{h_1, h_2\}$, by (2.5), (2.6), (3.1), and (2.2) we can infer that

$$\begin{aligned}
\hat{G}_{n+k}(h\beta_k) = & \| \tilde{y}_{n+k} - y(t_{n+k}) - h\beta_k[f(t_{n+k}, \tilde{y}_{n+k}, y^h(t_{n+k}-\tau)) \\
& -f(t_{n+k}, y(t_{n+k}), y^h(t_{n+k}-\tau))] \| \\
\le & \| \sum_{i=0}^{k} \alpha_i[y(t_{n+i}) - h\beta_i y'(t_{n+i})] \| + \\
& \| h\beta_k[f(t_{n+k}, y(t_{n+k}), y^h(t_{n+k}-\tau)) - f(t_{n+k}), y(t_{n+k})-\tau))] \\
\le & \hat{M}_2 h^{p+1} + h\beta\gamma \| y^h(t_{n+k}-\tau) - y(t_{n+k}-\tau) \|, \ h \in (0, h_0].
\end{aligned} \tag{3.5}$$

A combination of $(3.1), (3.3), (3.4)$ and (3.5) yields

$$\begin{aligned}
G_{n+k}(h\beta_k) \le & h\beta\gamma \max\nolimits_{i=0\sim k} \| y^h(t_{n+i}-\tau) - y(t_{n+i}-\tau) \| \\
& + \max_{i\in I_0} G_{n+i}(h\beta_k) + \hat{M}_2 h^{p+1}, \quad h \in (0, h_0]
\end{aligned} \tag{3.6}$$

Furthermore, with Lemma 2.2 we can conclude that

$$\| G_{n+k}(h\beta_k) \| \le (1 + Mh) \max_{i\in I_1} G_{n+i}(h\beta_k) + \Gamma h^{\min\{p,q+1\}+1}, h \in (0, h_0] \tag{3.7}$$

where

$$M = \beta\gamma \sup_{\delta\in[0,1]} \sum_{j=-r}^{s} | L_j(\delta) |,$$

$$\Gamma = \begin{cases} \beta\gamma\hat{M}_1 h_0^{q+1-p} + \hat{M}_2, & p \leq q+1, \\ \beta\gamma\hat{M}_1 + \hat{M}_2 h_0^{p-q-1.} & p \geq q+1. \end{cases}$$

In view of $1 + Mh > 1$, using the second induction to (3.7), we obtain

$$\| G_{n+k}(h\beta_k) \| \leq (1+Mh)^{n+1}[\max_{i\in I_0} G_i(h\beta_k) + (n+1)\Gamma h^{\min\{p,q+1\}+1}], \quad h \in (0, h_0] \tag{3.8}$$

From (2.4) and Definition 3.1 it yields that $G_i(h\beta_k) = 0$ whenever $i \in I_1$. Hence, combining(3.2) with (3.8) leads to

$$\| y_{n+k} - y(t_{n+k}) \| \leq (1+Mh)^{n+1}(n+1)\Gamma h^{\min\{p,q+1\}+1}, h \in (0, h_0].$$

Therefore

$$\begin{aligned} \| y_n - y(t_n) \| &\leq \Gamma(1+Mh)^n (nh) h^{\min\{p,q+1\}} \\ &\leq \Gamma e^{Mnh}(nh)h^{\min\{p,q+1\}} \\ &= C(t_n)h^{\min\{p,q+1\}}, \quad h \in (0, h_0]. \end{aligned}$$

where $C(t) = \Gamma e^{M(t-t_0)}(t - t_0)$. This completes the proof of Theorem 3.1.

In the following, we further present a result on generally stability of the method $(2.1) - (2.3)$.

Theorem 3.2. *A LMLM* $(2.1)-(2.3)$ *is generally stable with respect to the problem* (1.1) *of class* $D_{\sigma,\gamma}$.

Proof. Let $\{y_{n+k}\} and \{z_{n+k}\}$ be two solution sequences of the method $(2.1) - (2.3)$ for $(1.1a)$ with the different initial functions $\varphi(t), \psi(t)$ respectively. Moreover, we also write $H_n(\lambda) = G_{y_n, z_n, y^h(t_n-\tau), t, f}(\lambda)$

$$\| y_{n+k} - z_{n+k} \| = H_{n+k}(0) \leq H_{n+k}(h\beta_k). \tag{3.9}$$

Whereas, according to $(2.1), (2.2), (2.3), (1.3)$ and Lemma 2.1 it yields

$$\begin{aligned} H_{n+k}(h\beta_k) &\leq \sum_{i=0}^{k-1} | \alpha_i | H_{n+i}(h\beta_i) + h | \beta_i | \| f(t_{n+i}, z_{n+i}, y^h(t_{n+i} - \tau)) \\ &\qquad - f(t_{n+i}, z_{n+i}, z^h(t_{n+i} - \tau)) \| \\ &\leq \max_{i\in I_0} H_{n+i}(h\beta_k) + h\beta\gamma \max_{i\in I_0} \| y^h(t_{n+i} - \tau) - z^h(t_{n+i} - \tau) \| \\ &\leq \max_{i\in I_0} H_{n+i}(h\beta_k) + h\beta\gamma \sup_{\delta\in[0,1)} \sum_{j=-r}^{s} | L_j(\delta) | \max_{i\in I_0} H_{n+i}(0) \\ &\leq (1+Mh) \max_{i\in I_1} H_{n+i}(h\beta_k), \quad h \in (0, h_1], \end{aligned} \tag{3.10}$$

where

$$M = \beta\gamma \sup_{\delta\in[0,1)} | L_j(\delta) | .$$

Furthermore, with the second induction to (3.10) we get

$$H_{n+k}(h\beta_k) \le (1+Mh)^{n+1}[\max_{i\in I_1} H_i((h\beta_k)], \quad h \in (0,h_1] \tag{3.11}$$

From the definition of $H_i(\lambda)$ we can know that $H_i(\lambda) = 0$ whenever $i < 0$. So, a combination of (3.9), (3.11) and (2.2) follows

$$\begin{aligned} \| y_{n+k} - z_{n+k} \| &\le (1+Mh)^{n+1}[\max_{i\in I_0} H_i(h\beta_k)] \\ &\le e^{M(n+1)h} \max_{i\in I_0} H_i(h_1\beta) \\ &\le e^{M(T-t_0)} \max_{i\in I_0} H_i(h_1\beta), \qquad h\in(0,h_1]. \end{aligned} \tag{3.12}$$

which implies the method (2.1) − (2.3) is generally stable for the class $D_{\sigma,\gamma}$.

4. Some Examples

As the application of Theorem 3.1, 3.2, for the class $D_{\sigma,\gamma}$, we consider the following method with the linear interpolation of order one ($i.e. r = 0, s = 1$ in(2.3))

$$y^h(t) = \begin{cases} \dfrac{t-(t_0+nh)}{h} y_{n+1} + \dfrac{t_0+(n+1)h]-t}{h} y_n, t_0+nh \le t \le t_0+(n+1)h, \\ \qquad\qquad\qquad\qquad\qquad\qquad n = 0,1,2,\cdots \\ \varphi(t), \qquad\qquad\qquad t_0 - \tau \le t \le t_0. \end{cases} \tag{4.1}$$

Method I

$$y_{n+1} = y_n + \tan(\frac{h}{2})[f(t_{n+1}, y_{n+1}, y^h(t_{n+1}-\tau)) + f(t_n, y_n, y^h(t_n-\tau))], \tag{4.2}$$

which is of order two. Contrast to the method (2.1), $\alpha_1 = 1, \alpha_0 = -1, \beta_1 = \frac{1}{h}\tan(\frac{h}{2}), \beta_0 = -\frac{1}{h}\tan(\frac{h}{2})$. It is easy to verify that this method satisfies condition (2.2). Thus, by Theorem 3.1, 3.2 we know that the method (4.1) − (4.2) is D-convergent of order two and generally stable for the class $D_{\sigma,\gamma}$.

Method II

$$y_{n+2} - \frac{1}{2}y_{n+1} - \frac{1}{2}y_n = h[\frac{5}{2}f((t_{n+2}, y_{n+2}, y^h(t_{n+2}-\tau)) - f(t_n, y_n, y^h(t_n-\tau))], \tag{4.3}$$

which is of order one and conform to condition (2.2). With Theorem 3.1, 3.2 we infer that this method is D-convergint of order one and generally stable for the class $D_{\sigma,\gamma}$.

Method III

$$\begin{aligned} y_{n+2} - (1-h^2)y_{n+1} - h^2 y_n = \tfrac{1}{2}[&(\exp(h)-1)f(t_{n+2}, y_{n+2}, y^h(t_{n+2}-\tau)) + \\ &(1-\exp(-h))f(t_{n+1}, y_{n+1}, y^h(t_{n+1}-\tau))], \end{aligned} \tag{4.4}$$

where, contrast to the method (2.1), $\alpha_2 = 1, \alpha_1 = 1-h^2, \alpha_0 = h^2, \beta_2 = \frac{\exp(h)-1}{2h}, \beta_1 = \frac{1-\exp(-h)}{2h(1-h^2)}$ and $\beta_0 = 0$. It is easy to testify that this method satisfies condition (2.2). Therefore, in terms of Theorem 3.1, 3.2 we conclude that this method is D-convergent of order two and generally stable for the class $D_{\sigma,\gamma}$.

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