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PARALLEL ROSENBROCK METHODS FOR SOLVING STIFF SYSTEMS IN REAL-TIME SIMULATION*

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Abstract

In this paper parallel Rosenbrock methods in real-time simulation are presented on parallel computers. Their construction, their convergence and their numerical stability are studied, and the numerical simulation experiments are conducted on a personal computer and a parallel computer respectively.

Key words: Rosenbrock methods, stiff system, real-time simulation, parallel algorithms

1. Introduction

In the fields of astronautics engineering and continuous system simulation, many models are described by stiff ODE's. In order to simulate (especially in real-time) these systems we have to use speedy algorithms so as to complete the computation within the designated time. Papers [5,7,8] present parallel-iterated Runge-Kutta methods and implicit Runge-Kutta methods. Although these methods have higher stability, a heavier workload will be imposed by the iteration. And our inability to predict the number of iteration times in advance will cause us difficulties in the application of the above methods to real-time simulation.

Aimed at real-time simulation of stiffly large systems on parallel computers, parallel Rosenbrock methods(PRM) are constructed through a frontal approach. The different internal stages of Rosenbrock methods are calculated in parallel on different processors. The methods have higher stability and need no iteration. In each step only one Jacobian matrix and one LU-decomposition need to be computed.

2. A Class of Parallel Rosenbrock Methods

Consider the autonomous initial value problems

$$y'(x) = f(y(x)), \quad y(x_0) = y_0, \quad x \in [x_0, x_M], \quad y \in R^n \tag{1}$$

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here $f(y)$ is assumed to possess derivatives of all orders which will be needed in the following discussion.

The class of (sequential) Rosenbrock methods has the following form[4]:

$$\begin{aligned} y_{n+1} &= y_n + \sum_{i=1}^{s} c_i l_i \\ (I - h\gamma J) l_i &= hf(y_n + \sum_{j=1}^{i-1} \alpha_{ij} l_j) + hJ \sum_{j=1}^{i-1} \gamma_{ij} l_j \\ i &= 1, 2, \ldots, s \end{aligned} \tag{2}$$

where $\gamma,\ \alpha_{ij},\ \gamma_{ij},\ c_i$ are real coefficients, I denotes identity matrix, J is Jacobian matrix $f_y(y_n)$. Through a frontal approach, using the information before time point t_n, a class of s-stage parallel Rosenbrock methods is constructed by the following form:

$$\begin{aligned} y_{n+1} &= y_n + \sum_{i=1}^{s} c_i l_{in} \\ (I - h\gamma J) l_{in} &= hf(y_n + \sum_{j=1}^{i-1} \alpha_{ij} l_{jn-1}) + hJ \sum_{j=1}^{i-1} \gamma_{ij} l_{jn-1} \\ i &= 1, 2, \ldots, s \end{aligned} \tag{3}$$

As the quantities $y_n, l_{in-1},\ i = 1, \ldots, s-1$ are known, $l_{1n}, l_{2n}, \ldots, l_{sn}$ can be calculated on s processors in parallel. The information flow that describes the parallel execution of two-stage formula on two processors $P_1,\ P_2$ is shown in Fig.1.

Let t_l denote the CPU time needed to compute each l_{in}, and t_y denote the CPU time needed to compute y_{n+1} from l_{in} (or l_n), and let t_{syn} denote the CPU time needed to exchange information on s processors. Then in each step the sequential costs are $t_T = st_l + t_y$, and the parallel costs on an s-processor system are $t_P = t_l + t_y + t_{syn}$. So the speed-up is

$$S_P = \frac{st_l + t_y}{t_l + t_y + t_{syn}}$$

Particularly, when $t_l \gg t_{syn}$, $S_P \approx s$.

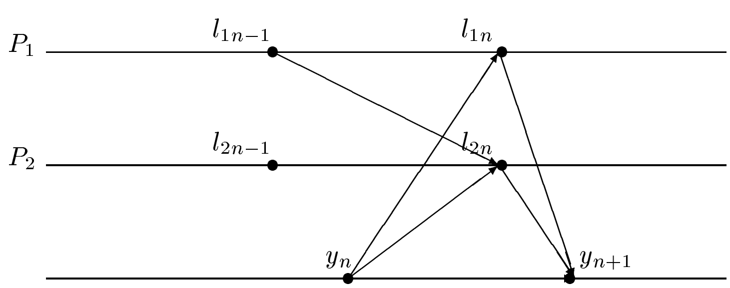


Fig 1. Information flow of parallel Rosenbrock formula

The following abbreviations are used:

$$\begin{aligned}
&\alpha_{ij} = 0 \qquad j \geq i, \qquad \gamma_{ij} = 0 \qquad j > i \\
&\beta'_{ij} = \alpha_{ij} + \gamma_{ij}, \qquad \gamma_{ii} = \gamma \\
&\beta_{ij} = \begin{cases} \beta'_{ij} & i > j \\ 0 & i = j \end{cases} \\
&\alpha_i = \sum_{j=1}^{s} \alpha_{ij}, \qquad \beta'_i = \sum_{j=1}^{s} \beta'_{ij} \\
&\beta_i = \sum_{j=1}^{s} \beta_{ij}, \qquad q_i = \sum_{j=1}^{s} \beta'_{ij}\beta'_j - \beta_i
\end{aligned}$$

3. Order Conditions for PRM

In the following the consistency conditions for a method of order p are given according to the Butcher series theory[1,3]. For the meaning of the denotations, please refer to [3].

Assume that $y(x)$ is the exact solution of the initial value problem (1), and denote

$$\begin{aligned}
&y^*_{n+1} = y(x_n) + \sum_{i=1}^{s} c_i L_{in} \\
&(I - h\gamma f_y(y(x_n)))L_{in} = hf(y(x_n) + \sum_{j=1}^{i-1} \alpha_{ij} L_{jn-1}) + hf_y(y(x_n)) \sum_{j=1}^{i-1} \gamma_{ij} L_{jn-1} \\
&\qquad i = 1, 2, \ldots, s
\end{aligned} \tag{4}$$

Then PRM (3) are of order p if and only if

$$y(x_n + h) - y^*_{n+1} = O(h^{p+1}) \tag{5}$$

To derive the order condition equations, we assume that $L_{in} = L_i(x_n, h)$ can be represented as a B-series:

$$L_{in} = B(\Phi_i, y(x_n)) \tag{6}$$

Hence one obtains

$$L_{in-1} = B(\Phi_i, y(x_n - h))$$

Since $y(x_n - h)$ can be represented as B-series at $y(x_n)$, namely

$$y(x_n - h) = B(\Psi, y(x_n)) = \sum_{t \in T} \frac{h^{\rho(t)}}{\rho(t)!} \alpha(t)(-1)^{\rho(t)} F(t)(y(x_n))$$

where $\Psi(t) = (-1)^{\rho(t)}, \quad \forall t \in T$, it follows from [3, Theorem 12.6] that

$$L_{in-1} = B(\Phi_i, B(\Psi, y(x_n))) = B(\Psi\Phi_i, y(x_n)) \tag{7}$$

Denote

$$\Gamma_i = \Psi\Phi_i$$

it leads that

$$y(x_n) + \sum_{j=1}^{i-1} \alpha_{ij} L_{jn-1} = B(\Gamma'_i, y(x_n))$$

with

$$\Gamma'_i(t) = \begin{cases} 1 & \rho(t) = 0 \\ \sum_{j=1}^{i-1} \alpha_{ij}\Gamma_j(t) & \rho(t) \geq 1 \end{cases}$$

It follows from [3, Corollary 12.7] due to $\Gamma'_i(\phi) = 1$ that

$$hf(y(x_n) + \sum_{j=1}^{i-1} \alpha_{ij} L_{jn-1}) = B(\Gamma^I_i, y(x_n)) \tag{8}$$

with

$$\begin{aligned} &\Gamma^I_i(\phi) = 0, \qquad \Gamma^I_i(\bullet) = 1 \\ &\Gamma^I_i(t) = \rho(t)\Gamma'_i(t_1)\cdots\Gamma'_t(t_m) \\ &\qquad = \rho(t)\sum_{j_1\cdots j_m} \alpha_{ij_1}\cdots\alpha_{ij_m}\Gamma_{j_1}(t_1)\cdots\Gamma_{j_m}(t_m) \end{aligned} \tag{9}$$

where $t = [t_1, t_2, \ldots, t_m]$.

From [2, Lemma 15] we obtain that

$$h\frac{\partial f}{\partial y}(y(x_n))\sum_{j=1}^{i-1} \gamma_{ij} L_{jn-1} = B(\Gamma^{II}_i, y(x_n)) \tag{10}$$

with

$$\Gamma^{II}_i(t) = \begin{cases} \rho(t)\sum_{j=1}^{i-1} \gamma_{ij}\Gamma_j(t_1) & t = [t_1], \qquad \rho(t_1) \geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

and

$$h\gamma\frac{\partial f}{\partial y}(y(x_n))L_{in} = B(\overline{\Phi}_i, y(x_n)) \tag{12}$$

with

$$\overline{\Phi}_i(t) = \begin{cases} \rho(t)\gamma\Phi_i(t_1) & t = [t_1], \qquad \rho(t_1) \geq 1 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Hence

$$L_{in} = B(\Gamma^I_i, y(x_n)) + B(\Gamma^{II}_i, y(x_n)) + B(\overline{\Phi}_i, y(x_n))B(\Gamma^I_i + \Gamma^{II}_i + \overline{\Phi}_i, y(x_n)) \tag{14}$$

From (9),(11),(13) it follows that

$$\Phi_i(\phi) = 0, \qquad \Phi_i(\bullet) = 1$$
$$\Phi_i(t) = \begin{cases} \rho(t) \sum\limits_{j_1 \cdots j_m} \alpha_{ij_1} \cdots \alpha_{ij_m} \Gamma_{j_1}(t_1) \cdots \Gamma_{j_m}(t_m) & t = [t_1, t_2, \ldots, t_m],\ m \geq 2 \\ \rho(t)(\sum\limits_j \beta_{ij} \Gamma_j(t_1) + \gamma \Phi_i(t_1)) & t = [t_1],\ \rho(t_1) \geq 1 \end{cases} \tag{15}$$

Thus, using (4) we obtain

$$y^*_{n+1} = B(\Phi, y(x_n)) \tag{16}$$

with

$$\Phi(t) = \begin{cases} 1 & \rho(t) = 0 \\ \sum\limits_{i=1}^{s} c_i \Phi_i(t) & \rho(t) \geq 1 \end{cases} \tag{17}$$

where $\Phi_i(t)$ is determined by (15). A comparison of (16) with the expansion of the exact solution

$$y(x_n + h) = B(\Phi, y(x_n)) = \sum_{t \in T} \frac{h^{\rho(t)}}{\rho(t)!} \alpha(t) F(t)(y(x_n))$$

yields

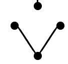
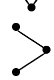
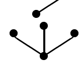
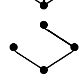
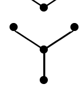
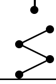
$$y(x_n + h) - y^*_{n+1} = \sum_{t \in T} \frac{h^{\rho(t)}}{\rho(t)!} \alpha(t)(1 - \Phi(t)) F(t)(y(x_n)) \tag{18}$$

From (5) and (18) one obtains directly the following theorem:

Theorem 1 (Consistency order of PRM). *The parallel Rosenbrock method* (3) *has order of consistency p, if* $\Phi(t) = 1$ $\forall t \in T$ *with* $\rho(t) \leq p$*,* $\Phi(t)$ *is defined by* (17).

The order condition equations are listed in Table 1.

Table 1. Order conditon equations for PRM up to 4

Order	Trees	Equation
1		$\sum c_i = 1$
2		$\sum c_i \beta'_i = \frac{1}{2}$
3		$\sum c_i \alpha_i^2 = \frac{1}{3}$
		$\sum c_i q_i = \frac{1}{6}$
4		$\sum c_i \alpha_i^3 = \frac{1}{4}$
		$\sum c_i \alpha_i \alpha_{ij} \beta'_j = \frac{11}{24}$
		$\sum c_i \beta_{ij} \alpha_j^2 = \frac{11}{12} - \frac{7}{3}\gamma$
		$\sum c_i \beta_{ij} q_j = \frac{11}{24} - \frac{7}{6}\gamma$

Remark. all summation indices $i,\ j$ are in range $1,\dots,s$.

4. Construction of PRM

We can construct PRM of two-stage third-order and three-stage fourth-order by virtue of the order condition equations.

Corollary 2. *There exist PRM of order $p=3$ with stage number $s=2$.*

In this case the order condition equations are:

$$\begin{aligned} &c_1+c_2=1\\ &c_1\gamma+c_2(\beta_{21}+\gamma)=\frac{1}{2}\\ &c_2\alpha_2^2=\frac{1}{3}\\ &c_1\gamma^2+c_2(\gamma^2+2\beta_{21}\gamma-\beta_{21})=\frac{1}{6}\end{aligned}\tag{19}$$

In (19), we take $\alpha_2=\alpha_{21}$ as a free parameter. A coefficient set of the real-time PRM with two-stage third-order is shown in Table 2.

Table 2. Coefficients of PRM with $p=3,\ s=2$

c_1	c_2	α_{21}	γ	γ_{21}
$\frac{1}{4}$	$\frac{3}{4}$	$\frac{2}{3}$	$1\pm\frac{1}{\sqrt{3}}$	$-\frac{4}{3}(1\pm\frac{1}{\sqrt{3}})$
$\frac{11}{27}$	$\frac{16}{27}$	$\frac{3}{4}$	$1\pm\frac{1}{\sqrt{3}}$	$\frac{3}{32}-\frac{27}{16}(1\pm\frac{1}{\sqrt{3}})$
$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{2}$	$1\pm\frac{1}{\sqrt{3}}$	$-\frac{3}{24}-\frac{3}{4}(1\pm\frac{1}{\sqrt{3}})$

Corollary 3. *There exist PRM of order $p=4$ with stage number $s=3$.*

The order condition equations are:

$$\begin{aligned} &c_1+c_2+c_3=1\\ &c_1\beta_1'+c_2\beta_2'+c_3\beta_3'=\frac{1}{2}\\ &c_1q_1+c_2q_2+c_3q_3=\frac{1}{6}\\ &c_2\alpha_2^2+c_3\alpha_3^2=\frac{1}{3}\\ &c_2\alpha_2^3+c_3\alpha_3^3=\frac{1}{4}\\ &c_2\alpha_2\alpha_{21}\beta_1'+c_3\alpha_3\alpha_{31}\beta_1'+c_3\alpha_3\alpha_{32}\beta_2'=\frac{11}{24}\\ &c_3\beta_{32}\alpha_2^2=\frac{11}{12}-\frac{7}{3}\gamma\\ &c_2\beta_{21}q_1+c_3\beta_{31}q_1+c_3\beta_{32}q_2=\frac{11}{24}-\frac{7}{6}\gamma\end{aligned}\tag{20}$$

In (20), we take $\alpha_2,\ \alpha_3$ as free parameters. A coefficient set of the real-time PRM with three-stage fourth-order is shown in Table 3.

Table 3. Coefficients of PRM with $p = 4,\ s = 3$

$\alpha_2 = \frac{1}{3}$	$\alpha_3 = \frac{2}{3}$	
$\gamma = 3.205737064E + 00$		
$\alpha_{21} = 3.333333333E - 01$	$\alpha_{31} = -1.205988612E + 01$	$\alpha_{32} = 1.272655279E + 01$
$\gamma_{21} = -4.100542740E - 01$	$\gamma_{31} = 7.212090006E + 01$	$\gamma_{32} = -7.573506302E + 01$
$c_1 = 8.125000000E - 01$	$c_2 = -7.500000000E - 01$	$c_3 = 9.375000000E - 01$

5. Convergence of PRM

Denote

$$l_{in} = l_i(y_n, y_{n-1}, \dots, y_{n-i+1}, h), \qquad L_{in} = l_i(y(x_n), \dots, y(x_{n-i+1}), h)$$
$$\Delta l_{in} = L_{in} - l_{in}, \qquad i = 1, 2, \dots, s$$

and

$$e_n = y(x_n) - y_n, \qquad \delta_n = \max\{\|e_n\|, \dots, \|e_{n-s+1}\|\}$$

Lemma 4. *There exist positive real numbers d_k $(k = 1, 2, \dots, s)$ such that*

$$\|\sum_{i=1}^{s} c_i L_{in} - \sum_{i=1}^{s} c_i l_{in}\| \le h \sum_{k=1}^{s} d_k \|e_{n-k+1}\| \tag{21}$$

Proof. It is easy to see that we only need to show that for any i $(i = 1, 2, \dots, s)$ there exist positive real number $d_{i1}, d_{i2}, \dots, d_{ii}$ such that

$$\|\Delta l_{in}\| \le h(\sum_{k=1}^{i} d_{ik} \|e_{n-k+1}\|) \tag{22}$$

First, we start with $i = 1$.

$$L_{1n} = hf(y(x_n)) + h\gamma f_y(y(x_n)) L_{1n}$$
$$l_{1n} = hf(y_n) + h\gamma f_y(y_n) l_{1n}$$

which yields

$$\Delta l_{1n} = hf_y e_n + h\gamma f_{yy} e_n L_{1n} + h\gamma f_y \Delta l_{1n}$$

Here the variables in partial derivatives $f_y,\ f_{yy}$ are omitted. Assuming $\|f_y\| \le L$, taking a small positive number h_0 such that $h_0|\gamma|L < 1$, we have that when $0 < h < h_0$,

$$\|(I - h\gamma f_y)^{-1}\| \le \frac{1}{1 - h_0|\gamma|L}$$

Furthermore, assuming $|\gamma|\|f_{yy}\|\|L_{in}\| \le M_1, \qquad i = 1, 2, \dots, s$, we obtain

$$\|\Delta l_{1n}\| \le h \frac{L + M_1}{1 - h_0|\gamma|L} \|e_n\| \tag{23}$$

Set $d_{11} = \dfrac{L+M_1}{1-h_0|\gamma|L}$, and it is showed that the formula (22) holds with $i = 1$.

Next, we assume (22) holds with $i < m$ ($m \le s$). When $i = m$

$$L_{mn} = hf(y(x_n) + \sum_{j=1}^{m-1} \alpha_{mj} L_{jn-1}) + hf_y(y(x_n)) \sum_{j=1}^{m-1} \gamma_{mj} L_{jn-1} + h\gamma f_y(y(x_n)) L_{mn}$$
$$l_{mn} = hf(y_n + \sum_{j=1}^{m-1} \alpha_{mj} l_{jn-1}) + hf_y(y_n) \sum_{j=1}^{m-1} \gamma_{mj} l_{jn-1} + h\gamma f_y(y_n) l_{mn}$$

Omitting the variables in partial derivetives $f_y,\ f_{yy}$, we obtain

$$\Delta l_{mn} = hf_y e_n + hf_y \sum_{j=1}^{m-1} \alpha_{mj} \Delta l_{jn-1} + hf_{yy} e_n \sum_{j=1}^{m-1} \gamma_{mj} L_{jn-1} + hf_y \sum_{j=1}^{m-1} \gamma_{mj} \Delta l_{jn-1}$$
$$+ h\gamma f_{yy} e_n L_{mn} + h\gamma f_y \Delta l_{mn}$$

Suppose $\|f_{yy}\| \sum_{j=1}^{m-1} |\gamma_{mj}| \|L_{jn-1}\| \le M_2, \quad m = 1, 2, \ldots, s$. it yields that when $0 < h < h_0$

$$\|\Delta l_{mn}\| \le h\Big(\frac{L+M_1+M_2}{1-h_0|\gamma|L} \|e_n\| + \frac{L}{1-h_0|\gamma|L} \sum_{j=1}^{m-1} (|\alpha_{mj}| + |\gamma_{mj}|) \|\Delta l_{jn-1}\| \Big) \qquad (24)$$

From the assumption that for $i < m$ there exist positive real numbers $d_{i1}, \ldots, d_{ii}$ such that

$$\|\Delta l_{in}\| \le h(\sum_{k=1}^{i} d_{ik} \|e_{n-k+1}\|)$$

it leads to

$$\|\Delta l_{jn-1}\| \le h(\sum_{k=1}^{j} d_{jk} \|e_{n-k}\|), \qquad j = 1, 2, \ldots, m-1$$

Substituting the above formula into (24), we obtain

$$\|\Delta l_{mn}\| \le h \frac{L+M_1+M_2}{1-h_0|\gamma|L} \|e_n\| + h \frac{h_0 L}{1-h_0|\gamma|L} \sum_{j=1}^{m-1} (|\alpha_{mj}| + |\gamma_{mj}|) \sum_{k=1}^{j} d_{jk} \|e_{n-k}\|$$
$$= h \frac{L+M_1+M_2}{1-h_0|\gamma|L} \|e_n\|$$
$$+ h \frac{h_0 L}{1-h_0|\gamma|L} \sum_{k=2}^{m} \Big(\sum_{j=k-1}^{m-1} (|\alpha_{mj}| + |\gamma_{mj}|) d_{jk-1} \Big) \|e_{n-k+1}\|$$

Setting

$$d_{m1} = \frac{L+M_1+M_2}{1-h_0|\gamma|L}$$
$$d_{mk} = \frac{h_0 L}{1-h_0|\gamma|L} \sum_{j=k-1}^{m-1} (|\alpha_{mj}| + |\gamma_{mj}|) d_{jk-1}, \qquad k = 2, \ldots, m$$

it follows that (22) holds when $i = m$. Hence for any i ($1 \leq i \leq s$) the formula (22) holds and Lemma 4 is proved.

For PRM with order p and stage number s, we have

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + \sum_{i=1}^{s} c_i L_{in} + T_{n+1} \\ (I - h\gamma f_y(y(x_n)))L_{in} &= hf(y(x_n) + \sum_{j=1}^{i-1} \alpha_{ij} L_{jn-1}) + hf_y(y(x_n)) \sum_{j=1}^{i-1} \gamma_{ij} L_{jn-1} \\ & i = 1, 2, \ldots, s \end{aligned} \tag{25}$$

where T_{n+1} denotes the local truncation error. Subtracting (3) from (25) we obtain

$$e_{n+1} = e_n + \sum_{i=1}^{s} c_i \Delta l_{in} + T_{n+1} \tag{26}$$

From Lemma 4 and (26) it can be easily verified that the following theorem holds.

Theorem 5. *For PRM with order p and stage number s, if the error of starting values satiesfies*

$$\delta_{s-1} = \max\{\|e_{s-1}\|, \ldots, \|e_0\|\} = O(h^p)$$

then for the global error it holds that

$$\|e_n\| = O(h^p), \qquad n \geq s$$

6. Numerical Stability of PRM

Appling two-stage third-order PRM in Table 2 to the test equation $y' = \lambda y$, denoting $z = \lambda h, \lambda \in C^-$, we have

$$y_{n+1} = (1 + \frac{z}{1-\gamma z})y_n + \frac{(\frac{1}{2} - \gamma)z^2}{(1-\gamma z)^2} y_{n-1}$$

Hence the corresponding stable polynomial is

$$\Lambda(r; z) = r^2 + \frac{-1 + (\gamma - 1)z}{1 - \gamma z} r + \frac{(\gamma - \frac{1}{2})z^2}{(1-\gamma z)^2}$$

When $\gamma = 1 + \frac{1}{\sqrt{3}}$, the two-stage third-order PRM is A-stable. In fact, it suffices to show that $\Pi(r) = (1-\gamma z)^2 \Lambda(r; z)$, namely

$$\Pi(r) = (1 - 2\gamma z + \gamma^2 z^2) r^2 - (1 - (2\gamma - 1)z + \gamma(\gamma - 1)z^2) r + (\gamma - \frac{1}{2}) z^2$$

is a Schur polynomial for all $Rez < 0$. Define the polynomial

$$\hat{\Pi}(r) = (\gamma - \frac{1}{2})\bar{z}^2 r^2 - (1 - (2\gamma - 1)\bar{z} + \gamma(\gamma - 1)\bar{z}^2)r + (1 - 2\gamma\bar{z} + \gamma^2\bar{z}^2)$$

where $\bar{z}$ is the complex conjugate of z, and

$$\Pi_1(r) = \frac{1}{r}[\hat{\Pi}(0)\Pi(r) - \Pi(0)\hat{\Pi}(r)]$$

By a theorem of Schur[6], $\Pi(r)$ is Schur polynomial if and only if

(1) $|\hat{\Pi}(0)| > |\Pi(0)|$,

(2) $\Pi_1(r)$ is a Schur polynomial.

Clerely, when $\gamma = 1 + \frac{1}{\sqrt{3}}$, $|\hat{\Pi}(0)| > |\Pi(0)|$ for all $Rez < 0$. $\Pi_1(r)$ has only root r. After simple computation we have

$$|r| \leq 1$$

on the whole imaginary axis. Since r is an analytic function in the left-half plane $C^- = \{z|Rez < 0\}$, by the maximum modulus princile, we have

$$|r| < 1, \qquad z \in C^-$$

That is, for all $Rez < 0, \Pi_1(r)$ is a Schur polynomail, and the PRM is A-stable.

Applying three-stage fourth-order PRM in Table 3 to $y' = \lambda y$, we get

$$y_{n+1} = (1 + \frac{z}{1 - \gamma z})y_n + \frac{\frac{1}{2} - \gamma}{(1 - \gamma z)^2} z^2 y_{n-1} + \frac{\gamma^2 - 2\gamma + \frac{2}{3}}{(1 - \gamma z)^3} z^3 y_{n-2}$$

The corresponding stability polynomial is

$$\Lambda(r; z) = r^3 + \frac{-1 + (\gamma - 1)z}{1 - \gamma z} r^2 + \frac{(\gamma - \frac{1}{2})}{(1 - \gamma z)^2} z^2 r + \frac{-\gamma^2 + 2\gamma - \frac{2}{3}}{(1 - \gamma z)^3} z^3$$

When $\gamma = 3.205737063$, three-stage fourth-order PRM is $A(\alpha)-$ stable with $\alpha \approx 87°$.

7. Numerical Experiment

Example 1. Stiffly linear ordinary differential equations

$$y_1' = -29998y_1 - 59994y_2, \quad y_1(0) = 1$$
$$y_2' = 9999y_1 + 19997y_2, \quad y_2(0) = 0$$

The eigenvalues of system are $\lambda_1 = -10000,\ \lambda_2 = -1$. The exact solution is

$$y_1(t) = \frac{1}{9999}(29997e^{-10000t} - 19998e^{-t}),\ y_2(t) = e^{-t} - e^{-10000t}.$$

Example 2. Stiffly nonlinear ordinary differential equations

$$y_1' = -(\epsilon^{-1} + 2)y_1 + \epsilon^{-1}y_2^2, \quad y_1(0) = 1$$
$$y_2' = y_1 - y_2 - y_2^2, \quad y_2(0) = 1$$

where $\epsilon = 10^{-6}$. The smaller ϵ is , the more serious the stiff of system is. The exact solution is

$$y_1(t) = y_2^2(t), \qquad y_2(t) = e^{-t}$$

Example 3. Weakly damped oscillatory differential equations

$$y' = Ay, \qquad y(0) = y_0$$

where

$$A = \begin{pmatrix} -0.01 & -1 & -1 \\ 2 & -100.005 & 99.995 \\ 2 & 99.995 & -100.005 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

The exact solution is

$$\begin{aligned} y_1(t) &= e^{-0.01t}(cos2t - sin2t) \\ y_2(t) &= e^{-0.01t}(cos2t + sin2t) + e^{-200t} \\ y_3(t) &= e^{-0.01t}(cos2t + sin2t) - e^{-200t} \end{aligned}$$

Applying PRM of two-stage third-order and three-stage fourth-order to examples 1–3, the computation results are listed in Tables 4–5. The error of $i-th$ component is given by

$$\text{err}i = \left| \frac{y_T^i - y_i(T)}{y_T^i} \right|$$

Here y_T^i, $y_i(T)$ are the numerical solution and exact solution at the endpoint T of the interval respectively.

Table 4. $\alpha_2 = 0.5,\ T = 10.0$

	h	err1	err2	err3
Example 1	0.1	1.079E-02	1.079E-02	
	0.01	1.270E-05	1.270E-05	
Example 2	0.1	4.389E-02	1.079E-02	
	0.01	2.280E-04	1.270E-05	
Example 3	0.1	3.457E-01	1.265E-01	1.265E-01
	0.01	2.402E-04	2.016E-04	2.016E-04

Table 5. $\alpha_2 = \frac{1}{3},\ \alpha_3 = \frac{2}{3},\ T = 10.0$

	h	err1	err2	err3
Example 1	0.1	1.259E-02	1.259E-02	
	0.01	2.349E-06	2.349E-06	
Example 2	0.1	7.283E-02	1.259E-02	
	0.01	4.076E-05	2.349E-06	
Example 3	0.1	3.888E-01	5.645E-01	5.645E-01
	0.01	1.923E-04	4.604E-05	4.604E-05

Because the parallel methods are constructed by aiming at large systems, we increase intentionally the number of equations in testing speed-up and efficiency. For instance, we shall use the following system instead of Example 1

$$\begin{aligned} &for(i=0;\ i<N;\ ++i)\ \{ \\ &\quad y_1' = -29998y_1 - 59994y_2 \\ &\quad y_2' = 9999y_1 - 19997y_2 \\ &\} \end{aligned}$$

where $N > 1$ is cycle index representing the computation complex of right-hand functions.

PRM for two-stage third-order and three-stage fourth-order are implemented on a parallel computer S10. The results are listed in Table 6 and Table 7 respectively.

Table 6. Two-stage third-order PRM for solving Example 1

N	500	1000	2000	5000
Speedup	1.08	1.14	1.45	1.76
Efficiency	54%	57%	77%	88%

Table 7. Three-stage fourth-order PRM for solving Example 2

N	500	1000	2000	5000
Speedup	1.47	1.64	2.16	2.51
efficiency	49%	55%	72%	84%

From these results we can conclude that PRM for solving stiffly large ODE's is efficient and applicable.

References

[1] J.C. Butcher, The Numerical Analysis of Ordinary Differential Equations: Runge-Kutta and General Linear Methods, JOHN WILEY & SONS, 1987.

[2] E. Hairer, Order Conditions for Numerical Methods for Partitioned Ordinary Differential Equations, *Numer. Math.* 36(1981), 431–445.

[3] E. Hairer, S. Nϕrsett, G. Wanner, Solving Ordinary Differential Equations I: Nonstiff Problems, Berlin, Heidelberg, Springer 1987.

[4] E. Hairer and G. Wanner, Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems, Berlin, Heidelberg, New York, Tokyo: Springer 1991.

[5] Huang Zili and Liu Degui, The A-stability Analysis for a Class of Parallel Implicit Runge-Kutta Methods, *J. Systems Engineering & Electronics (in Chinese)* Vol.13, No.3, 1991, 25–30.

[6] J.D. Lambert, Computational Methods in Ordinary Differential Equations, John Wiley and Sons, 1973.

[7] B.P. Sommeijer, Parallel-iterated Runge-Kutta methods for ordinary differential equations, *J.Comput. Appl. Math.* 45(1993) 151–168.

[8] P.J. van der Houwen, B.P. Sommeijer and W. Couzy, Embedded diagonally implicit Runge-Kutta algorithms on parallel computers, *Math. Comp.* 58(1992) 135–159.