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COMPOSITE LEGENDRE–LAGUERRE APPROXIMATION IN UNBOUNDED DOMAINS*

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

Composite Legendre–Laguerre approximation in unbounded domains is developed. Some approximation results are obtained. As an example, a composite spectral scheme is provided for the Burgers equation on the half line. The stability and convergence of proposed scheme are proved strictly. Two-dimensional exterior problems are discussed.

Key words: Composite spectral approximation, Unbounded domains, Exterior problems.

1. Introduction

Many problems in science and engineering are set in unbounded domains. There are several ways for their numerical simulations. We may restrict calculations to some bounded domains with certain artificial boundary conditions. But they induce errors. In particular, they affect the wave propagations in revolutionary problems. In opposite, if we use spectral methods associated with orthogonal systems of polynomials in unbounded domains, then we could avoid this trouble, e.g., see Maday, Pernaud-Thomas and Vandeven [1], Funaro [2], Funaro and Kavian [3], Boyd [4], Guo [5], and Guo and Shen [6]. Funaro and Kavian [3] proved the convergence for some linear problems, by using the Hermite functions. Recently Guo [5] proved the stability and the convergence of spectral approximations to nonlinear problems, using the Hermite polynomials. While Guo and Shen [6] analyzed the errors of the Laguerre spectral schemes for several nonlinear problems. But there are still some remaining problems. Firstly, in order to get the same accuracy, the Hermite and Laguerre methods need more regularities of solutions of differential equations, than the Legendre and Chebyshev methods for the same problems in the corresponding bounded subdomains. However, the solutions may change rapidly in certain bounded subdomains. For instance, the solutions might be less smooth near some corners. On the other hand, most of multiple-dimensional problems are set in non-rectangular domains, and so the standard Hermite and Laguerre approximations are not available for them. In particular, for exterior problems, the domains are never rectangular, and the solutions change very rapidly near the obstacles usually. One of reasonable ways for resolving such problems is to use spectral domain decomposition method, e.g., see Quarteroni[7], and Coulaud, Funaro and Kavian [8]. For instance, we may divide the domain into several subdomains, and then use the Legendre approximations in the bounded subdomains, and use the Laguerre approximations

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in the remaining parts. But so far, there is no theoretical results in this field. The aim of this paper is to investigate a new spectral domain decomposition method, called as composite spectral method. For simplicity of analysis, we first consider a one-dimensional model in detail, and then discuss two-dimensional problems in non-rectangular domains, and exterior problems. In the next section, we divide the half line to a finite subinterval and a infinite subinterval, and then construct a composite Legendre–Laguerre spectral scheme for the Burgers equation on the half line. We prove some composite imbedding inequalities and approximation results in Section 3, which play important roles in analysis of the composite spectral method. In section 4, we use the results in the previous section to prove the stability and the convergence of proposed scheme strictly. The final section is for two-dimensional problems. We consider a non-rectangular unbounded domain and an exterior problem.

2. The Composite Spectral Scheme

Let $I = (a,b), -\infty \le a < b \le \infty$, and $\chi(x)$ be certain weight function in the usual sense. For any $1 \le p \le \infty$, define

$$L^p_\chi(I) = \{v \mid \|v\|_{L^p_{\chi,I}} < \infty\}$$

where

$$\|v\|_{L^p_{\chi,I}} = \begin{cases} (\int_I |v(x)|^p \chi(x)dx)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x\in I} |v(x)|, & \text{if } p = \infty. \end{cases}$$

In particular, we denote by $(u,v)_{\chi,I}$ and $||v||_{\chi,I}$ the inner product and the norm of the space $L^2_\chi(I)$. Further let $\partial_x v(x) = \frac{\partial}{\partial_x} v(x), etc.$. For any non-negative integer m,

$$H^m_\chi(I) = \{v \mid \partial^k_x v \in L^2_\chi(I),\ 0 \le k \le m\}$$

equipped with the following semi-norm and norm

$$|v|_{m,\chi,I} = \|\partial^m_x v\|_{\chi,I}, \qquad \|v\|_{m,\chi,I} = \left(\sum_{k=0}^m |v|^2_{k,\chi,I}\right)^{\frac{1}{2}}.$$

For any $r > 0$, we define the space $H^r_\chi(I)$ with the norm $\|v\|_{r,\chi,I}$ by the space interpolation as in Adams [9]. Moreover let

$$H^1_{0,\chi}(I) = \{v \mid v \in H^1_\chi(I) \text{ and } v(a) = \lim_{x\to b} \chi^{\frac{1}{2}}(x)v(x) = \lim_{x\to b} \chi^{\frac{1}{2}}(x)\partial_x v(x) = 0\ \}.$$

For $\chi(x) \equiv 1$, we denote $H^r_\chi(I), H^r_{0,\chi}(I), |v|_{r,\chi,I}, \|v\|_{r,\chi,I}, (u,v)_{\chi,I}$ and $||v||_{\chi,I}$ by $H^r(I), H^r_0(I), |v|_{r,I}$, $\|v\|_{r,I}, (u,v)_I$ and $||v||_I$, respectively. In addition, $||v||_{\infty,I}$ stands for $||v||_{L^\infty(I)}$.

Let $\Lambda = (-1,\infty)$, and consider the Burgers equation on the half line as follows

$$\begin{cases} \partial_t U(x,t) + \frac{1}{2}\partial_x(U^2(x,t)) - \mu\partial^2_x U(x,t) = f(x,t), & x \in \Lambda, 0 < t \le T, \\ U(-1,t) = d(t), & 0 < t \le T, \\ \lim_{x\to\infty} U(x,t) = \lim_{x\to\infty} \partial_x U(x,t) = 0, & 0 < t \le T, \\ U(x,0) = U_0(x), & x \in \Lambda \end{cases} \tag{2.1}$$

where μ is a positive constant, $f(x,t), d(t)$ and $U_0(x)$ are given functions, and $U_0(x) \to 0, \partial_x U_0(x) \to 0$, as $x \to \infty$. For simplicity, assume $d(t) \equiv 0$. Let

$$a(u,v) = (\partial_x u, \partial_x v)_\Lambda.$$

A weak formulation of (2.1) is to find $U \in L^2(0,T;H_0^1(\Lambda)) \bigcap L^\infty(0,T;L^2(\Lambda))$ such that

$$\begin{cases} (\partial_t U(t), v)_\Lambda & + \dfrac{1}{2}(\partial_x(U^2(t)), v)_\Lambda + \mu a(U(t), v) \\ & = (f(t), v)_\Lambda, \quad \forall v \in H_0^1(\Lambda), \quad 0 < t \le T, \\ U(0) = U_0. \end{cases} \tag{2.2}$$

We can approximate (2.2) by the Laguerre spectral method. Whereas for the same accuracy, the Laguerre spectral scheme requires more regularities of $U(x,t)$ than the corresponding Legendre spectral scheme in bounded domains, see Remark 4.1 of this paper. On the other hand, it is difficult to generalize it to multiple-dimensional problems in non-rectangular unbounded domains. A reasonable way for solving (2.2) is to use the spectral domain decomposition method. To do this, let $\Lambda_1 = (-1,1), \Lambda_2 = [1,\infty)$, and $\Lambda = \Lambda_1 \bigcup \Lambda_2$. Furthermore let $U(x,t) = U_j(x,t)$ in $\Lambda_j, j = 1,2$. Then we approximate $U_1(x,t)$ by the Legendre spectral method in Λ_1, and approximate $U_2(x,t)$ by the Laguerre spectral method in Λ_2. In addition, $U_1(1,t) = U_2(1,t), 0 \le t \le T$. The main advantages of this method are as follows. Firstly, since the approximations to $U_1(x,t)$ and $U_2(x,t)$ are almost separate, we can decrease the degrees of the polynomials used in the expansions of the numerical solution. Also it benefits from the rapid convergence of the Legendre expansion in Λ_1, and keeps the spectral accuracy.

We now begin to construct the composite spectral scheme for (2.2). Let $\omega_1(x) \equiv 1$, and $L_l(x)$ be the Legendre polynomial of degree l, i.e.,

$$L_l(x) = \frac{(-1)^l}{2^l l!} \frac{d^l}{dx^l}(1-x^2)^l, \qquad l = 0,1,2,\cdots.$$

They satisfy the equation

$$\partial_x((1-x^2)\partial_x L_l(x)) + l(l+1)L_l(x) = 0 \tag{2.3}$$

and the recurrence relations

$$(2l+1)L_l(x) = \partial_x L_{l+1}(x) - \partial_x L_{l-1}(x), \quad l \ge 1. \tag{2.4}$$

The set of Legendre polynomials is the $L^2(\Lambda_1)$-orthogonal system,

$$(L_l, L_m)_{\Lambda_1} = \frac{2}{2l+1}\delta_{l,m} \tag{2.5}$$

where $\delta_{l,m}$ is the Kronecker function.

Let $\Lambda_0 = (0,\infty), \omega_0(x) = e^{-x}$, and $\mathcal{L}_l(x)$ be the Laguerre polynomial of degree l, defined by

$$\mathcal{L}_l(x) = \frac{1}{l!} e^x \partial_x^l \left(x^l e^{-x}\right).$$

They satisfy the equation

$$\partial_x(xe^{-x}\partial_x \mathcal{L}_l(x)) + le^{-x}\mathcal{L}_l(x) = 0 \tag{2.6}$$

and the recurrence relations

$$\mathcal{L}_l(x) = \partial_x \mathcal{L}_l(x) - \partial_x \mathcal{L}_{l+1}(x), \quad l \ge 0.$$

The set of Laguerre polynomials is the $L^2_{\omega_0}(\Lambda_0)$-orthogonal system ,

$$(\mathcal{L}_l, \mathcal{L}_m)_{\omega_0,\Lambda_2} = \delta_{l,m}. \tag{2.7}$$

Let $\omega_2(x) = e^{1-x}$, and define

$$\omega(x) = \begin{cases} \omega_1(x), & \text{if } x \in \Lambda_1, \\ \omega_2(x), & \text{if } x \in \Lambda_2. \end{cases}$$

Next, let $N = (N_1, N_2), N_j$ being any positive integers, $j = 1, 2$. Denote by $\mathcal{P}_{N_j,j}$ the sets of restrictions to Λ_j of all algebraic polynomials of degree at most N_j, $j = 1, 2$. Furthermore,

$$S_N = \{v \mid v = \omega^{\frac{1}{2}}\phi,\ \phi|_{\Lambda_j} \in \mathcal{P}_{N_j,j}\},$$

$$V_N = S_N \bigcap H^1(\Lambda), \quad V_N^0 = S_N \bigcap H_0^1(\Lambda).$$

We shall follow the idea in Guo and Shen [6] to approximate the function $\omega^{-\frac{1}{2}}(x)U_2(x,t)$ by the Laguerre approximation. Let $u_{N,0} \in V_N^0$ and u_N be the approximation to U. Let

$$b(u, v, w) = \frac{1}{2}(\partial_x(uv), w)_\Lambda.$$

For any $v \in H_0^1(\Lambda)$,

$$b(v, v, v) = 0.$$

The composite Legendre-Laguerre spectral scheme for (2.2) is to find $u_N \in V_N^0$ for all $0 \le t \le T$, such that

$$\begin{cases} (\partial_t u_N(t), \phi)_\Lambda + b(u_N(t), u_N(t), \phi) + \mu a(u_N(t), \phi) = (f(t), \phi)_\Lambda, \quad \forall \phi \in V_N^0, \quad 0 < t \le T, \\ u_N(0) = u_{N,0}. \end{cases} \tag{2.8}$$

3. Some Results on Composite Approximation

In order to analyze the errors, we need some imbedding inequalities and approximation results related to the composite approximation. In the sequel, we denote by c a generic positive constant independent of any function and N.

Lemma 3.1. *For any $v \in H_0^1(\Lambda)$,*

$$||v||_{\infty,\Lambda} \le \sqrt{2}||v||_\Lambda^{\frac{1}{2}}|v|_{1,\Lambda}^{\frac{1}{2}}. \tag{3.1}$$

Moreover for any $v \in H_0^1(\Lambda_2)$, let $v = \omega_2^{\frac{1}{2}}u$. Then

$$||v||_{1,\Lambda_2} \le 2|u|_{1,\omega_2,\Lambda_2}. \tag{3.2}$$

Proof. For any $v \in H_0^1(\Lambda)$ and $x \in \Lambda$,

$$v^2(x) = 2\int_{-1}^x v(y)\partial_y v(y)dy \le 2||v||_\Lambda |v|_{1,\Lambda}.$$

Thus (3.1) follows. Next, let $v \in H_0^1(\Lambda_2)$ and $v = \omega_2^{\frac{1}{2}}u$. By integration by parts,

$$\begin{aligned} |v|^2_{1,\Lambda_2} &= (\omega_2^{\frac{1}{2}}\partial_x u - \tfrac{1}{2}\omega_2^{\frac{1}{2}}u, \omega_2^{\frac{1}{2}}\partial_x u - \tfrac{1}{2}\omega_2^{\frac{1}{2}}u)_{\Lambda_2} \\ &= |u|^2_{1,\omega_2,\Lambda_2} - (\partial_x u, u)_{\omega_2,\Lambda_2} + \tfrac{1}{4}||v||^2_{\Lambda_2} \\ &= |u|^2_{1,\omega_2,\Lambda_2} - \tfrac{1}{4}||v||^2_{\Lambda_2} \end{aligned}$$

whence

$$|v|^2_{1,\Lambda_2}+\frac{1}{4}||v||^2_{\Lambda_2}=|u|^2_{1,\omega_2,\Lambda_2}.$$

We now turn to some orthogonal projections. The $L^2(\Lambda)$-orthogonal projection $P_N : L^2(\Lambda)\to S_N$, is such a mapping that for any $v\in L^2(\Lambda)$,

$$(P_N v-v,\phi)_\Lambda=0,\qquad \forall\phi\in S_N, \tag{3.3}$$

The $H^1(\Lambda)$-orthogonal projection $P^1_N : H^1(\Lambda)\to V_N$, is such a mapping that for any $v\in H^1(\Lambda)$,

$$a(P^1_N v-v,\phi)+(P^1_N v-v,\phi)_\Lambda=0,\qquad \forall\phi\in V_N. \tag{3.4}$$

The $H^1_0(\Lambda)$-orthogonal projection $P^{1,0}_N : H^1_0(\Lambda)\to V^0_N$, is such a mapping that for any $v\in H^1_0(\Lambda)$,

$$a(P^{1,0}_N v-v,\phi)=0,\qquad \forall\phi\in V^0_N. \tag{3.5}$$

We shall also use the $L^2(\Lambda)$-orthogonal projection $P^0_N : L^2(\Lambda)\to V^0_N$ such that for any $v\in L^2(\Lambda)$,

$$(P^0_N v-v,\phi)_\Lambda=0,\qquad \forall\phi\in V^0_N. \tag{3.6}$$

For technical reason, we introduce another space as in Bernardi and Maday [10]. Let $\chi=\omega_2$ or $\chi\equiv 1$. For any $\alpha>0$, define the space

$$H^{r_2}_\chi(\Lambda_2,\alpha)=\{v \mid v\in H^{r_2}_\chi(\Lambda_2),\quad (x-1)^{\frac{\alpha}{2}}v\in H^{r_2}_\chi(\Lambda_2)\ \}$$

with the norm

$$||v||_{r_2,\chi,\Lambda_2,\alpha}=||x^{\frac{\alpha}{2}}v||_{r_2,\chi,\Lambda_2}.$$

We shall drop the subscript χ whenever $\chi\equiv 1$. Moreover let

$$H^{r_1,r_2}(\Lambda,\alpha)=\{v \mid v|_{\Lambda_1}\in H^{r_1}(\Lambda_1),\ \ v|_{\Lambda_2}\in H^{r_2}(\Lambda_2,\alpha)\}$$

with the norm

$$||v||_{r_1,r_2,\Lambda,\alpha}=(||v||^2_{r_1,\Lambda_1}+||v||^2_{r_2,\Lambda_2,\alpha})^{\frac{1}{2}}.$$

Theorem 3.1. *Let $r_1,r_2\geq 0$ and α be the biggest integer for which $\alpha<r_2+1$. Then for any $v\in H^{r_1,r_2}(\Lambda,\alpha)$,*

$$||P_N v-v||_\Lambda\leq cN_1^{-r_1}||v||_{r_1,\Lambda_1}+cN_2^{-\frac{r_2}{2}}||v||_{r_2,\Lambda_2,\alpha}. \tag{3.7}$$

Proof. Let $P^{(1)}_{N_1} : L^2(\Lambda_1)\to\mathcal{P}_{N_1,1}$ be the $L^2(\Lambda_1)$-orthogonal projection, and $P^{(2)}_{N_2} : L^2_{\omega_2}(\Lambda_2)\to\mathcal{P}_{N_2,2}$ be the $L^2_{\omega_2}(\Lambda_2)$-orthogonal projection. Let $v_j=v|_{\Lambda_j}$, and

$$v^*_N=\begin{cases} P^{(1)}_{N_1}v_1, & \text{if } x\in\Lambda_1,\\ \omega_2^{\frac{1}{2}}P^{(2)}_{N_2}(\omega_2^{-\frac{1}{2}}v_2), & \text{if } x\in\Lambda_2.\end{cases}$$

Next, for any $\phi\in S_N$, let $\phi_1=\phi|_{\Lambda_1}\in\mathcal{P}_{N_1,1}, \phi_2=\phi|_{\Lambda_2}=\omega_2^{\frac{1}{2}}\psi_2, \psi_2\in\mathcal{P}_{N_2,2}$. It can be checked that

$$(v^*_N-v,\phi)_\Lambda=(P^{(1)}_{N_1}v_1-v_1,\phi_1)_{\Lambda_1}+(P^{(2)}_{N_2}(\omega_2^{-\frac{1}{2}}v_2)-\omega_2^{-\frac{1}{2}}v_2,\psi_2)_{\omega_2,\Lambda_2}=0.$$

Therefore $P_N v = v_N^*$. Now let $\mathcal{I}$ be the identity operator. By virtue of the approximation results of $P_{N_1}^{(1)}$ and $P_{N_2}^{(2)}$(see Bernardi and Maday [10]), we deduce that

$$\begin{aligned} ||P_N v - v||_\Lambda^2 &= ||P_{N_1}^{(1)} v_1 - v_1||_{\Lambda_1}^2 + ||(P_{N_2}^{(2)} - \mathcal{I})(\omega_2^{-\frac{1}{2}} v_2)||_{\omega_2,\Lambda_2}^2 \\ &\leq c N_1^{-2r_1} \|v_1\|_{r_1,\Lambda_1}^2 + c N_2^{-r_2} \|\omega_2^{-\frac{1}{2}} v_2\|_{r_2,\omega_2,\Lambda_2,\alpha}^2. \end{aligned}$$

Then (3.7) follows.

Theorem 3.2. *Let $r_1, r_2 \geq 1$ and α be the biggest integer for which $\alpha < r_2$. Then for any $v \in H^{r_1,r_2}(\Lambda,\alpha) \bigcap H^1(\Lambda)$,*

$$||P_N^1 v - v||_{1,\Lambda} \leq c N_1^{1-r_1} |v|_{r_1,\Lambda_1} + c N_2^{\frac{1}{2}-\frac{r_2}{2}} \|v\|_{r_2,\Lambda_2,\alpha}. \tag{3.8}$$

Proof. We have

$$||P_N^1 v - v||_{1,\Lambda} \leq \inf_{\phi \in V_N} ||\phi - v||_{1,\Lambda}.$$

Let

$$\phi(x) = \begin{cases} \phi_1(x), & \text{if } x \in \Lambda_1, \\ \phi_2(x), & \text{if } x \in \Lambda_2 \end{cases} \tag{3.9}$$

where

$$\phi_1(x) = \int_{-1}^x P_{N_1-1}^{(1)} \partial_y v(y) dy + v(-1), \quad x \in \Lambda_1,$$

$$\phi_2(x) = \omega^{\frac{1}{2}}(x)(\int_1^x P_{N_2-1}^{(2)} \partial_y (\omega^{-\frac{1}{2}}(y) v(y)) dy + v(1)), \quad x \in \Lambda_2.$$

Since $\phi_1(1) = \phi_2(1) = v(1)$, we assert that $\phi \in V_N$ and $(\phi - v)|_{\Lambda_j} \in H_0^1(\Lambda_j), j = 1, 2$. On the other hand,

$$v(x) = \omega^{\frac{1}{2}}(x)(\int_1^x \partial_y (\omega^{-\frac{1}{2}}(y) v(y)) dy + v(1)), \quad x \in \Lambda_2.$$

Thus we have from (3.2) and the approximation results of $P_{N_1}^{(1)}$ and $P_{N_2}^{(2)}$(see Bernardi and Maday [10]) that

$$\begin{aligned} ||P_N^1 v - v||_{1,\Lambda}^2 &\leq \|\phi - v\|_{1,\Lambda_1}^2 + \|\phi - v\|_{1,\Lambda_2}^2 \\ &\leq c|\phi - v|_{1,\Lambda_1}^2 + c|\omega^{-\frac{1}{2}}(\phi - v)|_{1,\omega_2,\Lambda_2}^2 \\ &= ||P_{N_1-1}^{(1)} \partial_x v - \partial_x v||_{\Lambda_1}^2 + c||(P_{N_2-1}^{(2)} - \mathcal{I}) \partial_x (\omega^{-\frac{1}{2}} v)||_{\omega_2,\Lambda_2}^2 \\ &\leq c N_1^{2-2r_1} |\partial_x v|_{r_1-1,\Lambda_1}^2 + c N_2^{1-r_2} \|\partial_x (\omega^{-\frac{1}{2}} v)\|_{r_2-1,\omega_2,\Lambda_2,\alpha}^2 \\ &\leq c N_1^{2-2r_1} |v|_{r_1,\Lambda_1}^2 + c N_2^{1-r_2} \|v\|_{r_2,\Lambda_2,\alpha}^2. \end{aligned}$$

Theorem 3.3. *Let $r_1, r_2 \geq 1$ and α be the biggest integer for which $\alpha < r_2$. Then for any $v \in H^{r_1,r_2}(\Lambda,\alpha) \bigcap H_0^1(\Lambda)$,*

$$||P_N^{1,0} v - v||_{1,\Lambda} \leq c N_1^{1-r_1} |v|_{r_1,\Lambda_1} + c N_2^{\frac{1}{2}-\frac{r_2}{2}} \|v\|_{r_2,\Lambda_2,\alpha}. \tag{3.10}$$

Proof. We reach the conclusion by an argument as in the proof of Theorem 3.2.

Theorem 3.4. *If the conditions of Theorem 3.3 hold, then*

$$||P_N^0 v - v||_\Lambda \leq c N_1^{-r_1} |v|_{r_1,\Lambda_1} + c N_2^{\frac{1}{2}-\frac{r_2}{2}} \|v\|_{r_2,\Lambda_2,\alpha}. \tag{3.11}$$

Proof. Let ϕ be the same as in (3.9). Then

$$||P_N^0 v - v||_\Lambda^2 \leq ||\phi - v||_\Lambda^2 \leq ||\phi - v||_{\Lambda_1}^2 + ||\phi - v||_{\Lambda_2}^2. \tag{3.12}$$

Set

$$G(x)=\int_{-1}^{x} g(x)dx, \quad g(x)=\phi(x)-v(x).$$

By integration by parts, we derive that

$$\begin{aligned}(\phi-v,g)_{\Lambda_1} &= (\partial_x\phi-\partial_x v,G)_{\Lambda_1}\\ &= (P^{(1)}_{N_1-1}\partial_x v-\partial_x v, G-P^{(1)}_{N_1-1}G)_{\Lambda_1}\\ &\le cN^{-r_1}|\partial_x v|_{r_1-1,\Lambda_1}|\partial_x G|_{1,\Lambda_1}\\ &\le cN^{-r_1}|v|_{r_1,\Lambda_1}||g||_{\Lambda_1}.\end{aligned}$$

Therefore by a duality argument,

$$||\phi-v||_{\Lambda_1}\le cN_1^{-r_1}|v|_{r_1,\Lambda_1}. \tag{3.13}$$

Finally we obtain from (3.12), (3.13) and the estimate for $||\phi-v||_{1,\Lambda_2}$ in the proof of Theorem 3.2 that

$$||P^0_N v-v||_\Lambda \le c(||\phi-v||_{\Lambda_1}+||\phi-v||_{1,\Lambda_2})\le cN_1^{-r_1}|v|_{r_1,\Lambda_1}+cN_2^{\frac{1}{2}-\frac{r_2}{2}}||v||_{r_2,\Lambda_2,\alpha}.$$

4. Error Estimations

In this section, we analyze the errors.

Lemma 4.1. *Let U and u_N be the solutions of (2.2) and (2.8). If $f\in L^2(0,T;L^2(\Lambda))$ and $U_0\in L^2(\Lambda)$, then for all $0\le t\le T$,*

$$||U(t)||^2_\Lambda+\mu\int_0^t |U(s)|^2_{1,\Lambda}ds\le c(||f||^2_{L^2(0,T;L^2(\Lambda))}+||U_0||^2_\Lambda), \tag{4.1}$$

$$||u_N(t)||^2_\Lambda+\mu\int_0^t |u_N(s)|^2_{1,\Lambda}ds\le c(||f||^2_{L^2(0,t;L^2(\Lambda))}+||U_0||^2_\Lambda). \tag{4.2}$$

Proof. By taking $v=2U$ in (2.2), we find that

$$\partial_t||U(t)||^2_\Lambda+2\mu|U(t)|^2_{1,\Lambda}\le 2(f(t),U(t))_\Lambda\le ||f(t)||^2_\Lambda+||U(t)||^2_\Lambda.$$

Then (4.1) follows from the Gronwell lemma. We can prove (4.2) similarly.

We now analyze the stability of scheme (2.8). Assume that f and $u_{N,0}$ have the errors $\tilde f$ and $\tilde u_{N,0}$, respectively, which induce the error of the numerical solution u_N, denoted by $\tilde u_N$. Then

$$\begin{cases}(\partial_t\tilde u_N(t),\phi)_\Lambda+b(\tilde u_N(t),\tilde u_N(t),\phi)+2b(\tilde u_N(t),u_N(t),\phi)\\ +\mu a(\tilde u_N(t),\phi)=(\tilde f(t),\phi)_\Lambda, \quad \forall\phi\in V^0_N, \ \ 0<t\le T,\\ \tilde u_N(0)=\tilde u_{N,0}.\end{cases} \tag{4.3}$$

By taking $\phi=2\tilde u_N$ in (4.3), it follows that

$$\partial_t||\tilde u_N(t)||^2_\Lambda+2\mu|\tilde u_N(t)|^2_{1,\Lambda}+4b(\tilde u_N(t),u_N(t),\tilde u_N(t))\le ||\tilde f(t)||^2_\Lambda+||\tilde u_N(t)||^2_\Lambda. \tag{4.4}$$

Let c^* be a positive constant depending only on $||f||^2_{L^2(0,T;L^2(\Lambda))}$ and $||U_0||^2_\Lambda$. By (3.1) and (4.2), we have that

$$\begin{aligned}|b(\tilde u_N(t),u_N(t),\tilde u_N(t))| &\le c^*||\tilde u_N(t)||_{\infty,\Lambda}||u_N(t)||_\Lambda|\tilde u_N(t)|_{1,\Lambda}\\ &\le c^*||\tilde u_N(t)||^{\frac{1}{2}}_\Lambda|\tilde u_N(t)|^{\frac{3}{2}}_{1,\Lambda}\le \mu|\tilde u_N(t)|^2_{1,\Lambda}+\tfrac{c^*}{\mu}||\tilde u_N(t)||^2_\Lambda.\end{aligned}$$

For description of errors, set

$$E(v,t)=||v(t)||_\Lambda^2+\mu\int_0^t |v(s)|_{1,\Lambda}^2 ds,$$

$$\rho(v,g,t)=||v_0||_\Lambda^2+\int_0^t \|g(s)\|_\Lambda^2 ds.$$

By integrating (4.4) for t, we get that

$$E(\tilde{u}_N,t)\le c\rho(\tilde{u}_{N,0},\tilde{f},t)+c^*\int_0^t E(\tilde{u}_N,s)ds.$$

Therefore we obtain the following result.

Theorem 4.1. *Let u_N be the solution of (2.8), and $\tilde{u}_N$ be its error induced by $\tilde{f}$ and $\tilde{u}_{N,0}$. Then for all $0\le t\le T$,*

$$E(\tilde{u}_N,t)\le c^*\rho(\tilde{u}_{N,0},\tilde{f},t). \tag{4.5}$$

We next turn to the convergence of (2.8). We may take $u_{N,0}=P_N^0 U_0$. Let $U_N=P_N^{1,0}U$. Then it follows from (2.2) and (3.5) that

$$\begin{cases} (\partial_t U_N(t),\phi)_\Lambda+b(U_N(t),U_N(t),\phi)+\mu a(U_N(t),\phi)\\ +G_1(t,\phi)+G_2(t,\phi)=(f(t),\phi)_\Lambda, \quad \forall \phi\in V_N^0, \quad 0<t\le T,\\ U_N(0)=P_N^{1,0}U_0 \end{cases} \tag{4.6}$$

where

$$\begin{aligned} G_1(t,\phi) &= (\partial_t U(t)-\partial_t U_N(t),\phi)_\Lambda,\\ G_2(t,\phi) &= b(U(t),U(t),\phi)-b(U_N(t),U_N(t),\phi). \end{aligned}$$

Furthermore let $\tilde{U}_N=u_N-U_N$. We derive from (2.8) and (4.6) that

$$(\partial_t\tilde{U}_N(t),\phi)_\Lambda+b(\tilde{U}_N(t),\tilde{U}_N(t),\phi)+2b(\tilde{U}_N(t),U_N(t),\phi)+\mu a(\tilde{U}_N(t),\phi)$$

$$=G_1(t,\phi)+G_2(t,\phi), \qquad \forall\phi\in V_N^0, \quad 0<t\le T. \tag{4.7}$$

In addition, $\tilde{U}_N(0)=P_N^0U_0-P_N^{1,0}U_0$. Comparing (4.7) with (4.3), we can derive an estimation like (4.5). But u_N and $\tilde{u}_N$ are now replaced by U_N and $\tilde{U}_N$, respectively. Thus it suffices to estimate $||P_N^0U_0-P_N^{1,0}U_0||_\Lambda$ and $|G_j(t,\tilde{U}_N(t))|, j=1,2$. Let $U_N(t)=U(t)+W_N(t)$. Then

$$G_1(t,\tilde{U}_N(t))=-(\partial_t W_N,\tilde{U}_N(t))_\Lambda,$$

$$G_2(t,\tilde{U}_N(t))=-b(W_N(t),W_N(t),\tilde{U}_N(t))-2b(W_N(t),U(t),\tilde{U}_N(t)).$$

Clearly

$$|G_1(t,\tilde{U}_N(t))|\le c||\tilde{U}_N(t)||_\Lambda^2+c||\partial_t W_N||_\Lambda^2.$$

Next, we have that

$$|b(W_N(t),W_N(t),\tilde{U}_N(t))|\le\mu|\tilde{U}_N(t)|_{1,\Lambda}^2+\frac{c}{\mu}||U(t)||_{\infty,\Lambda}^2||W_N(t)||_\Lambda^2.$$

We can estimate $|b(W_N(t),U(t),\tilde{U}_N(t))|$ similarly. Finally we use Theorems 3.3 and 3.4 , and the above estimates to obtain the following result.

Theorem 4.2. *Let $r_1, r_2 \geq 1$ and α be the biggest integer for which $\alpha < r_2$. If $U \in H^1(0,T; H^{r_1,r_2}(\Lambda,\alpha) \bigcap H_0^1(\Lambda))$, then for all $0 \leq t \leq T$,*

$$||U(t) - u_N(t)||_{L^\infty(0,T;L^2(\Lambda)) \cap L^2(0,T;H^1(\Lambda))} \leq d^*(N_1^{1-r_1} + N_2^{\frac{1}{2}-\frac{r_2}{2}})$$

where d^* is a positive constant depending only on μ, c^* and the norms of U in the mentioned spaces.

Remark 4.1. We can use the Laguerre spectral method with the mode N to solve (2.2), see Guo and Shen [6]. Let $r \geq 1$ and α be the biggest integer for which $\alpha < r$. If $U \in H^1(0,T; H^r(\Lambda,\alpha) \bigcap H_0^1(\Lambda))$, then for all $0 \leq t \leq T$,

$$||U(t) - u_N(t)||_{L^\infty(0,T;L^2(\Lambda)) \cap L^2(0,T;H^1(\Lambda))} \leq d^* N^{\frac{1}{2}-\frac{r}{2}}.$$

Comparing this fact with Theorem 4.2, we know that the composite Legendre-Laguerre spectral method improves the accuracy.

Remark 4.2. Theorem 4.2 shows the spectral accuracy of (2.8). But the error estimate is not optimal, since (3.11) and the estimates for $|G_1(t, \tilde{U}_N(t))|$ and $|G_2(t, \tilde{U}_N(t))|$ are not very precise. The main reason is that so fare, we can not use a duality argument to derive a better estimate for $||P_N^{1,0} v - v||_\Lambda$. This is an open problem in the Laguerre approximation. Indeed, it is caused by the index α in the definition of the space $H^r(\Lambda_2, \alpha)$. Recently Mostroianni and Monegato [11] introduced another space without the index α, and got nice approximation result. It was used for numerical solutions of certain integral equations. But it still seems difficult to use it in the duality argument for deriving the optimal estimation of $||P_N^{1,0} v - v||_\Lambda$.

5. Two-Dimensional Problems

As pointed out in the first section, a more important motivation of this work is to use it for multiple-dimensional problems. In particular, it is also available for exterior problems.

We first consider the domain $\Omega = \{(x,y) \mid -1 < x < \infty,\ 0 < y < \infty\}$. We divide it into two subdomains Ω_1 and Ω_2,

$$\Omega_1 = \{(x,y) \mid -1 < x < 1,\ 0 < y < \infty\}, \qquad \Omega_2 = \{(x,y) \mid 1 \leq x < \infty,\ 0 < y < \infty\}.$$

In this case, we take the composite weight function as

$$\omega(x,y) = \begin{cases} e^{-y}, & \text{in } \Omega_1, \\ e^{1-x-y}, & \text{in } \Omega_2. \end{cases}$$

Let $M = (M_1, M_2)$ and $N = (N_1, N_2)$. The approximation space $V_{M,N}$ is defined by

$$V_{M,N} = \{v \in H^1(\Lambda) \mid v|_{\Omega_j} = \sqrt{\omega} \sum_{m_j=0}^{M_j} \sum_{n_j=0}^{N_j} \hat{v}_{m_j,n_j}^{(j)} G_{m_j,n_j}^{(j)}(x,y), j = 1,2\}$$

where

$$G_{l,k}^{(j)}(x,y) = \begin{cases} L_l(x)\mathcal{L}_k(y), & j = 1, \\ \mathcal{L}_l(x-1)\mathcal{L}_k(y), & j = 2. \end{cases}$$

This approximation can be used for numerical solutions of differential equations in this domain.

We next consider a non-rectangular domain Ω which is divided into three subdomains Ω_1, Ω_2 and Ω_3,

$$\begin{aligned}\Omega_1 &= \{(x,y) \mid -1 < x < 1,\ 1 < y < \infty\},\\ \Omega_2 &= \{(x,y) \mid 1 \le x < \infty,\ 1 \le y < \infty\},\\ \Omega_3 &= \{(x,y) \mid 1 < x < \infty,\ 0 < y < 1\}.\end{aligned}$$

In this case, we take the composite weight function as

$$\omega(x,y) = \begin{cases} e^{1-y}, & \text{in } \Omega_1,\\ e^{2-x-y}, & \text{in } \Omega_2,\\ e^{1-x}, & \text{in } \Omega_3.\end{cases}$$

Let $M = (M_1, M_2, M_3)$ and $N = (N_1, N_2, N_3)$. The approximation space $V_{M,N}$ is defined by

$$V_{M,N} = \{v \in H^1(\Lambda) \mid v|_{\Omega_j} = \sqrt{\omega} \sum_{m_j=0}^{M_j} \sum_{n_j=0}^{N_j} \hat{v}^{(j)}_{m_j,n_j} G^{(j)}_{m_j,n_j}(x,y), j = 1,2,3\}$$

where

$$G^{(j)}_{l,k}(x,y) = \begin{cases} L_l(x)\mathcal{L}_k(y-1), & j=1,\\ \mathcal{L}_l(x-1)\mathcal{L}_k(y-1), & j=2,\\ \mathcal{L}_l(x-1)L_k(2y-1), & j=3.\end{cases}$$

This approximation can be used for numerical solutions of differential equations in this non-rectangular unbounded domain. We also can mix this method with the technique in Section 2 to develop some more precise algorithms. For instance, we divide Ω_1, Ω_2 and Ω_3 into several subdomains near the corner $(x,y) = (1,1)$, and so simulate the exact solutions more precisely.

We now deal with an exterior problem. Assume that the obstacle is a square $\Omega_0 = \{(x,y) \mid -1 \le x, y \le 1\}$, and the differential equation is defined in the domain $\Omega = \mathcal{R}^2 - \Omega_0$. We decompose it as $\Omega = \bigcup_{j=1}^{8} \Omega_j$ where

$$\begin{aligned}&\Omega_1 = \{(x,y)|\ -1 \le x \le 1, 1 < y < \infty\}, && \Omega_2 = \{(x,y)|\ -\infty < x < -1, 1 < y < \infty\},\\ &\Omega_3 = \{(x,y)|\ -\infty < x < -1, -1 \le y \le 1\}, && \Omega_4 = \{(x,y)|\ -\infty < x < -1, -\infty < y < -1\},\\ &\Omega_5 = \{(x,y)|\ -1 \le x \le 1, -\infty < y < -1\}, && \Omega_6 = \{(x,y)|\ 1 < x < \infty, -\infty < y < -1\},\\ &\Omega_7 = \{(x,y)|\ 1 < x < \infty, -1 \le y \le 1\}, && \Omega_8 = \{(x,y)|\ 1 < x < \infty, 1 < y < \infty\}.\end{aligned}$$

The corresponding composite weight function is

$$\omega(x,y) = \begin{cases} e^{-y+1}, & \text{in } \Omega_1,\\ e^{x-y+2}, & \text{in } \Omega_2,\\ e^{x+1}, & \text{in } \Omega_3,\\ e^{x+y+2}, & \text{in } \Omega_4,\\ e^{y+1}, & \text{in } \Omega_5.\\ e^{-x+y+2}, & \text{in } \Omega_6,\\ e^{-x+1}, & \text{in } \Omega_7,\\ e^{-x-y+2}, & \text{in } \Omega_8.\end{cases}$$

The approximation space $V_{M,N}$ is defined by

$$V_{M,N} = \{v \in H^1(\Lambda) \mid v|_{\Omega_j} = \sqrt{\omega} \sum_{m_j=0}^{M_j} \sum_{n_j=0}^{N_j} \hat{v}^{(j)}_{m_j,n_j} G^{(j)}_{m_j,n_j}(x,y), 1 \le j \le 8\}$$

where

$$G^{(j)}_{l,k}(x,y) = \begin{cases} L_l(x)\mathcal{L}_k(y-1), & j=1, \\ \mathcal{L}_l(-x-1)\mathcal{L}_k(y-1), & j=2, \\ \mathcal{L}_l(-x-1)L_k(y), & j=3, \\ \mathcal{L}_l(-x-1)\mathcal{L}_k(-y-1), & j=4, \\ L_l(x)\mathcal{L}_k(-y-1), & j=5, \\ \mathcal{L}_l(x-1)\mathcal{L}_k(-y-1), & j=6, \\ \mathcal{L}_l(x-1)L_k(y), & j=7, \\ \mathcal{L}_l(x-1)\mathcal{L}_k(y-1), & j=8. \end{cases}$$

If we want to describe the exact solution near the obstacle more precisely, then we can use the domain decomposition $\Omega = \bigcup_{j=1}^{12} \Omega_j$,

$$\begin{array}{ll} \Omega_1 = \{(x,y)|\ -1 \le x \le 1, 1 < y < 2\}, & \Omega_2 = \{(x,y)|\ -2 < x < -1, -2 < y < 2\}, \\ \Omega_3 = \{(x,y)|\ -1 \le x \le 1, -2 < y < -1\}, & \Omega_4 = \{(x,y)|\ 1 < x < 2, -2 < y < 2\}, \\ \Omega_5 = \{(x,y)|\ -2 \le x \le 2, 2 \le y < \infty\}, & \Omega_6 = \{(x,y)|\ -\infty < x \le -2, -2 \le y \le 2\}, \\ \Omega_7 = \{(x,y)|\ -2 \le x \le 2, -\infty < y \le -2\}, & \Omega_8 = \{(x,y)|\ 2 \le x < \infty, -2 \le y \le 2\} \\ \Omega_9 = \{(x,y)|\ -\infty < x < -2, 2 < y < \infty\}, & \Omega_{10} = \{(x,y)|\ -\infty < x < -2, -\infty < y < -2\}, \\ \Omega_{11} = \{(x,y)|\ 2 < x < \infty, -\infty < y < -2\}, & \Omega_{12} = \{(x,y)|\ 2 < x < \infty, 2 < y < \infty\}. \end{array}$$

We can construct the composite weight function and the corresponding approximation space in the same manner as in the previous paragraphs. Clearly we approximate the solution by the Legendre approximation near the obstacle Ω_0.

In practice, the obstacle Ω_0 may not be a square. Assume that it is contained in a square. In this case, we take the Legendre-like interpolation points as the nodes on the external boundary of this square. Then we use finite element method in this subdomain, and use the Legendre-Laguerre approximations and the two-dimensional Laguerre approximations in the remaining unbounded subdomains. We shall report the related results in the future.

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