Journal of Computational Mathematics, Vol.19, No.1, 2001, 67–76.

OPTIMAL MIXED $H-P$ FINITE ELEMENT METHODS FOR STOKES AND NON-NEWTONIAN FLOW*

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

Based upon a new mixed variational formulation for the three-field Stokes equations and linearized Non-Newtonian flow, an $h-p$ finite element method is presented with or without a stabilization. As to the variational formulation without stabilization, optimal error bounds in h as well as in p are obtained. As with stabilization, optimal error bounds are obtained which is optimal in h and one order deterioration in p for the pressure, that is consistent with numerical results in [9, 12] and therefore solved the problem therein. Moreover, we proposed a stabilized formulation which is optimal in both h and p.

Key words: Mixed hp- finite element method, Non-Newtonian flow, Stabilization, Scaled weak $B-B$ inequality.

1. Introduction

Motivated by some advantages of $h-p$ FEM over the classic FEM uncovered by recent computation works(see-[19]), Schwab and Süri [11] have considered the mixed $h-p$ finite element method for Non-Newtonian flow based upon a three-field Stokes formulation emanating from linearization of some different models of Non-Newtonian flow, in which, stress, velocity and pressure are coupled. Theoretical analysis and tailored numerical experiments show that the mixed $h-p$ finite element method exhibits an exponential convergence on geometrical graded meshes. However, optimal error bounds for both h and p are not available, and extra freedoms are needed for the stress if a continuous approximation is preferable, the latter is momentous when the equation has to be coupled with other equations in a big system or the problem is set up in high-dimensions. Though a modified EVSS method (Elastic Viscous Split Stress) [7] makes the drop of redundant freedoms possible, it gives rise to a non-symmetric system with an extra unknow which increases the complexity of computations.

Combined with the well-known stabilized FEM(see [8] for a survey), Schötzau, Gerdes and Schwab proposed a stabilized $h-p$ FEM in [13] and [9]. However, error bounds obtained therein are not consistent with numerical tests [9, 12]. It seems that such discrepancy is merely due to techniques employed.

The purpose of this paper is developing a unifying method, a stabilized $h-p$ FEM to resolve the above problem. Our method relies on a new variational formulation. The main advantage of this method is that the choice of finite element spaces for the stress is independent of those for the velocity and pressure. The ingredient in our analysis is the *scaled weak B-B inequality* proved in the $h-p$ setting and the *divide and conquer* principle.

Outline of the paper follows. In the next section, a new variational formulation is proposed and a finite element space pair is presented and analyzed. As a direct consequence of this variational formulation, an iterative algorithm is deduced in Section §3, convergence rate is also

* Received December 10, 1999.

estimated. In the last section, a stabilized $h-p$ FEM is formulated and the error bound is derived which is consistent with numerical results.

Throughout this paper, we assume that the constant C is independent of h and p.

2. New Variational Formulation for Upper Convected Maxwell Model

In the following, we only consider the upper convected Maxwell model, the simplest one in Non-Newtonian fluids, which can be described by the following equations:

$$-\operatorname{div}\boldsymbol{\sigma}+\boldsymbol{\nabla}p=\boldsymbol{f},\qquad \operatorname{div}\boldsymbol{u}=0,\qquad \boldsymbol{\sigma}+\lambda\frac{\delta\boldsymbol{\sigma}}{\delta t}=2\nu\mathcal{E}\boldsymbol{u},\tag{2.1}$$

where $\mathcal{E}\boldsymbol{u}$ is the strain rate tensor defined by the symmetric part of $\boldsymbol{\nabla u}$ as $\mathcal{E}\boldsymbol{u}=\frac{1}{2}(\boldsymbol{\nabla u}+(\boldsymbol{\nabla u})^T)$, $\delta\boldsymbol{\sigma}/\delta t$ is the upper convected derivative defined by

$$\frac{\delta\boldsymbol{\sigma}}{\delta t}=\frac{\partial\boldsymbol{\sigma}}{\partial t}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\sigma}-(\boldsymbol{\nabla u}\cdot\boldsymbol{\sigma}+\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla u})^T).\tag{2.2}$$

λ is the relaxation time of the material. Let $\lambda=0$, (2.1) reduces to

$$-\operatorname{div}\boldsymbol{\sigma}+\boldsymbol{\nabla}p=\boldsymbol{f},\qquad \operatorname{div}\boldsymbol{u}=0,\qquad \boldsymbol{\sigma}=2\nu\mathcal{E}\boldsymbol{u},\tag{2.3}$$

which is just the three-field Stokes problem.

We introduce some notations.

Let Ω be a bounded convex polygonal domain in $\mathcal{R}^2$ with the Lipschitz boundary Γ. $\mathcal{R}^2$ is equipped with Cartesian coordinates $x_i,\ i=1,2$. Denote by $(\cdot,\cdot)$ the $\mathcal{L}^2(\Omega)$ scalar product of functions, vectors or tensors. Defined the following Sobolev spaces: $\boldsymbol{T}=[\mathcal{L}^2(\Omega)]^4_{\text{sym}}=\{\tau=(\tau_{ij})\mid \tau_{ij}=\tau_{ji},\ \tau_{ij}\in\mathcal{L}^2(\Omega),\ i,j=1,2\}$ with the norm $\|\tau\|_{\boldsymbol{T}}=(\int_\Omega|\tau|^2)^{\frac{1}{2}}$, $\boldsymbol{X}=[H^1_0(\Omega)]^2, M=\mathcal{L}^2_0(\Omega)=\{q\in\mathcal{L}^2(\Omega)\mid \int_\Omega q=0\}$. X,M are equipped with the norm $\|\boldsymbol{v}\|_{\boldsymbol{X}}=(\int_\Omega|\mathcal{E}\boldsymbol{v}|^2)^{\frac{1}{2}}$, $\|q\|_M=(\int_\Omega|q|^2)^{\frac{1}{2}}$ respectively. It is easy to see that $\|\cdot\|_{\boldsymbol{X}}$ is an equivalent norm over $\boldsymbol{X}$.

With these notations, we state a new variational formulation as follows.

Find $(\boldsymbol{\sigma},\boldsymbol{u},p)\in\boldsymbol{T}\times\boldsymbol{X}\times M$ such that

$$\frac{\alpha}{2\nu}(\boldsymbol{\sigma},\boldsymbol{\tau})-\alpha(\boldsymbol{\tau},\mathcal{E}\boldsymbol{u})=0\qquad\forall\boldsymbol{\tau}\in\boldsymbol{T},\tag{2.4}$$

$$\alpha(\boldsymbol{\sigma},\mathcal{E}\boldsymbol{v})+2(1-\alpha)\nu(\mathcal{E}\boldsymbol{u},\mathcal{E}\boldsymbol{v})-(p,\operatorname{div}\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v})\qquad\forall\boldsymbol{v}\in\boldsymbol{X},\tag{2.5}$$

$$(\operatorname{div}\boldsymbol{u},q)=0\qquad\forall q\in M.\tag{2.6}$$

Remark 2.1. The above formulation is similar to the Oldroyd version of the Stokes problem [2] with $\nu=1$. However, a finite element discretization of the latter yields a non-symmetric algebraic system while the previous one gives rise to a symmetric system with a variant of α that accounts for the flexibility in applications.

To facilitate the analysis, we define two operators as follows

$$A_\alpha(\cdot,\cdot):\ \boldsymbol{T}\times\boldsymbol{X}\times\boldsymbol{T}\times\boldsymbol{X}\to\mathcal{R},$$

$$A_\alpha(\boldsymbol{\sigma},\boldsymbol{u};\boldsymbol{\tau},\boldsymbol{v})=\frac{\alpha}{2\nu}(\boldsymbol{\sigma},\boldsymbol{\tau})-\alpha(\boldsymbol{\tau},\mathcal{E}\boldsymbol{u})+\alpha(\boldsymbol{\sigma},\mathcal{E}\boldsymbol{v})+2(1-\alpha)\nu(\mathcal{E}\boldsymbol{u},\mathcal{E}\boldsymbol{v})\tag{2.7}$$

and

$$B(\cdot,\cdot):\ \boldsymbol{T}\times\boldsymbol{X}\times M\to\mathcal{R},$$

$$B(\boldsymbol{\tau},\boldsymbol{v};q)=-(p,\operatorname{div}\boldsymbol{v}).\tag{2.8}$$

Problem *H*: find $(\boldsymbol{\sigma},\boldsymbol{u},p)\in\boldsymbol{T}\times\boldsymbol{X}\times M$ such that

$$A_\alpha(\boldsymbol{\sigma},\boldsymbol{u};\boldsymbol{\tau},\boldsymbol{v})+B(\boldsymbol{\tau},\boldsymbol{v};p)=(\boldsymbol{f},\boldsymbol{v})\qquad\forall(\boldsymbol{\tau},\boldsymbol{v})\in(\boldsymbol{T},\boldsymbol{X}),\tag{2.9}$$

$$B(\boldsymbol{\sigma},\boldsymbol{u};q)=0 \quad \forall q\in M. \tag{2.10}$$

The existence and uniqueness of **Problem H** can be obtained by the classic saddle-point theory [5].

Let $(\boldsymbol{T}_N,\boldsymbol{X}_N,M_N)$ be the corresponding finite element space for $(\boldsymbol{\sigma},\boldsymbol{u},p)$ respectively, then the discrete approximation of **Problem H** is

Problem $\boldsymbol{H}_h$: find $(\boldsymbol{\sigma}_N,\boldsymbol{u}_N,p_N)\in\boldsymbol{T}_N\times\boldsymbol{X}_N\times M_N$ such that

$$A_\alpha(\boldsymbol{\sigma}_N,\boldsymbol{u}_N;\boldsymbol{\tau},\boldsymbol{v})+B(\boldsymbol{\tau},\boldsymbol{v};p_N)=(\boldsymbol{f},\boldsymbol{v}) \quad \forall(\boldsymbol{\tau},\boldsymbol{v})\in\boldsymbol{T}_N\times\boldsymbol{X}_N, \tag{2.11}$$

$$B(\boldsymbol{\sigma}_N,\boldsymbol{u}_N;q)=0 \quad \forall q\in M_N. \tag{2.12}$$

To guarantee the well-posedness of **Problem $\boldsymbol{H}_h$**, we need two assumptions:

1. *K-Ellipticity*: There exists a constant $\beta>0$ such that
$$A_\alpha(\boldsymbol{\sigma},\boldsymbol{u};\boldsymbol{\sigma},\boldsymbol{u})\geq\beta|||\boldsymbol{\sigma},\boldsymbol{u}|||^2 \tag{2.13}$$
for all
$$(\boldsymbol{\sigma},\boldsymbol{u})\in\boldsymbol{Z}_N=\{(\boldsymbol{\sigma},\boldsymbol{u})\in\boldsymbol{T}_N\times\boldsymbol{X}_N \mid B(\boldsymbol{\sigma},\boldsymbol{u};q)=0 \ \ \forall q\in M_N\}, \tag{2.14}$$
where $|||\boldsymbol{\sigma},\boldsymbol{u}|||^2:=\|\boldsymbol{\sigma}\|^2_{\boldsymbol{T}}+\|\boldsymbol{u}\|^2_{\boldsymbol{X}}$.

2. *B-B Inequality*: There exists a constant $\delta(N)$ such that
$$\sup_{(\boldsymbol{\tau},\boldsymbol{v})\in\boldsymbol{T}_N\times\boldsymbol{X}_N}\frac{B(\boldsymbol{\tau},\boldsymbol{v};p)}{|||\boldsymbol{\tau},\boldsymbol{v}|||}\geq\delta(N)\|p\|_M \quad \forall p\in M. \tag{2.15}$$

Under the above two assumptions, we have the following a-priori error estimates:

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_N\|_{\boldsymbol{T}}+\|\boldsymbol{u}-\boldsymbol{u}_N\|_{\boldsymbol{X}}\leq(1+\frac{\|A\|}{\beta})(1+\frac{\|B\|}{\delta(N)})\inf_{(\boldsymbol{\tau},\boldsymbol{v})\in\boldsymbol{T}_N\times\boldsymbol{X}_N}|||\boldsymbol{\sigma}-\boldsymbol{\tau},\boldsymbol{u}-\boldsymbol{v}|||$$
$$+\frac{\|B\|}{\beta}\inf_{q\in M_N}\|p-q\|_M, \tag{2.16}$$

$$\|p-p_N\|_M\leq\frac{\|A\|}{\delta(N)}(1+\frac{\|A\|}{\beta})(1+\frac{\|B\|}{\delta(N)})\inf_{(\boldsymbol{\tau},\boldsymbol{v})\in\boldsymbol{T}_N\times\boldsymbol{X}_N}|||\boldsymbol{\sigma}-\boldsymbol{\tau},\boldsymbol{u}-\boldsymbol{v}|||$$
$$+(1+\frac{\|A\|\|B\|}{\delta(N)\beta}+\frac{\|B\|}{\beta})\inf_{q\in M_N}\|p-q\|_M, \tag{2.17}$$

where $\|A\|,\|B\|$ is the norm of the bilinear form $A(\cdot,\cdot),B(\cdot,\cdot)$ respectively.

Since we only consider conforming finite element spaces, the first assumption holds automatically. As to the second one, there are many examples in [16] and [11]. We only list the following one.

Let $\mathcal{C}_n$ be a parallelogram or triangular mesh defined on Ω. Each element $K\in\mathcal{C}_n$ is affinely equivalent to either the reference square $\hat{K}=\hat{Q}=[-1,1]^2$ or the reference triangle $\hat{K}=\{(x,y)\mid 0<x<1, 0<y<x\}$. An affine mapping F_k maps $\hat{K}$ to K. The finite element spaces defined on the reference element are $\mathcal{L}^k=\text{span}\{\xi^{\alpha_1}\eta^{\alpha_2}, 0\leq\alpha_1,\alpha_2\leq k\}$ and $\mathcal{P}^k=\text{span}\{\xi^{\alpha_1}\eta^{\alpha_2}, 0\leq\alpha_1,\alpha_2,\alpha_1+\alpha_2\leq k\}$.

We define the finite element space for T_N, X_N and M_N as follows:

$$\boldsymbol{T}_N=\{\boldsymbol{\sigma}\in\boldsymbol{T}\mid \boldsymbol{\sigma}_{|_K}\circ F_k\in[\mathcal{L}^{k'}(\hat{K})]^4_{\text{sym}}, K\in\mathcal{C}_n\},$$

where $\mathcal{L}^{k'}(\hat{K})$ is the serendipity space for $k\geq 2$.

$$\boldsymbol{X}_N=\{\boldsymbol{v}\in\boldsymbol{X}\mid \boldsymbol{v}_{|_K}\circ F_k\in[\mathcal{L}^{k+1}(\hat{K})]^2, K\in\mathcal{C}_n\},$$

$$M_N = \{p \in M \mid p_{|_K} \circ F_k \in \mathcal{P}^k(\hat{K}), K \in \mathcal{C}_n\}.$$

As stated in the introduction, the choice of $\boldsymbol{T}_N$ is independent of $\boldsymbol{X}_N$ and M_N, therefore, how to chose $\boldsymbol{T}_N$, continuous or not is up to users, that is the main advantage of our method. The method in [18] is a special case of ours for $\alpha = 1/2$.

Lemma 2.1. [3] *The finite element space pairs defined above satisfy the B-B inequality (2.15) and admit the following estimate*

$$\delta(N) = C. \tag{2.18}$$

(2.16) and (2.17) together with Lemma 2.1 yield the following.

Theorem 2.1. *If* $(\boldsymbol{\sigma}, \boldsymbol{u}, p)$ *and* $(\boldsymbol{\sigma}_N, \boldsymbol{u}_N, p_N)$ *be the solutions of* **Problem $\boldsymbol{H}$** *and* **Problem $\boldsymbol{H}_h$** *respectively, then we have*

$$|||\boldsymbol{\sigma} - \boldsymbol{\sigma}_N, \boldsymbol{u} - \boldsymbol{u}_N||| + \|p - p_N\|_M \leq Ch^{\min(m-1,k+1)}k^{-m+1}(\|\boldsymbol{\sigma}\|_{m-1} + \|\boldsymbol{u}\|_m + \|p\|_{m-1}).$$

Remark 2.2. It is easy to see that this element is completely stable. Moreover, comparing to the so-called Modified EVSS, the resulting algebraic system is symmetric and no extra unknown is needed to introduce.

3. Iterative Algorithm

The algebraic system generated from **Problem $\boldsymbol{H}_h$** exhibits the following forms

$$\begin{aligned} C\Sigma - B_1^T U &= 0, \\ B_1\Sigma + AU + B_2^T P &= F, \\ B_2 U &= 0. \end{aligned} \tag{3.1}$$

Note that matrix C is positive and if a discontinuous approximation of the stress is assumed, then $\boldsymbol{\sigma}$ can be eliminated on each element level that yields

$$\begin{aligned} \hat{A}U + B_2^T P &= F, \\ B_2 U &= 0, \end{aligned} \tag{3.2}$$

where $\hat{A} = A + B_1 C^{-1} B_1^T$ is positive. Therefore, the system (3.2) is just the same as that from the two-field Stokes equations. It is well-known that there are many efficient algorithms for solving such a system.

Remark 3.1. The above elimination procedure is the discrete counterpart of the following argument.

In fact, as the discontinuous approximation of the stress tensor is exploited, we can define the following projection operator locally. Let $\Pi_{|_K} = \Pi_K$, and Π_K is defined as

$$\Pi_K : \boldsymbol{T}_K \to \boldsymbol{X}_K \quad \text{such that } (\Pi_K \boldsymbol{\sigma}, \boldsymbol{\tau}) = (\boldsymbol{\sigma}, \boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in \boldsymbol{T}_N.$$

Therefore, **Problem $\boldsymbol{H}_h$** reduces to the following one:

Find $(\boldsymbol{u}_N, p_N) \in (\boldsymbol{X}_N, M_N)$ such that

$$2\alpha\nu(\Pi\mathcal{E}\boldsymbol{u}_N, \Pi\mathcal{E}\boldsymbol{v}) + 2(1-\alpha)\nu(\mathcal{E}\boldsymbol{u}_N, \mathcal{E}\boldsymbol{v}) - (p_N, \operatorname{div}\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N, \tag{3.3}$$

$$(\operatorname{div}\boldsymbol{u}_N, q) = 0 \quad \forall q \in M_N \tag{3.4}$$

with $\boldsymbol{\sigma}_{N|_K} = \Pi_K \mathcal{E}\boldsymbol{u}_N$.

As to the continuous approximation of the stress tensor, we make use the iterative algorithm which naturally emanates from **Problem $\boldsymbol{H}$**.

Given $(\boldsymbol{\sigma}^n, \boldsymbol{u}^n, p^n) \in \boldsymbol{T}_N \times \boldsymbol{X}_N \times M_N$, then **Problem $\boldsymbol{H}_h$** can be solved by the following **Iterative Algorithm.** Find $(\boldsymbol{\sigma}^{n+1}, \boldsymbol{u}^{n+1}, p^{n+1}) \in \boldsymbol{T}_N \times \boldsymbol{X}_N \times M_N$ such that

$$\alpha(\boldsymbol{\sigma}^n, \mathcal{E}\boldsymbol{v}) + 2(1-\alpha)\nu(\mathcal{E}\boldsymbol{u}^{n+1}, \mathcal{E}\boldsymbol{v}) - (p^{n+1}, \operatorname{div}\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N, \tag{3.5}$$

$$(\operatorname{div} \boldsymbol{u}^{n+1}, q) = 0 \quad \forall q \in M_N, \tag{3.6}$$

$$\frac{\alpha}{2\nu}(\boldsymbol{\sigma}^{n+1}, \boldsymbol{\tau}) - \alpha(\boldsymbol{\tau}^{n+1}, \mathcal{E}\boldsymbol{u}) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{T}_N. \tag{3.7}$$

Theorem 3.1. *If $0 < \alpha < \frac{1}{2}$, then the above Iterative Algorithm converges to the solution of* **Problem $\boldsymbol{H}_h$**.

The proof is easy, we leave it(see also [10]).

Remark 3.2. Comparing to the **Iterative Algorithm** proposed in [10], our algorithm need not an accurate estimate of the constant involved into $B-B$ inequality, which is highly unrealistic in many cases.

4. Stabilized h-p Method

In this section, we will present a stabilized formulation for **Problem $\boldsymbol{H}$** in the context of $h-p$ FEM version. We introduce some more notations. The quasi-uniformity of $\mathcal{C}_n$ is assumed. Γ_n is the collection of all element boundaries of $\mathcal{C}_n, E$ the boundary of K, and $[,]$ is the jump across E. Let

$$R_m(K) = \begin{cases} \mathcal{P}_m(K), & \text{if K is a triangle,} \\ \mathcal{L}_m(K), & \text{if K is a quadrilateral,} \end{cases}$$

and

$$S(K) = \begin{cases} \mathcal{P}_2(K), & \text{if K is a triangle,} \\ \mathcal{L}_2(K), & \text{if K is a qudrilateral.} \end{cases}$$

$$\boldsymbol{T}_N = \{\boldsymbol{\sigma} \in \boldsymbol{T} \mid \boldsymbol{\sigma}_{|_K} \in [R_q(K)]^4_{\text{sym}}, K \in C_n\},$$

$$\boldsymbol{X}_N = \{\boldsymbol{u} \in \boldsymbol{X} \mid \boldsymbol{u}_{|_K} \in R_l(K)^2, K \in C_n\},$$

$$S_N = \{\boldsymbol{u} \in \boldsymbol{X} \mid \boldsymbol{v}_{|_K} \in S(K)^2, K \in C_n\}.$$

The presssure space M is approximated either continuously or discontinuously:

$$M_N = \{p \in M \cap \mathcal{C}^0(\overline{\Omega}) \mid p_{|_K} \in R_k(K), K \in C_n\}$$

or

$$M_N = \{p \in M \mid p_{|_K} \in R_k(K), K \in C_n\}.$$

It is reasonable to assume that $q = \mathcal{O}(k), l = \mathcal{O}(k)$.

Problem $\tilde{H}_h$: find $(\boldsymbol{\sigma}_N, \boldsymbol{u}_N, p_N) \in \boldsymbol{T}_N \times \boldsymbol{X}_N \times M_N$ such that

$$\mathcal{B}(\boldsymbol{\sigma}_N, \boldsymbol{u}_N, p_N; \boldsymbol{\tau}, \boldsymbol{v}, q) = F(\boldsymbol{\tau}, \boldsymbol{v}, q) \quad \forall (\boldsymbol{\tau}, \boldsymbol{v}, q) \in \boldsymbol{T}_N \times \boldsymbol{X}_N \times M_N \tag{4.1}$$

with

$$\begin{aligned} \mathcal{B}_\alpha(\boldsymbol{\sigma}, \boldsymbol{u}, p; \boldsymbol{\tau}, \boldsymbol{v}, q) = {} & \frac{\alpha}{2\nu}(\boldsymbol{\sigma}, \tau) - \alpha(\tau, \mathcal{E}\boldsymbol{u}) + \alpha(\boldsymbol{\sigma}, \mathcal{E}\boldsymbol{v}) + 2(1-\alpha)\nu(\mathcal{E}\boldsymbol{u}, \mathcal{E}\boldsymbol{v}) \\ & - (p, \operatorname{div} \boldsymbol{v}) + (\operatorname{div} \boldsymbol{u}, q) + \beta \sum_{K \in \mathcal{C}_n} h_K^2 k^{-4}(-\operatorname{div} \mathcal{E}\boldsymbol{u} + \boldsymbol{\nabla} p, -\operatorname{div} \mathcal{E}\boldsymbol{v} + \boldsymbol{\nabla} q) \\ & + \gamma \sum_{E \in \Gamma_n} h_E k^{-1} < [p], [q] >, \end{aligned} \tag{4.2}$$

and

$$F(\boldsymbol{\tau}, \boldsymbol{v}, q) = (\boldsymbol{f}, \boldsymbol{v}) + \beta \sum_{K \in \mathcal{C}_n} h_K^2 k^{-4}(\boldsymbol{f}, -\operatorname{div} \mathcal{E}\boldsymbol{v} + \boldsymbol{\nabla} q), \tag{4.3}$$

where $\alpha \in (0,1)$.

Remark 4.1. Let $\alpha = 0$ and $\gamma = 0$, the above formulation reduces to the previous work [4]. Our formulation includes an extra term $\gamma \sum_{E\in\Gamma_n} h_E k^{-1} < [p],[q] >$, which accomodates the discontinuous approximation of the pressure.

To prove the well-posedness and derive the error bounds of **Problem** $\tilde{H}_h$, we need a *scaled weak B-B inequality.*

Firstly we introduce the following semi-norm

$$|p|_{k,h}^2 = \beta \sum_{K\in\mathcal{C}_n} h_K^2 k^{-2} \|\boldsymbol{\nabla} p\|_{0,K}^2 + \gamma \sum_{E\in\Gamma_n} h_E k^{-1} \int_E |[p]|^2. \tag{4.4}$$

The scaled norm admits the optimal approximation property:

$$\inf_{q\in M_N} |p-q|_{k,h} \le C h^{m-1} k^{-m+1} \|p\|_{m-1}. \tag{4.5}$$

Theorem 4.1. *If $\boldsymbol{S}_N \subset \boldsymbol{X}_N$, then there exist constants C_1 and C_2 such that*

$$\sup_{\boldsymbol{v}\in\boldsymbol{X}_N} \frac{(\operatorname{div}\boldsymbol{v}, p)}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} \ge C_1 \|p\|_M - C_2 k |p|_{k,h} \qquad \forall p \in M_N. \tag{4.6}$$

If $M_N \subset C^0(\Omega)$ or $\gamma > 0$, then there exist constants C_1 and C_2 such that

$$\sup_{\boldsymbol{v}\in\boldsymbol{X}_N} \frac{(\operatorname{div}\boldsymbol{v}, p)}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} \ge C_1 \|p\|_M - C_2 |p|_{k,h} \qquad \forall p \in M_N. \tag{4.7}$$

Proof. We prove (4.6) firstly. Let Π be the L^2-projection from M_N onto the piecewise constant space, i.e.

$$\Pi q_{|K} = \frac{1}{|K|}\int_K q dx \qquad \forall K \in \mathcal{C}_n \ \text{ and } \ q \in M_N.$$

Since $\boldsymbol{S}_N \subset \boldsymbol{X}_N$, it is known that the pair $(\boldsymbol{X}_N, \Pi M_N)$ satisfies the $B-B$ inequality, i.e. there exists a constant C such that for all $p \in M_N$, there exists $\boldsymbol{v} \in \boldsymbol{X}_N$ with $\|\boldsymbol{v}\|_{\boldsymbol{X}} = 1$, and

$$(\operatorname{div}\boldsymbol{v}, \Pi p) \ge C_1 \|\Pi p\|_0.$$

$$\begin{aligned}
(\operatorname{div}\boldsymbol{v}, p) &= (\operatorname{div}\boldsymbol{v}, \Pi p) - (\operatorname{div}\boldsymbol{v}, \Pi p - p)\\
&\ge C_1\|\Pi p\|_M - \|\boldsymbol{v}\|_{\boldsymbol{X}} \|p - \Pi p\|_M\\
&\ge C_1\|p\|_M - (1+C_1)\|p-\Pi p\|_M\\
&\ge C_1\|p\|_M - (1+C_1)C_2 k (\sum_{K\in\mathcal{C}_h} h_K^2 k^{-2} \|\boldsymbol{\nabla} p\|_{0,K}^2)^{\frac12}\\
&\ge C_1\|p\|_M - Ck|p|_{k,h}.
\end{aligned} \tag{4.8}$$

We turn to the second case (4.7). Since $p \in M = \mathcal{L}_0^2(\Omega)$, then there exists $\boldsymbol{w} \in \boldsymbol{X}$ such that

$$(\operatorname{div}\boldsymbol{w}, p) \ge C_4 \|\boldsymbol{w}\|_{\boldsymbol{X}} \|p\|_M.$$

For $\boldsymbol{w}$, there exists an interpolation $r_h\boldsymbol{w} \in \boldsymbol{X}_N$ [15] such that

$$(\sum_{K\in\mathcal{C}_n} h_K^{-2} k^2 \|\boldsymbol{w} - r_h\boldsymbol{w}\|_{0,K}^2 + \sum_{E\in\Gamma_n} h_E^{-1} k \int_E |\boldsymbol{w} - r_h \boldsymbol{w}|^2)^{\frac12} \le C_5 \|\boldsymbol{w}\|_{\boldsymbol{X}},$$

and $\|r_h\boldsymbol{w}\|_{\boldsymbol{X}} \le C_6 \|\boldsymbol{w}\|_{\boldsymbol{X}}$. Integrating by parts on each $K \in \mathcal{C}_n$ and using the above estimate, we get

$$\begin{aligned}
(\operatorname{div} r_h \boldsymbol{w}, p) &= (\operatorname{div}(r_h \boldsymbol{w} - \boldsymbol{w}), p) + (\operatorname{div} \boldsymbol{w}, p) \\
&\geq (\operatorname{div}(r_h \boldsymbol{w} - \boldsymbol{w}), p) + C_4 \|\boldsymbol{w}\|_{\boldsymbol{X}} \|p\|_M \\
&\geq \sum_{K \in \mathcal{C}_n} (\boldsymbol{w} - r_h \boldsymbol{w}, \boldsymbol{\nabla} p) + \sum_{E \in \Gamma_n} \int_E (r_h \boldsymbol{w} - \boldsymbol{w}) \cdot \boldsymbol{n}[p] + C_4 \|\boldsymbol{w}\|_{\boldsymbol{X}} \|p\|_M \\
&\geq C_4 \|\boldsymbol{w}\|_{\boldsymbol{X}} \|p\|_M - (\sum_{K \in \mathcal{C}_n} h_K^{-2} k^2 \|\boldsymbol{w} - r_h \boldsymbol{w}\|_{0,K}^2 + \sum_{E \in \Gamma_n} h_E^{-1} k \int_E |\boldsymbol{w} - r_h \boldsymbol{w}|^2)^{\frac{1}{2}} \\
&\times (\sum_{K \in \mathcal{C}_n} h_K^2 k^{-2} \|\boldsymbol{\nabla} p\|_{0,K}^2 + \sum_{E \in \Gamma_n} h_E k^{-1} \int_E |[p]|^2)^{\frac{1}{2}} \\
&\geq \|\boldsymbol{w}\|_{\boldsymbol{X}} (C_4 \|p\|_M - C_7 |p|_{k,h}).
\end{aligned}$$

Finally, we have obtained the following estimate:

$$\begin{aligned}
\sup_{\boldsymbol{v} \in \boldsymbol{X}_N} \frac{(\operatorname{div} v, p)}{\|\boldsymbol{v}\|_{\boldsymbol{X}}} &\geq \frac{(\operatorname{div} r_h \boldsymbol{w}, p)}{\|r_h \boldsymbol{w}\|_{\boldsymbol{X}}} \\
&\geq \frac{\|\boldsymbol{w}\|_{\boldsymbol{X}}}{C_6 \|\boldsymbol{w}\|_{\boldsymbol{X}}} (C_4 \|p\|_M - C_7 |p|_{k,h}) \\
&\geq C_8 \|p\|_M - C_9 |p|_{k,h}.
\end{aligned}$$

Thus (4.7) is proved.

To get the well-posedness of **Problem** $\tilde{H}_h$, we need the following stability inequality. A special case of such stability inequality for **Problem** $\tilde{H}_h$ with $\alpha = 0$ and $\gamma = 0$ has been proved in [12], we will extend it to a more general case.

To faciliate our analysis, we introduce the following scaled norm

$$|u, p|_{k,h}^2 = \beta \sum_{K \in \mathcal{C}_n} h_K^2 k^{-4} \| - \operatorname{div} \mathcal{E} \boldsymbol{u} + \boldsymbol{\nabla} p\|_0^2.$$

Theorem 4.2. *If $(1-\alpha)\nu > \beta C_I^{-1}$, C_I is the inverse constant appeared in (4.12), and $\boldsymbol{S}_N \subset \boldsymbol{X}_N$, then there exists a constant C such that*

$$\sup_{(\boldsymbol{\tau}, \boldsymbol{v}, q) \in \boldsymbol{T}_N \times \boldsymbol{X}_N \times M_N} \frac{\mathcal{B}_\alpha(\boldsymbol{\sigma}, \boldsymbol{u}, p; \boldsymbol{\tau}, \boldsymbol{v}, q)}{(\|\boldsymbol{\sigma}\|_{\boldsymbol{T}} + \|\boldsymbol{u}\|_{\boldsymbol{X}} + \|p\|_M)(\|\boldsymbol{\tau}\|_{\boldsymbol{T}} + \|\boldsymbol{v}\|_{\boldsymbol{X}} + \|q\|_M)} \geq C k^{-4}. \tag{4.9}$$

If $M_N \subset C^0(\Omega)$ or $\gamma > 0$, then there exists a constant C_1 such that

$$\sup_{(\boldsymbol{\tau}, \boldsymbol{v}, q) \in \boldsymbol{T}_N \times \boldsymbol{X}_N \times M_N} \frac{\mathcal{B}_\alpha(\boldsymbol{\sigma}, \boldsymbol{u}, p; \boldsymbol{\tau}, \boldsymbol{v}, q)}{(\|\boldsymbol{\sigma}\|_{\boldsymbol{T}} + \|\boldsymbol{u}\|_{\boldsymbol{X}} + \|p\|_M)(\|\boldsymbol{\tau}\|_{\boldsymbol{T}} + \|\boldsymbol{v}\|_{\boldsymbol{X}} + \|q\|_M)} \geq C_1 k^{-2}. \tag{4.10}$$

Proof. We only prove (4.10). The proof of (4.9) is almost the same except we use (4.6) instead of (4.7). Note that

$$\begin{aligned}
\mathcal{B}_\alpha(\boldsymbol{\sigma}, \boldsymbol{u}, p; \boldsymbol{\sigma}, \boldsymbol{u}, p) &= \frac{\alpha}{2\nu} \|\boldsymbol{\sigma}\|_{\boldsymbol{T}}^2 + 2(1-\alpha)\nu \|\boldsymbol{u}\|_{\boldsymbol{X}}^2 \\
&\quad + |\boldsymbol{u}, p|_{k,h}^2 + \gamma \sum_{E \in \Gamma_n} h_E k^{-1} \int_E |[p]|^2.
\end{aligned} \tag{4.11}$$

Making use of the following inverse inequality: $\forall u \in \boldsymbol{X}_N$,

$$C_I \sum_{K \in \mathcal{C}_n} h_K^2 k^{-4} \| \operatorname{div} \mathcal{E} \boldsymbol{u}\|_0^2 \leq \|\mathcal{E} \boldsymbol{u}\|_0^2 = \|\boldsymbol{u}\|_{\boldsymbol{X}}^2, \tag{4.12}$$

the condition $(1-\alpha)\nu > \beta C_I^{-1}$ together with (4.11) give

$$\mathcal{B}_\alpha(\boldsymbol{\sigma},\boldsymbol{u},p;\boldsymbol{\sigma},\boldsymbol{u},p) \geq \frac{\alpha}{2\nu}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{T}} + (1-\alpha)\nu\|\boldsymbol{u}\|^2_{\boldsymbol{X}} + |\boldsymbol{u},p|^2_{k,h} + Ck^{-2}|p|^2_{k,h}. \tag{4.13}$$

By virtue of Theorem 4.1, there exists $\boldsymbol{z}\in\boldsymbol{X}_N$ such that

$$(\operatorname{div}\boldsymbol{z},p) = \|p\|^2_M - \|p\|_M|p|_{k,h}, \quad \text{with} \quad \|\boldsymbol{z}\|_{\boldsymbol{X}} \leq C\|p\|_M.$$

Thus

$$\begin{aligned}\mathcal{B}_\alpha(\boldsymbol{\sigma},\boldsymbol{u},p;0,-\boldsymbol{z},0) &= -\alpha(\boldsymbol{\sigma},\mathcal{E}\boldsymbol{z}) - 2(1-\alpha)(\mathcal{E}\boldsymbol{u},\mathcal{E}\boldsymbol{z}) + (\operatorname{div}\boldsymbol{z},p)\\ &\quad + \beta\sum_{K\in\mathcal{C}_n} h_K^2 k^{-4}(-\operatorname{div}\mathcal{E}\boldsymbol{u} + \boldsymbol{\nabla}p, -\operatorname{div}\mathcal{E}\boldsymbol{z})\\ &\geq -\alpha\|\boldsymbol{\sigma}\|_{\boldsymbol{T}}\|\boldsymbol{z}\|_{\boldsymbol{X}} - 2(1-\alpha)\|\boldsymbol{u}\|_{\boldsymbol{X}}\|\boldsymbol{z}\|_{\boldsymbol{X}}\\ &\quad + \|p\|^2_M - |p|_{k,h}\|p\|_M - \beta C_I^{-1/2}|u,p|_{k,h}\|p\|_M\\ &\geq \frac{1}{2}\|p\|^2_M - C_1\|\boldsymbol{\sigma}\|^2_{\boldsymbol{T}} - C_2\|\boldsymbol{u}\|^2_{\boldsymbol{X}} - C_3|\boldsymbol{u},p|^2_{k,h} - C_5|p|^2_{k,h}.\end{aligned} \tag{4.14}$$

For any $\delta > 0$, let $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}, \boldsymbol{v}_1 = \boldsymbol{u} - \delta\boldsymbol{z}, p_1 = p$, then

$$\begin{aligned}\mathcal{B}_\alpha(\boldsymbol{\sigma},\boldsymbol{u},p;\boldsymbol{\sigma}_1,\boldsymbol{v}_1,p_1) &= \mathcal{B}_\alpha(\boldsymbol{\sigma},\boldsymbol{u},p;\boldsymbol{\sigma},\boldsymbol{u}-\delta\boldsymbol{z},p)\\ &= \mathcal{B}_\alpha(\boldsymbol{\sigma},\boldsymbol{u},p;\boldsymbol{\sigma},\boldsymbol{u},p) - \delta\mathcal{B}_\alpha(\boldsymbol{\sigma},\boldsymbol{u},p;0,\boldsymbol{z},0)\\ &\geq \frac{\alpha}{2\nu}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{T}} + (1-\alpha)\nu\|\boldsymbol{u}\|^2_{\boldsymbol{X}} + |\boldsymbol{u},p|^2_{k,h} + Ck^{-2}|p|^2_{k,h}\\ &\quad + \delta(\frac{1}{2}\|p\|^2_M - C_1\|\boldsymbol{\sigma}\|^2_{\boldsymbol{T}} - C_2\|\boldsymbol{u}\|^2_{\boldsymbol{X}} - C_3|\boldsymbol{u},p|^2_{k,h} - C_5|p|^2_{k,h})\\ &\geq Ck^{-2}(\|\boldsymbol{\sigma}\|^2_{\boldsymbol{T}} + \|\boldsymbol{u}\|^2_{\boldsymbol{X}} + \|p\|^2_M)\end{aligned} \tag{4.15}$$

with $\delta = Ck^{-2}$. It is easy to see that

$$\|\boldsymbol{\tau}\|_{\boldsymbol{T}} + \|\boldsymbol{u}-\delta\boldsymbol{z}\|_{\boldsymbol{X}} + \|q\|_M \leq C(\|\boldsymbol{\sigma}\|_{\boldsymbol{T}} + \|\boldsymbol{u}\|_{\boldsymbol{X}} + \|p\|_M).$$

We come to the conclusion.

For all $(\boldsymbol{\tau},\boldsymbol{v},q)\in\boldsymbol{T}_N\times\boldsymbol{X}_N\times M_N$, define the norm

$$|||\boldsymbol{\tau},\boldsymbol{v},q|||^2 = \|\boldsymbol{\tau}\|^2_{\boldsymbol{T}} + \|\boldsymbol{v}\|^2_{\boldsymbol{X}} + |\boldsymbol{v},q|^2_{k,h} + |q|^2_{k,h}.$$

The following theorem is the main result of this section.

Theorem 4.3. *If $(1-\alpha)\nu > \beta C_I^{-1}$ and assume $(\boldsymbol{\sigma},\boldsymbol{u},p)$,$(\boldsymbol{\sigma}_N,\boldsymbol{u}_N,p_N)$ be the solution of* **Problem H**,**Problem** $\tilde{H}_h$ *respectively, then for the case $\gamma = 0$ and $\boldsymbol{S}_N\subset\boldsymbol{X}_N$, we get*

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_N\|_{\boldsymbol{T}} + \|\boldsymbol{u}-\boldsymbol{u}_N\|_{\boldsymbol{X}} + k^{-1}\|p-p_N\|_M &\leq C(h^{l_1}q^{-(m-1)}\|\boldsymbol{\sigma}\|_{m-1}\\ &\quad + h^{l_2}l^{-(m-1)}\|\boldsymbol{u}\|_m + h^{l_3}k^{-(m-1)}\|p\|_{m-1}).\end{aligned} \tag{4.16}$$

Otherwise

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_N\|_{\boldsymbol{T}} + \|\boldsymbol{u}-\boldsymbol{u}_N\|_{\boldsymbol{X}} + \|p-p_N\|_M &\leq C(h^{l_1}q^{-(m-1)}\|\boldsymbol{\sigma}\|_{m-1}\\ &\quad + h^{l_2}l^{-(m-1)}\|\boldsymbol{u}\|_m + h^{l_3}k^{-(m-1)}\|p\|_{m-1}),\end{aligned} \tag{4.17}$$

where $l_1 = \min(m-1,q+1)$, $l_2 = \min(m-1,l)$, $l_3 = \min(m-1,k+1)$.

Proof. As before, we only prove (4.17). By (4.13), for any $(\boldsymbol{\tau},\boldsymbol{v},q)\in \boldsymbol{T}_N\times\boldsymbol{X}_N\times M_N$, we have

$$\begin{aligned}&\mathcal{B}_\alpha(\boldsymbol{\tau}-\boldsymbol{\sigma}_N,\boldsymbol{v}-\boldsymbol{u}_N,q-p_N;\boldsymbol{\tau}-\boldsymbol{\sigma}_N,\boldsymbol{v}-\boldsymbol{u}_N,q-p_N)\\ \geq\ & C(\alpha,\nu)|||\boldsymbol{\tau}-\boldsymbol{\sigma}_N,\boldsymbol{v}-\boldsymbol{u}_N,q-p_N|||^2.\end{aligned}\tag{4.18}$$

Note that

$$\begin{aligned}&\mathcal{B}_\alpha(\boldsymbol{\tau}-\boldsymbol{\sigma}_N,\boldsymbol{v}-\boldsymbol{u}_N,q-p_N;\boldsymbol{\tau}-\boldsymbol{\sigma}_N,\boldsymbol{v}-\boldsymbol{u}_N,q-p_N)\\ &=\mathcal{B}_\alpha(\boldsymbol{\tau}-\boldsymbol{\sigma},\boldsymbol{v}-\boldsymbol{u},q-p;\boldsymbol{\tau}-\boldsymbol{\sigma}_N,\boldsymbol{v}-\boldsymbol{u}_N,q-p_N)\\ &\quad+\mathcal{B}_\alpha(\boldsymbol{\sigma}-\boldsymbol{\sigma}_N,\boldsymbol{u}-\boldsymbol{u}_N,p-p_N;\boldsymbol{\tau}-\boldsymbol{\sigma}_N,\boldsymbol{v}-\boldsymbol{u}_N,q-p_N).\end{aligned}$$

Since our stabilized problem, i.e. **Problem** $\tilde{H}_h$ is consistent, the second term in the above equation vanishes. Using the continuity of the bilinear form $\mathcal{B}(\cdot\,;\,\cdot)$, we obtain

$$\begin{aligned}&\mathcal{B}_\alpha(\boldsymbol{\sigma}-\boldsymbol{\tau},\boldsymbol{u}-\boldsymbol{v},p-q;\boldsymbol{\sigma}_N-\boldsymbol{\tau},\boldsymbol{u}_N-\boldsymbol{v},p_N-q)\\ \leq\ & C|||\boldsymbol{\sigma}-\boldsymbol{\tau},\boldsymbol{u}-\boldsymbol{v},p-q|||\ |||\boldsymbol{\tau}-\boldsymbol{\sigma}_N,\boldsymbol{v}-\boldsymbol{u}_N,q-p_N|||.\end{aligned}\tag{4.19}$$

A combination of (4.18), (4.19) and the usual interpolation estimate yield the error bounds for $\boldsymbol{\sigma},\boldsymbol{u}$. Moreover,

$$|p-p_N|_{k,h}\leq C(h^{l_1}q^{-(m-1)}\|\boldsymbol{\sigma}\|_{m-1}+h^{l_2}l^{-(m-1)}\|\boldsymbol{u}\|_m+h^{l_3}k^{-(m-1)}\|p\|_{m-1}).\tag{4.20}$$

As to the pressure p, by virtue of the *scaled B-B inequality* (4.7), we have

$$\begin{aligned}\|q-p_N\|_M-|q-p_N|_{k,h}&\leq C\sup_{\boldsymbol{v}\in\boldsymbol{X}_h}\frac{(\operatorname{div}\boldsymbol{v},q-p_N)}{\|\boldsymbol{v}\|_{\boldsymbol{X}}}\\ &\leq C(\|p-q\|_M+|||\boldsymbol{\sigma}-\boldsymbol{\sigma}_N,\boldsymbol{u}-\boldsymbol{u}_N,p-p_N|||),\end{aligned}\tag{4.21}$$

which together with the triangle inequality

$$\|p-p_N\|_M\leq\|p-q\|_M+\|q-p_N\|_M\tag{4.22}$$

and the usual interpolation estimate give the error bound for the pressure.

The problem considered in [9], [13] and [14] is just a special case of **Problem** $\tilde{H}_h$ with $\alpha=0$. Thus, we have given a mathematical analysis for the numerical results reported in those papers, and even give an interpretation for the cause of the discrepancy.

Remark 4.2. Our results also hold for the case $\alpha=0$(because in this case, all terms related to the stress vanish), thus, all results in [4] could be recovered, moreover, the main results such as Theorem 4.2 and 4.3 have been improved.

Remark 4.3. It is easy to see that the error bounds for $\boldsymbol{\sigma},\boldsymbol{u}$ are optimal not only in the mesh size h but also in the polynomial order p provided we choose $l=k,q=k+1$. With such choice, one order deterioration with respect to p for the pressure is observed, which is consistent with numerical tests.

Remark 4.4. With $M_N\in\mathcal{C}^0(\overline{\Omega})$ or $\gamma>0$, we have an optimal mixed $h-p$ stabilized finite element method provided we choose $l=k,q=k+1$.

Remark 4.5. Note that the error bounds of our stabilized method for the stress $\boldsymbol{\sigma}$ and the velocity $\boldsymbol{u}$ are not polluted by the error bound for the pressure p as the usual mixed $h-p$ finite element approximation(c.f. (2.1)). This is a merit of the stabilized method.

Remark 4.6. It seems clear that the high-orer deterioration of error bounds(up to 4 orders with respect to p) in [13] is due to techniques. [13] employed a *stability inequality* similar to that in Theorem 4.2 (c.f. (4.9)) to prove the well-posedness as well as to derive the error bounds. Here we only use that inequality to obtain the well-posedness, and exploit the *scaled weak B-B inequality* to get the error bounds.

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