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CONVERGENCE OF AN EXPLICIT UPWIND FINITE ELEMENT METHOD TO MULTI-DIMENSIONAL CONSERVATION LAWS*[1)]

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

An explicit upwind finite element method is given for the numerical computation to multi-dimensional scalar conservation laws. It is proved that this scheme is consistent to the equation and monotone, and the approximate solution satisfies discrete entropy inequality. To guarantee the limit of approximate solutions to be a measure valued solution, we prove an energy estimate. Then the L^p strong convergence of this scheme is proved.

Key words: Conservation law, Finite element method, Convergence.

1. Introduction

The convergence problem for the numerical schemes to one dimensional conservation laws has been extensively studied. By tensor product one dimensional schemes can be applied to multi-dimensional equations. However the convergence of many of those schemes is still unknown even if it is true for one dimensional cases. Besides, for those physical domains with complicated geometry unstructured grids are more flexible. In recent years the convergence problem for unstructured grids has called the attention of some authors. In [2] [3] [10] [11] the convergence of finite volume methods was proved. In [9] [8] [13] [14] the convergence of a streamline diffusion finite element method was proved. In [6] [5] the discontinuous Galerkin method to multi-dimensional conservation laws was studied. We refer the readers to a survey by Shu [12].

In this paper we prove the convergence of an explicit upwind finite element method, the edge-averaged finite element method, given in [15] for convection-diffusion equations, to multi-dimensional scalar conservation laws. Our technique is the new approach using measure valued solutions (see [4] [13]). However since artificial viscosity is employed in the scheme, the estimates are quite different, and we believe that our technique can be applied to more general schemes. Numerical experiments have shown that this scheme gives satisfactory results, which will be reported later on.

We consider the equation

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0, \tag{1}$$

where $f : \mathsf{R} \to \mathsf{R}^N (N \geq 1), f \in C^1$. Without loss of generality we assume that $f(0) = 0$, otherwise $f(u)$ can be replaced by $f(u) - f(0)$.

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2. The Finite Element Scheme and Main Convergence Theorem

Let T be a simplicial finite element with linear shape functions interpolated by $N+1$ vertices, $q_1, \cdots, q_{N+1}$. Then it is easy to prove the following lemmas[15].

Lemma 2.1. *If $u, v \in P_1(T)$, then*

$$\int_T \nabla u \cdot \nabla v \, dx = \sum_{i<j} a_{ij}^T (u_i - u_j)(v_j - v_i),$$

where u_i, u_j, v_i, v_j are the values of u, v at nodes q_i and q_j,

$$a_{ij}^T = \int_T \nabla \chi_i \cdot \nabla \chi_j \, dx,$$

and $\chi_i \in P_1(T), i = 1, \cdots, N+1$, $\chi_i(x_j) = \delta_i^j$.

Lemma 2.2. *If $v \in P_1(T)$ and c is a constant vector, then*

$$\int_T c \cdot \nabla v \, dx = \sum_{i<j} a_{ij}^T c \cdot \tau_E h_{ij} (v_i - v_j),$$

where E is the edge connecting q_i, q_j, τ_E is the unit vector pointing from q_i to q_j, and h_{ij} is the length of E.

By adding an artificial viscosity term equation (1) becomes

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = \varepsilon \triangle u, \tag{2}$$

where $\varepsilon > 0$.

The weak formulation of the initial value problem of the equation (2) is: given u^0, find $u : [0, T] \to H^1(\mathsf{R}^N)$ such that

$$\frac{d}{dt} \int_{\mathsf{R}^N} uv \, dx - \int_{\mathsf{R}^N} f(u) \cdot \nabla v \, dx + \varepsilon \int_{\mathsf{R}^N} \nabla u \cdot \nabla v \, dx = 0, \quad \forall v \in H^1(\mathsf{R}^N), \tag{3}$$

$$\int_{\mathsf{R}^N} u|_{t=0} v \, dx = \int_{\mathsf{R}^N} u^0 v \, dx, \quad \forall v \in H^1(\mathsf{R}^N).$$

Let

$$J = \varepsilon \nabla u - f(u),$$

then the projection of J on E is

$$J \cdot \tau_E = \varepsilon \frac{\partial u}{\partial \tau_E} - f(u) \cdot \tau_E.$$

We define an integral along E with $x \in E$,

$$\psi = -\int_{q_i}^{x} \frac{f(u) \cdot \tau_E}{\varepsilon u} \, ds, \tag{4}$$

then

$$J \cdot \tau_E = \varepsilon e^{-\psi} \frac{\partial}{\partial \tau_E} (e^{\psi} u).$$

Integrating on E one has

$$(e^{\psi} u)_j - (e^{\psi} u)_i = \int_{q_i}^{q_j} \frac{J \cdot \tau_E}{\varepsilon} e^{\psi} \, ds,$$

then approximately

$$J \cdot \tau_E = \frac{(e^{\psi}u)_j - (e^{\psi}u)_i}{\int_{q_i}^{q_j} \frac{e^{\psi}}{\varepsilon}\, ds}.$$

By Lemma 2.2 we have the approximation,

$$\int_T J \cdot \nabla v \, dx \doteq \sum_{i<j} a_{ij}^T \frac{(e^{\psi}u)_j - (e^{\psi}u)_i}{\int_{q_i}^{q_j} \frac{e^{\psi}}{\varepsilon}\, ds} h_{ij}(v_i - v_j). \tag{5}$$

The space R^N is triangulized into simplicial finite elements. The set of all nodes is denoted by $\{x_i\}$. In the paper we use q_i for vertices in a single element T, otherwise we use x_i. Let x_i be one node on the grid and Ω_i be the union of all elements neighboring x_i. Let ϕ_i be the shape function with $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$ for all $j \neq i$. We take a time step Δt. By the weak formulation (3) and the approximation (5) we have the finite element scheme,

$$\int_{\Omega_i} u^{n+1} v \, dx - \int_{\Omega_i} u^n v \, dx + \Delta t \sum_{T \in \mathcal{T}} \sum_{j \in I_T} a_{ij}^T \frac{(e^{\psi}u)_j^n - (e^{\psi}u)_i^n}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon}\, ds} h_{ij} = 0,$$

where $\mathcal{T}$ denotes the set of elements neighboring x_i and I_T the index set of the nodes of T beside x_i, and $u^n = u(x, n\Delta t)$. Moreover, making it an explicit scheme, we use the standard "mass lumping" quadrature and obtain

$$u_i^{n+1} - u_i^n + \frac{\Delta t}{A_i} \sum_{j \in I_i} a_{ij} J(u_i^n, u_j^n) h_{ij} = 0, \tag{6}$$

where $u_i^n = u(x_i, n\Delta t)$, I_i is the index set of the nodes neighboring x_i, and $A_i = \int_{\Omega_i} v \, dx$, and

$$a_{ij} = \sum_{T \in \mathcal{T}} a_{ij}^T,$$

and

$$J(u_i, u_j) = \frac{(e^{\psi}u)_j - (e^{\psi}u)_i}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon}\, ds}.$$

We note that, since $\sum_j a_{ij} = 0$ and $a_j \leq 0$ for $i \neq j$, we have

$$a_{ii} = \sup_j a_{ij}.$$

We are going to prove the convergence of the scheme (6) in the remaining of this paper. For all cases we denote by C a generic constant.

Basic assumptions:

We assume that the triangulation is regular, that is, $h_T/\rho_T \leq C$ for all elements T, where h_T is the diameter of T and ρ_T is the supremum of the diameters of all balls contained in T, and[15].

$$\sum_{T \supset E} |\kappa_E^T| \cot \theta_E^T \geq 0, \tag{7}$$

where θ_E^T is the angle between the faces F_i and F_j, F_j is the $N-1$ dimensional simplex opposite to the vertex q_j, and κ_E^T is the $N-2$ dimensional simplex opposite to E. Condition (7) implies $a_{ij} \leq 0$ for all i, j. Besides we assume the mesh sizes to satisfy

$$h \leq \delta\varepsilon, \quad \varepsilon h^{N-2} \Delta t \leq \delta \min_i A_i, \quad \delta > 0, \tag{8}$$

where $h = \max h_T$.

We consider the initial condition

$$u|_{t=0} = u^0, \tag{9}$$

where $u^0 \in L^\infty(\mathsf{R}^N)$. then we set the discrete initial data,

$$u_i^0 = \int_{\Omega_i} u^0 \phi_i \, dx / A_i. \tag{10}$$

Let us extend the solution to (6) to the whole domain $\mathsf{R}_+^{N+1} = \{(x,t); x \in \mathsf{R}^N, t \in [0,\infty)\}$ such that it keeps constant on $[n\Delta t, (n+1)\Delta t), \forall n$, which is denoted by u_h.

Theorem 2.1. *We assume that*

(a) $m \le u^0 \le M$.

(b) *The triangulation is regular and (7) holds.*

(c) *(8) holds with δ sufficiently small.*

Then as $\Delta t \to 0$, $h \to 0$, $\varepsilon \to 0$ the solution u_h converges, on each compact subdomain in the L^p-norm, $1 \le p < \infty$, to a function u, which is the weak solution to the equation (1) and initial condition (9). Moreover u satisfies the entropy condition

$$\int \left(U(u) \frac{\partial \chi}{\partial t} + F(u) \cdot \nabla \chi \right) dxdt \ge 0, \qquad \forall \chi \in C_0^\infty(\mathsf{R}_+^{N+1}), \chi \ge 0,$$

for all entropy-entropy flux pair $\{U, F\}$ satisfying $U'' \ge 0$.

The proof of this theorem depends on a few technical lemmas.

The following lemma implies the consistency.

Lemma 2.3. *If u is a constant function, namely $u_i = u_j = u$, then*

$$J(u,u) = -f(u) \cdot \tau_E. \tag{11}$$

The proof of this lemma is straightforward since if $u_i = u_j = u$, then

$$e^\psi = e^{-\frac{f(u) \cdot \tau_E}{\varepsilon u} |x - x_i|},$$

which yields (11) by direct computation.

As a consequence of the above result, if $u_i = u_j$ for all i and j, by Lemma 2.2,

$$\sum_{j \in I_i} a_{ij} J(u_i, u_j) h_{ij} = \sum_{j \in I_i} a_{ij} f(u) \cdot \tau = \frac{\Delta t}{A_i} \int_{\Omega_i} f(u) \cdot \nabla \phi_i \, dx = 0.$$

This means that the constant function is a solution to the scheme (6).

The following lemma states the monotonicity.

Lemma 2.4. *Assume that $u_i, u_j \in [m, M]$ and (7) and (8) holds, then*

$$\frac{\partial J}{\partial u_i} = O(\frac{\varepsilon}{h_{ij}}), \tag{12}$$

$$\frac{\partial J}{\partial u_j} = \frac{\varepsilon e^{\psi_j}}{\int_{x_i}^{x_j} e^\psi \, ds} \left\{ 1 + O\left(\frac{h_{ij}}{\varepsilon} \right) \right\}, \tag{13}$$

consequently if $u^n \in [m, M]$ and δ is sufficiently small (depending on m, M), then

$$\frac{\partial u_i^{n+1}}{\partial u_i^n} \ge 0, \qquad \frac{\partial u_i^{n+1}}{\partial u_j^n} \ge 0,$$

where u_i^{n+1} is given in (6).

Lemma 2.3 and Lemma 2.4 imply maximum principle as the following:

Lemma 2.5. *Under the assumptions of Lemma 2.4, it holds that* $u_i^{n+1} \in [m, M]$.

We shall now describe a discrete entropy inequality. Let the entropy $U \in C^2(\mathsf{R})$ such that $U'' \geq 0$. Without loss of generality we may assume that $U(0) = U'(0) = 0$, otherwise $U(u)$ can be replaced by $U(u) - U(0) - U'(0)u$. We define a discrete entropy flux

$$\begin{aligned} F(u_i, u_j) &= \frac{1}{2}\int_{-\infty}^{+\infty} U''(z)\{J(z \vee u_i, z \vee u_j) - J(z \wedge u_i, z \wedge u_j) \\ &+ \operatorname{sign} z(J(u_i, u_j) + f(z) \cdot \tau_E)\}\, dz, \end{aligned}$$

where $a \wedge b = \min(a, b)$, and $a \vee b = \max(a, b)$.

Lemma 2.6. *We have the following:*

(a) *Consistency*

$$F(u, u) = -F(u) \cdot \tau_E, \tag{14}$$

where $F(u) = \int_0^u f'U'\, du$ is the entropy flux.

(b) *Under the assumptions of Lemma 2.4 it holds that*

$$U(u_i^{n+1}) \leq U(u_i^n) - \frac{\Delta t}{A_i} \sum_{j \in I_i} a_{ij} F(u_i^n, u_j^n) h_{ij}, \tag{15}$$

which is the discrete entropy inequality.

Energy estimate:

Lemma 2.7. *We assume that the initial data u^0 are compactly supported. Under the assumptions of Lemma 2.4, if δ is small enough then it holds that*

$$\frac{1}{2}\sum_k (u_k^{\mathcal{N}+1})^2 A_k - \frac{\varepsilon \Delta t}{2} \sum_{n=0}^{\mathcal{N}} \sum_E a_{ij} \frac{e^{\psi_j} h_{ij}}{\int_{x_i}^{x_j} e^{\psi}\, ds} (u_j^n - u_i^n)^2 \leq \frac{1}{2}\sum_k (u_k^0)^2 A_k. \tag{16}$$

Proof of the Theorem. We take $\chi \in C_0^\infty(\overline{\mathsf{R}_+^{N+1}})$, and let $\chi_i^n = \chi(x_i, n\Delta t)$. Multiplying the equation (6) by χ_i^n and summing up we get

$$\begin{aligned} \sum_n \sum_i u_i^{n+1} \frac{\chi_i^n - \chi_i^{n+1}}{\Delta t} \Delta t A_i \quad &+ \quad \Delta t \sum_n \sum_E a_{ij} J(u_i^n, u_j^n) h_{ij} (\chi_i^n - \chi_j^n) \\ &= \quad \sum_i u_i^0 \chi_i^0 A_i. \end{aligned} \tag{17}$$

By Lemma 2.3

$$\begin{aligned} &\Delta t \sum_n \sum_E a_{ij} J(u_i^n, u_j^n) h_{ij} (\chi_i^n - \chi_j^n) \\ = \quad &-\Delta t \sum_n \sum_E a_{ij} f(u_i^n) \cdot \tau_E h_{ij} (\chi_i^n - \chi_j^n) + R_1, \end{aligned} \tag{18}$$

where

$$R_1 = \Delta t \sum_n \sum_E a_{ij} \frac{\partial J}{\partial u_j} (u_j^n - u_i^n) h_{ij} (\chi_i^n - \chi_j^n),$$

where $\frac{\partial J}{\partial u_j}$ is a mean value. In view of Lemma 2.4 we see that

$$\begin{aligned} |R_1| \quad \leq \quad & C\left\{\Delta t\sum_n\sum_E \varepsilon(-a_{i\,j})\frac{e^{\psi_j}h_{ij}}{\int_{x_i}^{x_j} e^{\psi}\,ds}(u_j^n-u_i^n)^2\right\}^{1/2} \\ & \cdot\left\{\Delta t\sum_n\sum_E \varepsilon(-a_{i\,j})\frac{\int_{x_i}^{x_j} e^{\psi}\,ds}{e^{\psi_j}h_{ij}}(\chi_i^n-\chi_j^n)^2\right\}^{1/2}, \end{aligned}$$

where the sum $\sum_n\sum_E$ is taken on the support of χ. Since the scheme is explicit, u^n depends only on the values of u^0 on a compact set, so Lemma 2.7 can be applied here to get

$$|R_1| \leq C\left\{\Delta t\sum_n\sum_E \varepsilon(-a_{i\,j})\frac{\int_{x_i}^{x_j} e^{\psi}\,ds}{e^{\psi_j}h_{ij}}(\chi_i^n-\chi_j^n)^2\right\}^{1/2}.$$

We notice that $\chi \in C_0^\infty$, so $\Delta t\sum_n\sum_E(-a_{i\,j})(\chi_i^n-\chi_j^n)^2 \leq C$. By Lemma 2.5 u is bounded, so there is a constant $\beta > 0$ such that

$$\frac{e^{\psi_j}h_{ij}}{\int_{x_i}^{x_j} e^{\psi}\,ds} \geq \beta. \tag{19}$$

Consequently $R_1 \to 0(\varepsilon \to 0)$.

Let P be the interpolation operator on T, then by Lemma 2.2, we have

$$\begin{aligned} \int_T f(u^n)\cdot\nabla(P\chi^n)\,dx \quad &= \quad \int_T f(u(x_T,n\Delta t))\cdot\nabla(P\chi^n)\,dx \\ &= \quad \sum_{i<j} a_{i\,j}^T f(u_T)\cdot\tau_E h_{i\,j}(\chi_i^n-\chi_j^n), \end{aligned}$$

where $x_T \in T$ and $u_T = u(x_T, n\Delta t)$. It follows that

$$\begin{aligned} & -\Delta t\sum_n\sum_E a_{i\,j}f(u_i^n)\cdot\tau_E h_{i\,j}(\chi_i^n-\chi_j^n) \\ = \quad & -\Delta t\sum_n\sum_T\sum_{i<j} a_{i\,j}^T f(u_i^n)\cdot\tau_E h_{i\,j}(\chi_i^n-\chi_j^n) \\ = \quad & -\Delta t\sum_n\sum_T\int_T f(u^n)\cdot\nabla(P\chi^n)\,dx + R_2, \end{aligned} \tag{20}$$

where

$$R_2 = \Delta t\sum_n\sum_T\sum_{i<j} a_{i\,j}^T(f(u_T)-f(u_i^n))\cdot\tau_E h_{i\,j}(\chi_i^n-\chi_j^n).$$

By Lemma 2.1

$$\begin{aligned} |f(u_T)-f(u_i^n)| \quad &\leq \quad C|u_T-u_i^n| \leq Ch_T|\nabla u^n| \\ &\leq \quad Ch_T^{1-\frac{N}{2}}|u^n|_{1,T} \\ &= \quad Ch_T^{1-\frac{N}{2}}\left(-\sum_{l<m} a_{l\,m}^T(u_l^n-u_m^n)^2\right)^{\frac{1}{2}}. \end{aligned} \tag{21}$$

Analogously we have

$$|\chi_i^n-\chi_j^n| \leq Ch_T^{1-\frac{N}{2}}\left(-\sum_{l<m} a_{l\,m}^T(\chi_l^n-\chi_m^n)^2\right)^{\frac{1}{2}}.$$

The regularity assumption implies[1]

$$|a_{i\,j}^T| \le C h_T^{N-2}. \tag{22}$$

We use the Schwarz inequality to obtain

$$\begin{aligned}
|R_2| &\le C\Delta t \sum_n \sum_T \sum_{i<j} h_T \left(-\sum_{l<m} a_{l\,m}^T (u_l^n - u_m^n)^2\right)^{\frac{1}{2}} \\
&\quad \cdot \left(-\sum_{l<m} a_{l\,m}^T (\chi_l^n - \chi_m^n)^2\right)^{\frac{1}{2}} \\
&\le C\left\{\Delta t \sum_n \sum_T \sum_{i<j} h_T \left(-\sum_{l<m} a_{l\,m}^T (u_l^n - u_m^n)^2\right)\right\}^{\frac{1}{2}} \\
&\quad \cdot \left\{\Delta t \sum_n \sum_T \sum_{i<j} h_T \left(-\sum_{l<m} a_{l\,m}^T (\chi_l^n - \chi_m^n)^2\right)\right\}^{\frac{1}{2}} \\
&\le C\left\{\Delta t \sum_n \sum_T h \left(-\sum_{l<m} a_{l\,m}^T (u_l^n - u_m^n)^2\right)\right\}^{\frac{1}{2}} \\
&\quad \cdot \left\{\Delta t \sum_n \sum_T h \left(-\sum_{l<m} a_{l\,m}^T (\chi_l^n - \chi_m^n)^2\right)\right\}^{\frac{1}{2}} \\
&= C\left\{\Delta t \sum_n \sum_E h(-a_{i\,j})(u_i^n - u_j^n)^2\right\}^{\frac{1}{2}} \\
&\quad \cdot \left\{\Delta t \sum_n \sum_E h(-a_{i\,j})(\chi_i^n - \chi_j^n)^2\right\}^{\frac{1}{2}}.
\end{aligned}$$

Taking (8) and (19) into account, then following the same lines as estimating R_1 we get $R_2 \to 0 (h \to 0)$.

By virtue of Lemma 2.5 we have the uniform estimate in L^∞, therefore for any sequence $\{u_h\}$ with $h \to 0$ there exists Young measure $\nu_{x,t}$ and a subsequence, still denoted by $\{u_h\}$, such that

$$u_h \stackrel{*}{\rightharpoonup} <\nu_{x,t}, \lambda> (L^\infty_{\rm loc}(\mathsf{R}_+^{N+1})),$$

$$f(u_h) \stackrel{*}{\rightharpoonup} <\nu_{x,t}, f(\lambda)> (L^\infty_{\rm loc}(\mathsf{R}_+^{N+1})).$$

We take the limit of (17) and notice (18) (20) to get

$$-\int \left\{<\nu_{x,t}, \lambda> \frac{\partial \chi}{\partial t} + <\nu_{x,t}, f(\lambda)> \cdot \nabla \chi\right\} dxdt = \int u^0 \chi(x,0)\, dx,$$

which implies that $\nu_{x,t}$ is a measure valued solution. Following the same lines, we derive from Lemma 2.6 that

$$\int \left\{<\nu_{x,t}, U(\lambda)> \frac{\partial \chi}{\partial t} + <\nu_{x,t}, F(\lambda)> \cdot \nabla \chi\right\} dxdt \ge 0,$$

for the χ, U, F defined above, therefore $\nu_{x,t}$ satisfies entropy condition. We employ DiPerna's theory (see [7]) to see that $\nu_{x,t} = \delta_{u(x,t)}$, u_h converges to u strongly in L^p-norm, $p \in [1, \infty)$, and u is the admissible solution.

3. Proofs of the Technical Lemmas

3.1. Proof of Lemmas 2.4 and 2.5

Proof of Lemma 2.4. We have $\psi(x_i) = 0$, and let $\psi_j = \psi(x_j)$, then

$$\begin{aligned}\frac{\partial J}{\partial u_i} &= \frac{e^{\psi_j} \int_{x_i}^{x_j} \left(-\frac{f(u)}{\varepsilon u}\right)' \frac{|x_j - x|}{h_{ij}} \cdot \tau_E \, ds u_j - 1}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon} \, ds} \\ &- \frac{(e^{\psi_j} u_j - u_i) \int_{x_i}^{x_j} e^{\psi} \left(\int_{x_i}^{x} \left(-\frac{f(u)}{\varepsilon u}\right)' \frac{|x_j - x|}{h_{ij}} \cdot \tau_E \, ds \right) ds}{\left(\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon} \, ds \right)^2},\end{aligned}$$

where for notational convenience we omit the superscript n. By the assumption we have (12), which implies

$$\frac{\partial u_i^{n+1}}{\partial u_i} \geq 1 - \frac{C \Delta t \varepsilon}{A_i} a_{ii}.$$

In view of (22) we get $\frac{\partial u_i^{n+1}}{\partial u_i} \geq 0$ for small δ.

For $x_j \in I_i$ we have

$$\begin{aligned}\frac{\partial J}{\partial u_j} &= \frac{e^{\psi_j}}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon} \, ds} + \frac{e^{\psi_j} \int_{x_i}^{x_j} \left(-\frac{f(u)}{\varepsilon u}\right)' \frac{|x - x_i|}{h_{ij}} \cdot \tau_E \, ds u_j}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon} \, ds} \\ &- \frac{(e^{\psi_j} u_j - u_i) \int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon} \left(\int_{x_i}^{x} \left(-\frac{f(u)}{\varepsilon u}\right)' \frac{|x - x_i|}{h_{ij}} \cdot \tau_E \, ds \right) ds}{\left(\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon} \, ds \right)^2} \\ &= \frac{\varepsilon e^{\psi_j}}{\int_{x_i}^{x_j} e^{\psi} \, ds} \left\{ 1 + \int_{x_i}^{x_j} \left(-\frac{f(u)}{\varepsilon u}\right)' \frac{|x - x_i|}{h_{ij}} \cdot \tau_E \, ds u_j \right. \\ &\left. - \frac{(u_j - e^{-\psi_j} u_i) \int_{x_i}^{x_j} e^{\psi} \left(\int_{x_i}^{x} \left(-\frac{f(u)}{\varepsilon u}\right)' \frac{|x - x_i|}{h_{ij}} \cdot \tau_E \, ds \right) ds}{\int_{x_i}^{x_j} e^{\psi} \, ds} \right\},\end{aligned}$$

which implies (13) and

$$\frac{\partial u_i^{n+1}}{\partial u_j} \geq -\frac{\Delta t \varepsilon}{A_i} a_{ij} \frac{e^{\psi_j} h_{ij}}{\int_{x_i}^{x_j} e^{\psi} \, ds} \left\{ 1 + O\left(\frac{h_{ij}}{\varepsilon}\right) \right\} \geq 0$$

for small δ.

Proof of Lemma 2.5. By Lemma 2.4

$$u_i^{n+1} \geq m - \frac{\Delta t}{A_i} \sum_{j \in I_i} a_{ij} J(m, m) h_{ij}.$$

Then by Lemma 2.3 and Lemma 2.2

$$\begin{aligned} u_i^{n+1} &\geq m + \frac{\Delta t}{A_i} \sum_{j \in I_i} a_{i\,j} f(m) \cdot \tau_E h_{ij} \\ &= m + \frac{\Delta t}{A_i} \sum_{T \in \mathcal{T}} \int_T f(m) \cdot \nabla v \, dx \\ &= m + \frac{\Delta t}{A_i} \int_{\Omega_i} f(m) \cdot \nabla v \, dx = m. \end{aligned}$$

By the same reasoning we get $u_i^{n+1} \leq M$.

3.2. Proof of Lemma 2.6

Proof. For the first part, we have

$$\begin{aligned} F(u,u) &= \frac{1}{2} \int_{-\infty}^{+\infty} U''(z) \{ J(z \vee u, z \vee u) - J(z \wedge u, z \wedge u) \\ &+ \ \operatorname{sign} z(-f(u) \cdot \tau_E + f(z) \cdot \tau_E) \} \, dz, \end{aligned}$$

If $u > 0$, then

$$\begin{aligned} F(u,u) &= \frac{1}{2} \int_0^u U''(z) \{ J(u,u) - J(z,z) + (-f(u) \cdot \tau_E + f(z) \cdot \tau_E) \} \, dz \\ &= \int_0^u U''(z) \{ -f(u) + f(z) \} \cdot \tau_E \, dz \\ &= -\int_0^u U'(z) f'(z) \cdot \tau_E \, dz = -F(u) \cdot \tau_E. \end{aligned}$$

The deduction for the case of $u \leq 0$ is the same.

For the second part, we denote $\mathbf{u}_i^n = \{u_j^n\}_{j \in I_i}$, and

$$H(u_i^n, \mathbf{u}_i^n) = u_i^n - \frac{\Delta t}{A_i} \sum_{j \in I_i} a_{i\,j} J(u_i^n, u_j^n) h_{ij},$$

then

$$H(z \vee u_i^n, \mathbf{z} \vee \mathbf{u}_i^n) = z \vee u_i^n - \frac{\Delta t}{A_i} \sum_{j \in I_i} a_{i\,j} J(z \vee u_i^n, z \vee u_j^n) h_{ij},$$

and

$$H(z \wedge u_i^n, \mathbf{z} \wedge \mathbf{u}_i^n) = z \wedge u_i^n - \frac{\Delta t}{A_i} \sum_{j \in I_i} a_{i\,j} J(z \wedge u_i^n, z \wedge u_j^n) h_{ij},$$

where $\mathbf{z} = \{z\}$. By subtracting we have

$$\begin{aligned} & H(z \vee u_i^n, \mathbf{z} \vee \mathbf{u}_i^n) - H(z \wedge u_i^n, \mathbf{z} \wedge \mathbf{u}_i^n) \\ = \ & |u_i^n - z| - \frac{\Delta t}{A_i} \sum_{j \in I_i} a_{i\,j} \{ J(z \vee u_i^n, z \vee u_j^n) - J(z \wedge u_i^n, z \wedge u_j^n) \} h_{ij}. \end{aligned} \tag{23}$$

On the other hand by Lemma 2.4

$$\begin{aligned} H(z \vee u_i^n, \mathbf{z} \vee \mathbf{u}_i^n) &\geq H(z, \mathbf{z}) \vee H(u_i^n, \mathbf{u}_i^n) \\ &= z \vee u_i^{n+1}, \end{aligned}$$

and

$$H(z \wedge u_i^n, \mathbf{z} \wedge \mathbf{u}_i^n) \le z \wedge u_i^{n+1},$$

therefore it holds that

$$\begin{aligned} H(z \vee u_i^n, \mathbf{z} \vee \mathbf{u}_i^n) - H(z \wedge u_i^n, \mathbf{z} \wedge \mathbf{u}_i^n) &\ge z \vee u_i^{n+1} - z \wedge u_i^{n+1} \\ &= |u_i^{n+1} - z|. \end{aligned} \tag{24}$$

By (23) and (24) we have

$$\begin{aligned} U(u_i^{n+1}) &= \frac{1}{2}\int_{-\infty}^{+\infty} U''(z)\{|u_i^{n+1} - z| + \text{sign } z(u_i^{n+1} - z)\}\, dz \\ &\le \frac{1}{2}\int_{-\infty}^{+\infty} U''(z)\{H(z \vee u_i^n, \mathbf{z} \vee \mathbf{u}_i^n) - H(z \wedge u_i^n, \mathbf{z} \wedge \mathbf{u}_i^n) \\ &+ \text{sign } z(u_i^{n+1} - z)\}\, dz \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} U''(z)\{|u_i^n - z| - \frac{\Delta t}{A_i}\sum_{j \in I_i} a_{i\,j}\{J(z \vee u_i^n, z \vee u_j^n) \\ &- J(z \wedge u_i^n, z \wedge u_j^n)\}h_{ij} + \text{sign } z(u_i^{n+1} - z)\}\, dz. \end{aligned}$$

We substitute the equation (6) in it to obtain

$$\begin{aligned} U(u_i^{n+1}) &\le \frac{1}{2}\int_{-\infty}^{+\infty} U''(z)\{|u_i^n - z| + \text{sign } z(u_i^n - z)\}\, dz \\ &- \frac{1}{2}\frac{\Delta t}{A_i}\sum_{j \in I_i} a_{i\,j}\int_{-\infty}^{+\infty} U''(z)\{J(z \vee u_i^n, z \vee u_j^n) \\ &- J(z \wedge u_i^n, z \wedge u_j^n) + \text{sign } z J(u_i^n, u_j^n)\}h_{ij}\, dz. \end{aligned}$$

Being the same as the proof of Lemma 2.5, one has

$$\sum_{j \in I_i} a_{i,j} f(z) \cdot \tau_E h_{ij} = 0,$$

therefore (15) holds.

3.3. Proof of Lemma 2.7

To prove the energy estimate, we need the following auxiliary result.

Lemma 3.1. *Under the assumptions of Lemma 2.4, it holds that*

$$|u_i^{n+1} - u_i^n| \le -\frac{C\varepsilon\Delta t}{A_i}\sum_{j \in I_i} a_{i\,j}|u_j^n - u_i^n|. \tag{25}$$

Proof. We have

$$J(u_i, u_j) = \frac{e^{\psi_j}(u_j - u_i)}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon}\, ds} + \frac{(e^{\psi_j} - 1)u_i}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon}\, ds}. \tag{26}$$

Let

$$Q(u_i, u_j) = \frac{(e^{\psi_j} - 1)u_i}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon}\, ds},$$

then by Lemma 2.3

$$Q(u_i, u_i) = J(u_i, u_i) = -f(u_i) \cdot \tau_E. \tag{27}$$

Let $\widetilde{\frac{\partial Q}{\partial u_j}}$ be a mean value, then

$$Q(u_i,u_j)-Q(u_i,u_i)=(u_j-u_i)\widetilde{\frac{\partial Q}{\partial u_j}}. \tag{28}$$

u is linear on E, therefore

$$\begin{aligned}\frac{\partial Q}{\partial u_j} &= \frac{u_i}{\int_{x_i}^{x_j}\frac{e^{\psi}}{\varepsilon}\,ds}\left\{-e^{\psi_j}\int_{x_i}^{x_j}\left(\frac{f(u)}{\varepsilon u}\right)'\frac{|x-x_i|}{h_{ij}}\cdot\tau_E\,ds\right.\\ &+ \left.\frac{e^{\psi_j}-1}{\int_{x_i}^{x_j}\frac{e^{\psi}}{\varepsilon}\,ds}\int_{x_i}^{x_j}\frac{e^{\psi}}{\varepsilon}\left(\int_{x_i}^{x}\left(\frac{f(u)}{\varepsilon u}\right)'\frac{|x-x_i|}{h_{ij}}\cdot\tau_E\,ds\right)ds\right\}.\end{aligned}$$

It is easy to see that

$$\frac{\partial Q}{\partial u_j}=O(1)u_i.$$

By (26), (27), (28) we get

$$J(u_i,u_j)=\frac{e^{\psi_j}(u_j-u_i)}{\int_{x_i}^{x_j}\frac{e^{\psi}}{\varepsilon}\,ds}-f(u_i)\cdot\tau_E+u_iO(u_j-u_i). \tag{29}$$

We define a linear function $p(x)=f(u_i)\cdot(x-x_i)$, then apply the Laplace operator to obtain $\triangle p=0$. Let ϕ_i be the shape function on Ω_i such that $\phi_i(x_i)=1$ and $\phi_i(x_l)=0$, $l\in I_i$, then by Lemma 2.1

$$0=\int_{\Omega_i}\nabla p\cdot\nabla v\,dx=\sum_{j\in I_i}a_{ij}f(u_i)\cdot\tau_E h_{ij}.$$

In view of the equation (6) and (29)

$$u_i^{n+1}-u_i+\frac{\Delta t}{A_i}\sum_{j\in I_i}a_{ij}\left\{\frac{e^{\psi_j}(u_j-u_i)}{\int_{x_i}^{x_j}\frac{e^{\psi}}{\varepsilon}\,ds}+u_iO(u_j-u_i)\right\}h_{ij}=0.$$

Taking into account that a_{ij} are non-positive we obtain (25).

We are now in a position to prove Lemma 2.7.

Proof of Lemma 2.7. Multiplying the equation (6) by u_i^{n+1} and taking sum we obtain

$$\sum_k(u_k^{n+1}-u_k)A_ku_k^{n+1}+\Delta t\sum_E a_{ij}J(u_i,u_j)h_{ij}(u_i^{n+1}-u_j^{n+1})=0, \tag{30}$$

where we have noticed that i and j are symmetric in $J(u_i,u_j)$. Substituting (29) into (30) gives

$$\begin{aligned}&\sum_k(u_k^{n+1}-u_k)A_ku_k^{n+1}+\Delta t\sum_E a_{ij}\{\frac{e^{\psi_j}(u_j-u_i)}{\int_{x_i}^{x_j}\frac{e^{\psi}}{\varepsilon}\,ds}-f(u_i)\cdot\tau_E\\ +\ &u_iO(u_j-u_i)\}h_{ij}(u_i^{n+1}-u_j^{n+1})=0,\end{aligned} \tag{31}$$

Let us study the terms in the brace in turn. First of all

$$\frac{e^{\psi_j}(u_j-u_i)}{\int_{x_i}^{x_j}\frac{e^{\psi}}{\varepsilon}\,ds}h_{ij}(u_i^{n+1}-u_j^{n+1})=-\frac{e^{\psi_j}(u_j-u_i)^2}{\int_{x_i}^{x_j}\frac{e^{\psi}}{\varepsilon}\,ds}h_{ij}+R_3,$$

where

$$R_3 = \frac{e^{\psi_j} h_{ij}}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon}\, ds}(u_j - u_i)(u_i^{n+1} - u_i - u_j^{n+1} + u_j).$$

By Lemma 3.1

$$|a_{i\,j} R_3| \le \frac{e^{\psi_j} h_{ij}}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon}\, ds} |a_{i\,j}(u_j - u_i)| \left\{ \frac{C\varepsilon\Delta t}{A_i} \sum |a_{i\,l}(u_l - u_i)| + \frac{C\varepsilon\Delta t}{A_j} \sum |a_{j\,l}(u_l - u_j)| \right\}.$$

By using the Schwarz inequality we obtain

$$\sum_E a_{i\,j} \frac{e^{\psi_j}(u_j - u_i)}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon}\, ds} h_{ij}(u_i^{n+1} - u_j^{n+1}) \ge (-1 + C\delta) \sum_E a_{i\,j} \frac{e^{\psi_j}(u_j - u_i)^2}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon}\, ds} h_{ij}. \tag{32}$$

We notice that $h_{ij}/\varepsilon \le \delta$, so the term $u_i O(u_j - u_i)$ in (31) can be estimated in the same way. We turn now to the term $-f(u_i) \cdot \tau_E$. Let $F(u)$ be the entropy flux with respect to $U(u) = \frac{u^2}{2}$, then

$$\begin{aligned} 0 &= \int_{\mathsf{R}^N} \nabla \cdot F(u^n)\, dx = \int_{\mathsf{R}^N} u^n \nabla \cdot f(u^n)\, dx \\ &= \int_{\mathsf{R}^N} u^{n+1} \nabla \cdot f(u^n)\, dx + \int_{\mathsf{R}^N} (u^n - u^{n+1}) \nabla \cdot f(u^n)\, dx \\ &= \int_{\mathsf{R}^N} u^{n+1} \nabla \cdot f(u^n)\, dx + R_4, \end{aligned}$$

where

$$R_4 = \sum_T \int_T (u^n - u^{n+1}) \nabla \cdot f(u^n)\, dx,$$

hence

$$|R_4| \le \sum_T \max_T |u^n - u^{n+1}| \max |f'| \int_T |\nabla u^n|\, dx.$$

We take $c = \nabla u^n / |\nabla u^n|$, then by Lemma 2.2 we get

$$|R_4| \le C \sum_T \max_T |u^n - u^{n+1}| \sum_{i<j} |a_{i\,j}^T| h_{i\,j} |u_i - u_j|.$$

Moreover, by Lemma 2.2

$$\begin{aligned} \int_{\mathsf{R}^2} u^{n+1} \nabla \cdot f(u^n)\, dx &= -\sum_T \int_T f(u^n) \cdot \nabla u^{n+1}\, dx \\ &= -\sum_T f(u(x_T, n\Delta t)) \cdot \nabla u^{n+1} \int_T dx \\ &= -\sum_T \sum_{i<j} a_{i\,j}^T f(u_T) \cdot \tau_E h_{ij} (u_i^{n+1} - u_j^{n+1}), \end{aligned}$$

where $x_T \in T$ and $u_T = u(x_T, n\Delta t)$. Therefore

$$\begin{aligned} &- \sum_E a_{i\,j} f(u_i) \cdot \tau_E h_{ij} (u_i^{n+1} - u_j^{n+1}) \\ &= -\sum_T \sum_{i<j} a_{i\,j}^T f(u_i) \cdot \tau_E h_{ij} (u_i^{n+1} - u_j^{n+1}) = R_5 - R_4, \end{aligned}$$

where

$$R_5 = \sum_T \sum_{i<j} a_{i\,j}^T (f(u_T) - f(u_i)) \cdot \tau_E h_{ij} (u_i^{n+1} - u_j^{n+1}).$$

In view of Lemma 3.1, (21) and (22) we have

$$\begin{aligned} |R_5| \quad &\leq \quad C \sum_T \sum_{i<j} |a_{i\,j}^T| |h_{i\,j}| f(u_T) - f(u_i) | \Big(|u_i - u_j| \\ &\quad + \sum_{l \in I_i} \frac{\varepsilon \Delta t |a_{i\,l}|}{A_i} |u_i - u_l| + \sum_{l \in I_j} \frac{\varepsilon \Delta t |a_{j\,l}|}{A_j} |u_j - u_l| \Big) \\ &\leq \quad C \sum_T h_T^{N-1} \sum_{i<j} \left\{ h_T^{1-\frac{N}{2}} \left(- \sum_{l<m} a_{l\,m}^T (u_l - u_m)^2 \right)^{\frac{1}{2}} \right. \\ &\quad \left. \cdot \sum_{T' \subset \cup_j \Omega_j \cup_i \Omega_i} h_T^{1-\frac{N}{2}} \left(- \sum_{l<m} a_{l\,m}^{T'} (u_l - u_m)^2 \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

We use the Schwarz inequality to obtain

$$\begin{aligned} |R_5| \quad &\leq \quad C\delta\varepsilon \left\{ \sum_T \sum_{i<j} \left(- \sum_{l<m} a_{l\,m}^T (u_l - u_m)^2 \right) \right\}^{\frac{1}{2}} \\ &\quad \cdot \left\{ \sum_T \sum_{i<j} \sum_{T' \subset \cup_j \Omega_j \cup_i \Omega_i} \left(- \sum_{l<m} a_{l\,m}^{T'} (u_l - u_m)^2 \right) \right\}^{\frac{1}{2}} \\ &\leq \quad C\delta\varepsilon \sum_T \left(- \sum_{l<m} a_{l\,m}^T (u_l - u_m)^2 \right) = C\delta\varepsilon \sum_E (-a_{i\,j})(u_i - u_j)^2 \\ &\leq \quad C\delta \sum_E (-a_{i\,j}) \frac{e^{\psi_j} (u_j - u_i)^2}{\int_{x_i}^{x_j} \frac{e^{\psi}}{\varepsilon}\, ds} h_{ij}. \end{aligned} \tag{33}$$

By the same way we get the same upper bound of $|R_4|$.
In addition we have

$$(u_k^{n+1} - u_k) u_k^{n+1} \geq \frac{1}{2} (u_k^{n+1})^2 - \frac{1}{2} (u_k)^2.$$

We take δ small enough, then by (32), (33) and (31) we obtain

$$\frac{1}{2} \sum_k (u_k^{n+1})^2 A_k - \frac{\varepsilon \Delta t}{2} \sum_E a_{i\,j} \frac{e^{\psi_j} h_{ij}}{\int_{x_i}^{x_j} e^{\psi}\, ds} (u_j^n - u_i^n)^2 \leq \frac{1}{2} \sum_k (u_k^n)^2 A_k.$$

By summing them up with respect to n, (16) follows finally.

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