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AN ASYMPTOTICAL $O((k+1)n^3L)$ AFFINE SCALING ALGORITHM FOR THE $P_*(k)$-MATRIX LINEAR COMPLEMENTRITY PROBLEM*

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Abstract

Based on the generalized Dikin-type direction proposed by Jansen et al in 1997, we give out in this paper a generalized Dikin-type affine scaling algorithm for solving the $P_*(\kappa)$-matrix linear complementarity problem (LCP). Form using high-order correctors technique and rank-one updating, the iteration complexity and the total computational turn out asymptotically $O((\kappa+1)\sqrt{n}L)$ and $O((\kappa+1)n^3L)$ respectively.

Key words: linear complementarity problem, $P_*(\kappa)$-matrix, affine scaling algorithm

1. Introduction

An LCP is normally for finding vectors $x, s \in \Re^n$ such that:

$$s = Mx + q, \quad x^T s = 0, \quad (x,s) \geq 0. \tag{1}$$

where $q \in \Re^n$ and $M \in \Re^{n\times n}$. An LCP is called monotonic if M is positive semi-definite. In this paper, M is assumed to be a $P_*(\kappa)$-matrix[6][9] i.e. for a $\kappa \geq 0$, M satisfies:

$$(1+4\kappa)\sum_{u_i(Mu)_i\geq 0} u_i(Mu)_i + \sum_{u_i(Mu)_i\leq 0} u_i(Mu)_i \geq 0$$

for any $u \in \Re^n$. Obviously, positive semi-definite matrix is a $P_*(0)$-matrix. It was proved in [10] that M is a $P_*(\kappa)$-matrix iff M is a sufficient[1].

Based on Dikin's approach, Monteiro and Adler proposed in [8] an affine scaling algorithm of primal-dual type for LP whose iteration complexity is $O(nL^2)$, and Jansen et al gave out lately in [3] a primal-dual algorithm whose iteration complexity is $O(nL)$. Later, Jansen et al[5] made an improvement on the complexity of the algorithm given in [3] such that the iteration complexity obtained is asymptotical $O(\sqrt{n}L)$, and the total computational complexity is asymptotical $O(n^{3.5}L)$. Recently, Jansen et al made an unified generalization in [4] of the primal-dual affine scaling directions and, starting from an arbitrary feasible pair (x^0, s^0), produced a generalized Dikin-type affine scaling algorithm for the monotone LCP, of which the iteration complexity is $O(\frac{n}{\rho^2(1-\rho^2)}\log\frac{(x^0)^\top s^0}{\varepsilon})$.

In this paper, we consider the $P_*(\kappa)$-matrix LCP. Based on the generalized Dikin-type direction given in [5], we give out an r-order generalized Dikin-type affine scaling algorithm by using the high-order correctors technique and the rank-one updating, where r is an integer in $[1, \sqrt{n}]$. The iteration complexity of our algorithm is $O((\kappa+1)n^{(r+1)/(2r)}\log\frac{(x^0)^\top s^0}{\varepsilon})$, and the total computational complexity is $O((\kappa+1)(n^{2.5}+rn^2)n^{(r+1)/(2r)}\log\frac{(x^0)^\top s^0}{\varepsilon})$. If $r = \lfloor\sqrt{n}\rfloor$ in

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particular, then the iteration complexity becomes asymptotically $O((\kappa+1)\sqrt{n}\log\frac{(x^0)^\top s^0}{\varepsilon})$, and the total computational complexity bound becomes asymptotically $O((\kappa+1)n^3\log\frac{(x^0)^\top s^0}{\varepsilon})$.

2. An r-Order Algorithm

In this paper, the following notations are adopted: For $u, v \in \Re^n_+$, let $\min(u)$ and $\max(u)$ denote reapectively $\min_{1\le i\le n} u_i$ and $\max_{1\le i\le n} u_i$, and let uv and $u^h (h \in \Re)$ represent respentively vectors of $\Re^n$ that $(uv)_i = u_i v_i$ and $(u^h)_i = (u_i)^h$.

Denote the set of strict feasible solution $\{(x,s) \in \Re^n \times \Re^n : s = Mx + q, (x,s) > 0\}$ by $\mathcal{F}$, and let

$$\mathcal{N}_\infty(\beta) = \{(x,s) \in \mathcal{F} : \|xs - \mu e\|_\infty \le \beta\mu\}$$

where $\mu = x^T s/n$ and $\beta \in (0,1)$.

In this paper, we assume $\mathcal{F} \ne \emptyset$; thus, the system (1) is solvable[6].

Our algorithm is as follows:

The algorithm is to be initiated from a given pair (x^0, s^0) that satisfies $(x^0, s^0) \in \mathcal{N}_\infty(\beta)$.

Step 0: Set $k := 0$.

Step 1: Set $(x,s) := (x^k, s^k)$. If $x^T s \le \varepsilon$ ($\varepsilon > 0$ is a pre-set tolerance error), stop.

Step 2: Let $\gamma \in (0,1)$, and choose $(\tilde{x}, \tilde{s}) \in \Re^n_+ \times \Re^n_+$ such that

$$(\tilde{x}_i)^{-1}|x_i - \tilde{x}_i| \le \gamma \ \text{ and } \ (\tilde{s}_i)^{-1}|s_i - \tilde{s}_i| \le \gamma \ \text{ for } \ i = 1, 2, \cdots, n. \tag{2}$$

Step 3: Let $w = xs$ and $\ell \ge 1$. compute $(d_x^{(1)}, d_s^{(1)})$ from

$$d_s^{(1)} = M d_x^{(1)}, \quad \tilde{s} d_x^{(1)} + \tilde{x} d_s^{(1)} = -\frac{w^{\ell+1}}{\|w^\ell\|}. \tag{3}$$

Step 4: For $j = 2, \cdots, r$, compute $d_x^{(j)}, d_s^{(j)})$ from

$$d_s^{(j)} = M d_x^{(j)}, \quad \tilde{s} d_x^{(j)} + \tilde{x} d_s^{(j)} = -\sum_{t=1}^{j-1} d_x^{(t)} d_s^{(j-t)}. \tag{4}$$

Step 5: Choose a step length $\bar{\alpha} > 0$ such that the new $(x(\bar{\alpha}), s(\bar{\alpha}))$,

$$x(\bar{\alpha}) = x + (1+\gamma)\sum_{j=1}^r \bar{\alpha}^j d_x^{(j)}, \quad s(\bar{\alpha}) = s + (1+\gamma)\sum_{j=1}^r \bar{\alpha}^j d_s^{(j)},$$

is in $\mathcal{N}_\infty^-(\beta)$.

Step 6: Set $(x^{k+1}, s^{k+1}) := (x(\bar{\alpha}), s(\bar{\alpha}))$, $k := k+1$ and go to Step 1.

The quantity $-\frac{w^{\ell+1}}{\|w^\ell\|}$ given in the step 3 (which was first introduced by Jansen et al[4]) is a generalized Dikin-type affine scaling; when $\ell = 0$, this quantity turns out a classical primal-dual affine scaling[10]; when $\ell = 1$, it becomes a primal-dual Dikin affine scaling[3][5].

For the sake of notational simplicity, we omit in the following discussion the superscript k unless otherwise specified.

Let $w = xs$ and $\tilde{w} = \tilde{x}\tilde{s}$. It is not diffcult to obtain the following results by (2).

$$(1+\gamma)^{-2} w_i \le \tilde{w}_i \le (1-\gamma)^{-2} w_i; \tag{5}$$

$$(1-\gamma)\tilde{x} \le x \le (1+\gamma)\tilde{x}, \quad (1-\gamma)\tilde{s} \le s \le (1+\gamma)\tilde{s}; \tag{6}$$

$$0 < 1-\gamma \le x_i(\tilde{x}_i)^{-1} \le 1+\gamma, \quad 0 < 1-\gamma \le s_i(\tilde{s}_i)^{-1} \le 1+\gamma. \tag{7}$$

Let $x(\alpha) = x + (1+\gamma)\sum_{j=1}^r \alpha^j d_x^{(j)}, s(\alpha) = s + (1+\gamma)\sum_{j=1}^r \alpha^j d_s^{(j)}$, where α is a certain step length. Suppose $w(\alpha)=x(\alpha)s(\alpha)$, we have

$$\begin{aligned} w(\alpha)=x(\alpha)s(\alpha) &= \left\{x+(1+\gamma)\sum_{j=1}^{r}\alpha^j d_x^{(j)}\right\}^T \left\{s+(1+\gamma)\sum_{j=1}^{r}\alpha^j d_s^{(j)}\right\} \\ &\le xs+(1+\gamma)^2\sum_{j=1}^{r}\alpha^j(\tilde{x}d_s^{(j)}+\tilde{s}d_x^{(j)})+(1+\gamma)^2\sum_{j=2}^{2r}\alpha^j\left\{\sum_{t=1}^{j-1}d_x^{(t)}d_s^{(j-t)}\right\} \\ &= w-(1+\gamma)^2\alpha\frac{w^{\ell+1}}{\|w^\ell\|}+(1+\gamma)^2\sum_{j=r+1}^{2r}\alpha^j\left\{\sum_{t=j-r}^{r}d_x^{(t)}d_s^{(j-t)}\right\} \end{aligned} \quad (8)$$

where the inequality comes from (6), and the last equality from (3) and (4).

Let D denote $\tilde{x}^{-1/2}\tilde{s}^{1/2}$. Using (3) and (4), it is not difficult to verify that

$$\|Dd_x^{(j)}\|^2+\|D^{-1}d_s^{(j)}\|^2+2(d_x^{(j)})^Td_s^{(j)}=\|q^{(j)}\|^2, \quad (9)$$

where $q^{(1)}=-\tilde{w}^{-1/2}\frac{w^{\ell+1}}{\|w^\ell\|}$, and $q^{(j)}=\tilde{w}^{-1/2}\Big\{-\sum_{t=1}^{j-1}d_x^{(t)}d_s^{(j-t)}\Big\}$ for $j=2,\cdots,r$.

It can be verified (see to Lemma 3.4 and 4.20 in [6]) that

$$-\kappa\|q^{(j)}\|^2\le(d_x^{(j)})^Td_s^{(j)}\le(1/4)\|q^{(j)}\|^2$$

$j=1,2,\cdots,r$; hence, for $j=1,2,\cdots,r$ we have

$$\|Dd_x^{(j)}\|\le\sqrt{2\kappa+1}\|q^{(j)}\|^2 \text{ and } \|D^{-1}d_s^{(j)}\|\le\sqrt{2\kappa+1}\|q^{(j)}\|. \quad (10)$$

Without loss of generality we assume $\mu=1$(otherwise, an scaling can be performed to achieve this). The following Lemma provides an upperbound of $\|q^{(j)}\| \quad for \quad j=1,\cdots,r.$

Lemma 1. *Let $\phi(j)$ be defined as: $\phi(1)=1$ and $\phi(j)=\sum_{t=1}^{j-1}\phi(t)\phi(j-t) \quad for \quad j=2,\cdots,r$. We have*

(i) $\|q^{(1)}\|^2\le(1+\gamma)^2(1+\beta)$;

(ii) $\|q^{(j)}\|\le\frac{(1+\gamma)^{j-1}\phi(j)(2\kappa+1)^{j-1}}{(1-\beta)^{(j-1)/2}}\|q^{(1)}\|^j$ *for* $j=1,\cdots,r$.

Proof. For the (i): Inequality (5) implies $(1-\gamma)w_i^{-1/2}\le\tilde{w}_i^{-1/2}\le(1+\gamma)w_i^{-1/2}$; hence, from $(x,s)\in\mathcal{N}_\infty(\beta)$ and the definition of $q^{(1)}$, we obtain

$$\begin{aligned} \|q^{(1)}\|^2 &\le \left\|(1+\gamma)w_i^{-1/2}\frac{w^{\ell+1}}{\|w^\ell\|}\right\|^2\le(1+\gamma)^2\frac{\|w^{\ell+1/2}\|^2}{\|w^\ell\|^2} \\ &\le(1+\gamma)^2\frac{\|w^\ell\min(w^{1/2})\|^2}{\|w^\ell\|^2}\le(1+\gamma)^2(1+\beta). \end{aligned}$$

For the (ii): We prove this by induction on j. For $j=1$ the inequality is obviously trivial.

Now, we assume it holds for $1 \le p < j$. By taking noticing of (10) we have

$$\|q^{(j)}\| = \left\| \tilde{w}^{-1/2} \sum_{t=1}^{j-1} d_x^{(t)} d_s^{(j-t)} \right\| \le (1+\gamma)\|w^{-1/2}\| \left\| \sum_{t=1}^{j-1} d_x^{(t)} d_s^{(j-t)} \right\|$$

$$\le \frac{(1+\gamma)}{(1-\beta)^{1/2}} \sum_{t=1}^{j-1} \|Dd_x^{(t)}\| \, \|D^{-1}d_s^{(j-t)}\| \le \frac{(1+\gamma)(2\kappa+1)}{(1-\beta)^{1/2}} \sum_{t=1}^{j-1} \|q^{(t)}\| \, \|q^{(j-t)}\|$$

$$\le \frac{(1+\gamma)(2\kappa+1)}{(1-\beta)^{1/2}} \sum_{t=1}^{j-1} \left\{ \frac{(1+\gamma)^{t-1}(2\kappa+1)^{t-1}\phi(t)}{(1-\beta)^{(t-1)/2}} \|q^{(1)}\|^t \right\} \bullet$$

$$\left\{ \frac{(1+\gamma)^{j-t-1}(2\kappa+1)^{j-t-1}\phi(j-t)}{(1-\beta)^{(j-t-1)/2}} \|q^{(1)}\|^{j-t} \right\}$$

$$= \frac{(1+\gamma)^{j-1}(2\kappa+1)^{j-1}\phi(j)}{(1-\beta)^{(j-1)/2}} \|q^{(1)}\|^j.$$

This completes the proof of the lemma 1.

3. The Iteration Complexity

Theorem 1. *The algorithm produces in $O((\kappa+1)n^{(r+1)/(2r)} \log \frac{(x^0)^T s^0}{\varepsilon})$ iterations an ε-approximate solution (x,s) to (1) i.e. $(x,s) \in \{(x,s) \in \mathcal{F} : x^T s < \varepsilon\}$.*

The proof uses two following lemmas.

Lemma 2. *Let $\theta = (2\kappa+1)\alpha$. If $\theta \le 1$, then*

$$\sum_{j=r+1}^{2r} \alpha^j \left\{ \sum_{t=j-r}^{r} \|Dd_x^{(t)}\| \, \|D^{-1}d_s^{(j-t)}\| \right\} \le \frac{\theta^{r+1}}{(2\kappa+1)} (1+\gamma)^{4r-2}(1-\beta) \left(\frac{1+\beta}{1-\beta} \right)^r \frac{16^r}{8r}; \quad (11)$$

if $\theta > 1$, then

$$\sum_{j=r+1}^{2r} \alpha^j \left\{ \sum_{t=j-r}^{r} \|Dd_x^{(t)}\| \, \|D^{-1}d_s^{(j-t)}\| \right\} \le \frac{\theta^{2r}}{(2\kappa+1)} (1+\gamma)^{4r-2}(1-\beta) \left(\frac{1+\beta}{1-\beta} \right)^r \frac{16^r}{8r}; \quad (12)$$

Proof. From Lemma 1 we have

$$\sum_{t=j-r}^{r} \|Dd_x^{(t)}\| \, \|D^{-1}d_s^{(j-t)}\| \le (2\kappa+1) \sum_{t=1}^{j-1} \|q^{(t)}\| \, \|q^{(j-t)}\|$$

$$\le (2\kappa+1) \sum_{t=j-r}^{r} \left\{ \frac{(1+\gamma)^{t-1}(2\kappa+1)^{t-1}\phi(t)}{(1-\beta)^{(t-1)/2}} \|q^{(1)}\|^t \right\} \bullet$$

$$\left\{ \frac{(1+\gamma)^{j-t-1}(2\kappa+1)^{j-t-1}\phi(j-t)}{(1-\beta)^{(j-t-1)/2}} \|q^{(1)}\|^{j-t} \right\}$$

$$= (1+\gamma)^{2j-2}(2\kappa+1)^{j-1}(1-\beta) \left(\frac{1+\beta}{1-\beta} \right)^{j/2} \phi(2r).$$

Since the function $(1+\gamma)^{2j-2}((1+\beta)/(1-\beta))^{j/2}$ is increasing in j and $\phi(j) \le 2^{2j-2}/j$, so

$$\sum_{j=j-r}^{r} \|Dd_x^{(t)}\| \, \|D^{-1}d_s^{(j-t)}\| \le (2\kappa+1)^{j-1}(1+\gamma)^{4r-2}(1-\beta) \left(\frac{1+\beta}{1-\beta} \right)^r \frac{16^r}{8r}.$$

Therefore, (11) follows from $\alpha^j(2\kappa+1)^{j-1} \le \theta^{r+1}/(2\kappa+1) \quad j=r+1,\cdots,2r$ and (12) follows from $\alpha^j(2\kappa+1)^{j-1} \le \theta^{2r}/(2\kappa+1) \quad j=r+1,\cdots,2r$.

Lemma 3. *Assume* $\sigma = \min\{2\beta, \frac{(1-\beta)^{\ell+1}}{(\ell+1)(1+\beta)^{\ell+1}}\}$ *and* $\alpha \le \frac{\sqrt{n}\sigma}{(1+\gamma)^2(1+\beta)}$. *For* $i=1,\cdots,n$ *we have*

(i) $w_i - (1+\gamma)^2\alpha\frac{w_i^{\ell+1}}{\|w^\ell\|} - (1-\beta)\left(1 - \frac{(1+\gamma)^2\alpha}{n}\frac{\sum_{i=1}^n w_i^{\ell+1}}{\|w^\ell\|}\right) \ge (1+\gamma)^2\alpha\beta\left(\frac{1-\beta}{1+\beta}\right)^\ell \frac{1}{\sqrt{n}}$;

(ii) $(1+\beta)\left(1 - \frac{(1+\gamma)^2\alpha}{n}\frac{\sum_{i=1}^n w_i^{\ell+1}}{\|w^\ell\|}\right) - \left(w_i - (1+\gamma)^2\alpha\frac{w_i^{\ell+1}}{\|w^\ell\|}\right) \ge (1+\gamma)^2\alpha\beta\left(\frac{1-\beta}{1+\beta}\right)^\ell \frac{1}{\sqrt{n}}$.

Proof. For the (i): Since the formula $w_i - (1+\gamma)^2\alpha w_i^{\ell+1}/\|w^\ell\|$ attains its minimum at $\bar{w}_i = 1-\beta$ or $1+\beta$ when being viewed as a function of w_i; so, in the case of $\bar{w}_i = 1-\beta$, we have

$$\begin{aligned}
&1-\beta-(1+\gamma)^2\alpha\frac{(1-\beta)^{\ell+1}}{\|w^\ell\|} - (1-\beta)\left(1-\frac{(1+\gamma)^2\alpha}{n}\frac{\sum_{i=1}^n w_i^{\ell+1}}{\|w^\ell\|}\right)\\
=&(1+\gamma)^2\alpha(1-\beta)\left\{\frac{\sum_{i=1}^n w_i^{\ell+1}}{n} - (1-\beta)^\ell\right\}\frac{1}{\|w^\ell\|}\\
\ge&(1+\gamma)^2\alpha(1-\beta)\{(1-\beta)^{\ell+1}-(1-\beta)^\ell\}\frac{1}{(1+\beta)^\ell\sqrt{n}}\\
=&(1+\gamma)^2\alpha\beta\left(\frac{1-\beta}{1+\beta}\right)^\ell\frac{1}{\sqrt{n}},
\end{aligned}$$

where the inequality follows from

$$\sum_{i=1}^n w_i^{\ell+1} \ge \min(w^{\ell-1})\sum_{i=1}^n w_i^2 \ge (1-\beta)^{\ell-1}\|w\|^2 \ge (1-\beta)^{\ell-1}n$$

(the last inequality above comes from $\|w\| \ge \sqrt{n}$, which holds because of $e^Tw = n$ and the Cauchy-Schwartz inequality) and

$$\|w^\ell\| = \sqrt{\sum_{i=1}^n w_i^{2\ell}} \le \sqrt{\max(w^{2\ell-2})\sum_{i=1}^n w_i^2} \le (1+\beta)^\ell\sqrt{n}$$

(the last inequality above holds because $\|w\| \le (1+\beta)\sqrt{n}$ (See to Proposition 3.1 in [5])).

In the case of $\bar{w}_i = 1+\beta$, we have

$$\begin{aligned}
&1+\beta-(1+\gamma)^2\alpha\frac{(1+\beta)^{\ell+1}}{\|w^\ell\|} - (1-\beta)\left\{1-\frac{(1+\gamma)^2\alpha}{n}\frac{\sum_i^n w_i^{\ell+1}}{\|w^\ell\|}\right\}\\
=&2\beta+(1+\gamma)^2\alpha\left\{(1-\beta)\frac{\sum_{i=1}^n w_i^{\ell+1}}{n} - (1+\beta)^{\ell+1}\right\}\frac{1}{\|w^\ell\|}\\
\ge&2\beta+(1+\gamma)^2\alpha\{(1-\beta)^\ell-(1+\beta)^{\ell+1}\}\frac{1}{(1+\beta)^\ell\sqrt{n}}\\
=&(1+\gamma)^2\alpha\beta\left(\frac{1-\beta}{1+\beta}\right)^\ell\frac{1}{\sqrt{n}}\left\{\frac{2\sqrt{n}}{(1+\gamma)^2\alpha((1-\beta)/(1+\beta))^\ell} + \frac{1}{\beta} - \left(\frac{1+\beta}{1-\beta}\right)^\ell\frac{1+\beta}{\beta}\right\}\\
\le&(1+\gamma)^2\alpha\beta\left(\frac{1-\beta}{1+\beta}\right)^\ell\frac{1}{\sqrt{n}}
\end{aligned}$$

where the last inequality comes from $\alpha \le \frac{2\beta\sqrt{n}}{(1+\gamma)^2(1+\beta)}$ and $\frac{1}{\beta} \ge 1$.

For the (ii): It is obvious that the formula $w_i - (1+\gamma)^2\alpha w_i^{\ell+1}/\|w^\ell\|$ attains its maximum at $\hat{w}_i = \frac{\|w\|^{1/\ell}}{[(1+\gamma)^2\alpha(\ell+1)]^{1/\ell}}$ when being viewed as a function of w_i. Now, we have either $\hat{w}_i \notin (1-\beta, 1+\beta)$ or $\hat{w}_i \in (1-\beta, 1+\beta)$. In the case that $\hat{w}_i \notin (1-\beta, 1+\beta)$, then $w_i - (1+\gamma)^2\alpha w_i^{\ell+1}/\|w^\ell\|$ attains its maximum in $w_i \in (1-\beta, 1+\beta)$ at $\bar{w}_i = 1-\beta$ or $1+\beta$. Now, the proof for the (ii) is similar to that of (i). In the case that $\hat{w}_i \in (1-\beta, 1+\beta)$, then $w_i - (1+\gamma)^2\alpha w_i^{\ell+1}/\|w^\ell\|$ attains its maximum at $\bar{w}_i = \hat{w}_i$. For $i = 1, \cdots, n$ we have

$$\begin{aligned}
&(1+\beta)\left\{1 - \frac{(1+\gamma)^2\alpha}{n}\frac{\sum_{i=1}^n w_i^{\ell+1}}{\|w^\ell\|}\right\} - \left\{\hat{w}_i - (1+\gamma)^2\alpha\frac{\hat{w}_i^{\ell+1}}{\|w^\ell\|}\right\} \\
\geq &(1+\gamma)^2\alpha\frac{\hat{w}_i^{\ell+1}}{\|w^\ell\|} - \frac{(1+\gamma)^2\alpha(1+\beta)}{n}\frac{\sum_{i=1}^n w_i^{\ell+1}}{\|w^\ell\|} \\
= &(1+\gamma)^2\alpha\left\{\frac{\hat{w}_i\|w^\ell\|}{(1+\gamma)^2\alpha(\ell+1)} - (1+\beta)\frac{\sum_{i=1}^n w_i^{\ell+1}}{\|w^\ell\|}\right\}\frac{1}{\|w^\ell\|} \\
\geq &(1+\gamma)^2\alpha\left\{\frac{(1-\beta)^\ell\sqrt{n}}{(1+\gamma)^2\alpha(\ell+1)} - (1+\beta)^{\ell+2}\right\}\frac{1}{(1+\beta)^\ell\sqrt{n}} \\
\geq &\frac{(1+\gamma)^2\alpha}{(1+\beta)^\ell\sqrt{n}}\frac{(1+\beta)^{\ell+2}\beta}{1-\beta} \\
\geq &(1+\gamma)^2\alpha\beta\left(\frac{1-\beta}{1+\beta}\right)\frac{1}{\sqrt{n}}
\end{aligned}$$

where the second inequality comes from $\bar{w}_i \geq 1-\beta$, $\|w^\ell\| \geq (1-\beta)^{\ell-1}\sqrt{n}$ and $\sum_{i=1}^n w^{\ell+1} \leq (1+\beta)^{\ell+1}n$, the third inequality from $\alpha \leq \frac{\sqrt{n}(1-\beta)^{\ell+1}}{(1+\gamma)^2(1+\beta)^{\ell+2}(\ell+1)}$, and the last inequality from $\frac{(1+\beta)^2}{1-\beta} > 1 > \left(\frac{1-\beta}{1+\beta}\right)^\ell$.

This completes the proof of the lemma 2.

Now, we prove the theorem 1 as follows:

The key to the estimation of the iteration complexity lies in determining the step length $\hat{\alpha}$. For a pair $(x, s) \in \mathcal{N}_\infty(\beta)$, we then set to find out an α such that $(x(\alpha), s(\alpha)) \in \mathcal{N}_\infty(\beta)$ i.e.

$$(1-\beta)\mu(\alpha)e \leq w(\alpha) \leq (1+\beta)\mu(\alpha)e \tag{13}$$

where $\mu(\alpha) = e^T w(\alpha)/n$. Let $\xi = \sum_{j=r+1}^{2r}\alpha^j\left\{\sum_{t=j-r}^{r}(d_x^{(t)})^T d_s^{(j-t)}\right\}$. From (8), we have

$$\mu(\alpha) = 1 - \frac{(1+\gamma)^2\alpha}{n}\frac{\sum_{i=1}^n w_i^{\ell+1}}{\|w^\ell\|} + \frac{(1+\gamma)^2}{n}\xi. \tag{14}$$

From (13), it can be seen that if

$$|\xi| \leq (1+\beta)\left(1 - \frac{(1+\gamma)^2\alpha}{n}\frac{\sum_{i=1}^n w_i^{\ell+1}}{\|w^\ell\|} + \frac{(1+\gamma)^2}{n}\xi\right) - \left(w_i - (1+\gamma)^2\alpha\frac{w_i^{\ell+1}}{\|w^\ell\|}\right) \tag{15}$$

and

$$|\xi| \leq w_i - (1+\gamma)^2\alpha\frac{w_i^{\ell+1}}{\|w^\ell\|} - (1-\beta)\left(1 - \frac{(1+\gamma)^2\alpha}{n}\frac{\sum_{i=1}^n w_i^{\ell+1}}{\|w^\ell\|} + \frac{(1+\gamma)^2}{n}\xi\right) \tag{16}$$

hold for $i = 1, \cdots, n$, then (13) follows. Now, by taking notice of $n/(n + (1+\gamma)^2(1+\beta)) \geq 1/5$, we have: if

$$|\xi| \leq \frac{1}{5}\left\{(1+\beta)\left(1 - \frac{(1+\gamma)^2\alpha}{n}\frac{\sum_{i=1}^n w_i^{\ell+1}}{\|w^\ell\|}\right) - \left(w_i - (1+\gamma)^2\alpha\frac{w_i^{\ell+1}}{\|w^\ell\|}\right)\right\} \tag{17}$$

and

$$|\xi| \leq \frac{1}{5}\left\{w_i - (1+\gamma)^2\alpha\frac{w_i^{\ell+1}}{\|w^\ell\|} - (1-\beta)\left(1 - \frac{(1+\gamma)^2\alpha}{n}\frac{\sum_{i=1}^n w_i^{\ell+1}}{\|w^\ell\|}\right)\right\} \tag{18}$$

hold for $i = 1, \cdots, n$, then (15) and (16) hold also.

By the definition of ξ, it is easy to verify that

$$|\xi| \leq \sum_{j=r+1}^{2r} \alpha^j \left\{\sum_{t=j-r}^{r} \|Dd_x^{(t)}\| \, \|D^{-1}d_s^{(j-t)}\|\right\}. \tag{19}$$

For the sake of simplity, we only consider the case of $\theta \leq 1$. (In fact, for the case of $\theta > 1$, discussion can be carried out similarly.)

From (13)–(18), the lemma 2, and the lemma 3, it can be seen that if

$$\alpha \leq \frac{\sqrt{n}\sigma}{(1+\gamma)^2(1+\beta)}$$

and

$$\frac{\theta^{r+1}}{2\kappa+1}(1+\gamma)^{4r-2}(1-\beta)\left(\frac{1+\beta}{1-\beta}\right)^r \frac{16^r}{8r} \leq \frac{1}{5}(1+\gamma)^2\alpha\beta\left(\frac{1-\beta}{1+\beta}\right)^\ell \frac{1}{\sqrt{n}}$$

hold, then $(x(\alpha), s(\alpha)) \in \mathcal{N}_\infty(\beta)$. According to the second inequality above, we have

$$\theta \leq \frac{1-\beta}{16(1+\gamma)^4(1+\beta)}\left(\frac{8r(1+\gamma)^4\beta}{5(1-\beta)}\right)^{1/r}\left(\frac{1-\beta}{1+\beta}\right)^{\ell/r}\frac{1}{n^{1/(2r)}}. \tag{20}$$

Therefore, the step length $\bar{\alpha}$ in the k-th iteration can be choosen as

$$\bar{\alpha} = \min\left\{\frac{1-\beta}{16(1+\gamma)^4(1+\beta)}\left(\frac{8r(1+\gamma)^4\beta}{5(1-\beta)}\right)^{1/r}\left(\frac{1-\beta}{1+\beta}\right)^{\ell/r}\frac{1}{(2\kappa+1)n^{1/(2r)}}, \frac{\sqrt{n}\sigma}{(1+\gamma)^2(1+\beta)}\right\}. \tag{21}$$

Obviously, $n \geq \frac{1}{5}(1+\gamma)^2\beta\left(\frac{1-\beta}{1+\beta}\right)$. Thus, from (20) and (21), teh following loose upperbound of $\bar{\alpha}$ can be obtained,

$$\bar{\alpha} \leq \frac{1-\beta}{16(1+\gamma)^4(1+\beta)}\left(\frac{8r(1+\gamma)^2(1+\beta)}{(1-\beta)^2}\right)^{1/r}\left(\frac{1-\beta}{1+\beta}\right)^{\ell/r}\frac{n^{1/(2r)}}{2\kappa+1}.$$

Substituting the above bound into (11) and taking notice of (19), we obtain

$$|\xi| \leq \left(\frac{1-\beta}{1+\beta}\right)^{\ell-1}\sqrt{n}\bar{\alpha}.$$

Using (14), we have

$$\begin{aligned}
\mu(\bar{\alpha}) &\leq 1 - \frac{(1+\gamma)^2\bar{\alpha}}{n}\frac{e^\top w^{\ell+1}}{\|w^\ell\|} + \frac{(1+\gamma)^2}{n}|\xi| \\
&\leq 1 - (1+\gamma)^2\frac{(1-\beta)^{\ell-1}}{(1+\beta)^\ell}\frac{\bar{\alpha}}{\sqrt{n}} + 2(1+\gamma)^2\left(\frac{1-\beta}{1+\beta}\right)^{\ell-1}\frac{\bar{\alpha}}{\sqrt{n}} \\
&= 1 - \frac{\delta}{(2\kappa+1)n^{(r+1)/(2r)}}
\end{aligned} \tag{22}$$

where $\delta = (1+\gamma)^2\frac{1+2\beta}{1+\beta}\left(\frac{1-\beta}{1+\beta}\right)^{\ell-1}\bar{\alpha}$ (see to (21)).

Now, from the key inequality (22) we have

$$(x^{k+1})^T s^{k+1} \leq \left\{1 - \frac{\delta}{(2\kappa+1)n^{(r+1)/(2r)}}\right\}(x^k)^T s^k \leq \left\{1 - \frac{\delta}{(2\kappa+1)n^{(r+1)/(2r)}}\right\}^k \varepsilon_0;$$

therefore, the rest of the proof can be carried out routinely as follows. Obviously

$$\left\{1-\frac{\delta}{(2\kappa+1)n^{(r+1)/(2r)}}\right\}^k \varepsilon_0 \le \varepsilon$$

is equivalent to

$$k\log\left\{1-\frac{\delta}{(2\kappa+1)n^{(r+1)/(2r)}}\right\} \le \log(\varepsilon/\varepsilon_0),$$

hence, to

$$k \ge \frac{1}{\delta}(2\kappa+1)n^{(r+1)/(2r)}\log(\varepsilon_0/\varepsilon)$$

(because $\log(1-\eta) \le -\eta$ for $\eta < 1$); so, after no less than $\lceil \frac{1}{\delta}(2\kappa+1)n^{(r+1)/(2r)}\log(\varepsilon_0/\varepsilon)\rceil$ iterations, an ε-approximate solution to (1) is then obtained i.e. the iteration complexity of our algorithm is $O((\kappa+1)n^{(r+1)/(2r)}\log(\varepsilon_0/\varepsilon))$.

By taking $r = \lfloor\sqrt{n}\rfloor$, the iteration complexity then turns out asymptotcally $O((\kappa+1)\sqrt{n}\log\frac{(x^0)^\top s^0}{\varepsilon})$.

4. The Total Computational Complexity

In this section, we specify the choice of $(\tilde{x},\tilde{s})$ in our algorithm by using the following scheme of rank-one updating.

The Rank-One Updating.

For $k=0$, set $\tilde{x}^0; = x^0, \tilde{s}^0; = s^0$.

For $k > 0$, let $(x^k, s^k) \in \mathcal{N}_\infty(\beta)$ and let $(\tilde{x}^k,\tilde{s}^k)$ satisfy (2). Compute (x^{k+1}, s^{k+1}) as indicated in the step 6 given in the section 2. Set

$$(\tilde{x}_i^{k+1}, (\tilde{s}_i^{k+1}) := \begin{cases} (x_i^{k+1}, s_i^{k+1}), & \text{if } (\tilde{x}_i^k)^{-1}|x_i^{k+1}-\tilde{x}_i^k| > \gamma, \text{ or } (\tilde{s}_i^k)^{-1}|s_i^{k+1}-\tilde{s}_i^k| > \gamma, \\ (\tilde{x}_i^k, (\tilde{s}_i^k) & \text{otherwise.} \end{cases}$$

Now, the β and γ are required to meet additionally the following assumptions:

(i) $\beta \in [0.2, 1)$,

(ii) $\beta^2(1+\gamma)(2r/5)^{2/r} < 16(1-\gamma)[(1-\beta^2)(1+\ell)-\beta^2(2r/5)^{2/r}]$.

It is obvious that such assumptions can be met, for instance let $\gamma = 0.1$ and $\beta = 0.25$.

In this section, we only discuss the case of $\theta \le 1$. The case of $\theta > 1$ can be dealt with similarly.

Since from the assumption (i) above it can be proved that $\frac{1}{\ell+1}\left(\frac{1-\beta}{1+\beta}\right)^{\ell+1} < 2\beta$; therefore, if we take

$$\bar{\alpha} = \frac{1-\beta}{16(1+\gamma)^4(1+\beta)(\ell+1)}\left(\frac{8r(1+\gamma)^4\beta}{5(1-\beta)}\right)^{1/r}\left(\frac{1-\beta}{1+\beta}\right)^{\ell+1}\frac{1}{(2\kappa+1)n^{1/(2r)}}, \tag{23}$$

the theorem 1 still holds.

Theorem 2. *Let the algorithm be specified with the above β,γ and the rank-one updating; then it has asymptotically a total computational complexity of $O((\kappa+1)n^3\log\frac{(x^0)^T s^0}{\varepsilon})$.*

Lemma 4. *Assume that $(x,s) \in \mathcal{N}_\infty(\beta), (\tilde{x},\tilde{s}) \in \Re^n_+ \times \Re^n_+$ where $(\tilde{x},\tilde{s})$ satisfies (2), and assumptions (i) and (ii) are met. Denote the new iterated point by $(\hat{x},\hat{s})$ and let*

$$\rho^* = \frac{(1+\gamma)^3}{(1-\gamma)(1-\beta)}\sum_{j=1}^{r}\frac{(1+\gamma)^{4j-2}(1+\beta)^j\phi^2(j)(2\kappa+1)^{j-1}}{(1-\beta)^{j-1}}.$$

Then, we have

$$\|x^{-1}(\hat{x}-x)\|^2 \le \rho^* < 1 \text{ and } \|s^{-1}(\hat{s}-s)\|^2 \le \rho^* < 1.$$

Proof. We only need to prove the first inequality above (the second can be proved similarly). Let D_{ii} be the i-th diagonal element in $D = \tilde{x}^{-1/2}\tilde{s}^{1/2}$, it is easy to see

$$\|x^{-1}(\hat{x}-x)\|^2 = (1+\gamma)^2 \sum_{i=1}^{n}\sum_{j=1}^{r}(x_i^{-1}d_{x_i}^{(j)})^2\hat{\alpha}^{2j}$$

$$= (1+\gamma)^2 \sum_{i=1}^{n}\sum_{j=1}^{r}(D_{ii}^{-1}x_i^{-1})^2(D_{ii}d_{x_i}^{(j)})^2(\hat{\alpha})^{2j}. \quad (24)$$

From $(x,s) \in \mathcal{N}_\infty(\beta)$ and (7), we have

$$(x_i^{-1}D_{ii}^{-1})^2 = (\tilde{x}_i^{-1}x_i)(s_i\tilde{s}_i^{-1})(w_i^{-1}) \le \frac{1+\gamma}{(1-\gamma)(1-\beta)}.$$

Form the above inequality and (24), it follows

$$\|x^{-1}(\hat{x}-x)\|^2 \le \frac{(1+\gamma)^3}{(1-\gamma)(1-\beta)}\sum_{j=1}^{r}\|Dd_x^{(j)}\|^2(\hat{\alpha})^{2j}$$

$$\le \frac{(1+\gamma)^3}{(1-\gamma)(1-\beta)}\sum_{j=1}^{r}(2\kappa+1)\|q^{(j)}\|^2\hat{\alpha}^{2j}$$

$$\le \frac{(1+\gamma)^3}{(1-\gamma)(1-\beta)}\sum_{i=1}^{r}\frac{(1+\gamma)^{4j-2}(1+\beta)^j\phi^2(j)(2\kappa+1)^{2j-1}}{(1-\beta)^{j-1}}\bar{\alpha}^{2j} = \rho^*.$$

Obviously, $4\beta > 1-\beta$ for $\beta \in [0.2, 1)$; hence, from (23) we have

$$\bar{\alpha} < \frac{(2r/5)^{1/r}\beta}{4(1+\gamma)^2(1+\beta)(\ell+1)}\frac{1}{(2\kappa+1)n^{1/(2r)}}.$$

Since $\phi(j) \le 2^{2j-2}/j < 2^{2j-2}, j = 1, \cdots, r$; so,

$$\rho^* < \frac{(1+\gamma)}{16(1-\gamma)(1-\beta)}\sum_{i=1}^{r}\frac{\beta^{2j}(2r/5)^{2j/r}}{(1+\beta)^j(1-\beta)^{j-1}(\ell+1)^{2j}}$$

$$< \frac{(1+\gamma)}{16(1-\gamma)(1-\beta)}\frac{\frac{\beta^2(2r/5)^{2/r}}{(1+\beta)(\ell+1)}[(1+\beta)^r(1-\beta)^r(\ell+1)^r - \beta^{2r}(2r/5)^2]}{(1+\beta)^{r-1}(1-\beta)^{r-1}[(1+\beta)(1-\beta)(\ell+1) - \beta^2(2r/5)^{2/r}]}.$$

By the assumption (i) i.e. $\beta \in [0.2, 1)$, it can be specified that $(1+\beta)(1-\beta)(\ell+1) > \beta^2(2r/5)^{2/r}$; therefore,

$$\rho^* < \frac{(1+\gamma)\beta^2(2r/5)^{2/r}}{16(1-\gamma)(1-\beta)(1+\beta)} \times$$

$$\frac{(1+\beta)^r(1-\beta)^r(\ell+1)^r}{(1+\beta)^{r-1}(1-\beta)^{r-1}(\ell+1)^{r-1}[(1+\beta)(1-\beta)(\ell+1) - \beta^2(2r/5)^{2/r}]}$$

$$= \frac{(1+\gamma)\beta^2(2r/5)^{2/r}}{16(1-\gamma)[(1-\beta^2)(\ell+1) - \beta^2(10r)^{2/r}]} < 1$$

where the last inequality is due to the assumption (ii).

Proof of Theorem 2. Let

$$S_i^K = \{k : |\tilde{x}_i^k|^{-1}|x_i^{k+1} - \tilde{x}_i^k| > \gamma, 1 \le k \le K\},$$
$$T_i^K = \{k : |\tilde{s}_i^k|^{-1}|s_i^{k+1} - \tilde{s}_i^k| > \gamma, 1 \le k \le K\},$$

and $\bar{\rho} = \sqrt{\rho^*}$. From the lemma 4 and the proposition 5.2 given in [7], it can be derived that

$$\max\left\{\sum_{i=1}^{n}|S_i^K|, \sum_{i=1}^{n}|S_i^K|\right\} \le -\frac{\bar{\rho}\sqrt{n}K}{(1-\bar{\rho})\ln(1-\gamma)}.$$

This demonstates that the total number of rank-one updating occurred within K major steps of iteration is bounded by $O(\sqrt{n}K)$. Since the algorithm finds out an ϵ-paaroximate solution to the problem (1) in $O((\kappa+1)n^{(r+1)/(2r)}\log(\varepsilon_0/\varepsilon))$ iterations, and each rank-one updating involves $O(n^2)$ arithmetic operations, therefore the total number of arithmetic operations is bounded by $O((\kappa+1)(n^{2.5}+rn^2)n^{(r+1)/(2r)}\log(\varepsilon_0/\varepsilon))$. Now, when taking $r=\lfloor\sqrt{n}\rfloor$ the total computational complexity bound turns out asymptotically $O((\kappa+1)n^3\log\frac{(x^0)^Ts^0}{\varepsilon})$.

Now, suppose the data of the problem are given in rational numbers. Let L be the data size[7], $\varepsilon=2^{-L}$ and $\varepsilon_0\leq 2^L$ (hence $\log(\varepsilon_0/\varepsilon)\leq 2L$); it can be concluded from the theorem 1 and 2 that the iteration complexity of the algorithm tends to be $O((\kappa+1)\sqrt{n}L)$, and the total computational complexity becomes asymtotically $O((\kappa+1)n^3L)$.

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