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ATTRACTORS FOR DISCRETIZATION OF GINZBURG-LANDAU-BBM EQUATIONS*[1)]

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Abstract

In this paper, Ginzburg-Landau equation coupled with BBM equation with periodic initial boundary value conditions are discreted by the finite difference method in spatial direction. Existence of the attractors for the spatially discreted Ginzburg-Landau-BBM equations is proved. For each mesh size, there exist attractors for the discretized system. Moreover, finite Hausdorff and fractal dimensions of the discrete attractors are obtained and the bounds are independent of the mesh sizes.

Key words: Attractor, Spatially discreted, Ginzburg-Landau-BBM equations, Hausdorff and fractal dimensions.

1. Introduction

In this paper, we consider the following periodic initial value problem for the system of Ginzburg-Landau equation coupled with BBM equation

$$\varepsilon_t + \nu\varepsilon - (\alpha_1 + i\alpha_2)\varepsilon_{xx} + (\beta_1 + i\beta_2)|\varepsilon|^2\varepsilon - i\delta n\varepsilon = g_1(x), \tag{1.1}$$

$$n_t + nn_x + \gamma n - \alpha n_{xx} - n_{xxt} + |\varepsilon|^2_x = g_2(x), \tag{1.2}$$

$$\varepsilon(x + 2\pi, t) = \varepsilon(x, t), \quad n(x + 2\pi, t) = n(x, t), \tag{1.3}$$

$$\varepsilon(x, 0) = \varepsilon_0(x), \quad n(x, 0) = n_0(x). \tag{1.4}$$

where $\varepsilon(x,t)$ is a complex function, $n(x,t)$ is a real scalar function, ν, α, δ, γ, α_1, α_2, β_1, β_2 are real constants, and $g_1(x), g_2(x)$ are given real functions.

This problem describes the nonlinear interactions between Langmuir wave and ion acoustic wave in plasma physics, $\varepsilon(x,t)$ denotes electric field, $n(x,t)$ the perturbation of density (see [1, 9, 2]). In [3] Guo proved the global existence of smooth solution ($\alpha_1 = \nu = 0$), and in [6], Guo and Jiang considered the periodic initial value problem with the weak dissipative case and obtained the upper bounds of Hausdorff and fractal dimensions for the global attractors, both are on a bounded domain. In [7] Guo and Jiang studied the existence of the global attractors of the problem (1.1)-(1.2) and (1.4) on an unbounded domain.

Although the existence and uniqueness of the global smooth solution for the problem (1.1)-(1.4) in one dimension have been obtained, but we still do not know if the existence of the attractors for the spatially finite difference equations of the problem (1.1)-(1.4) is available and the attractors independent of the mesh sizes. Moreover, we still need to know if the finite

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dimensionality of the attractors still hold for the spatially finite difference equations. On the other hand, in order to simulate the properties of the solution numerically, we need to discrete the equations. These are the main questions we would like to consider. In this paper, we discrete Ginzburg-Landau equation coupled with BBM equation with the periodic initial value conditions by the finite difference method in spatial directions. It is proved that for each mesh size, there exist attractors for the discretized systems. The bounds for the Hausdorff and fractal dimensions of the discrete attractors are obtained, and the various bounds are independent of the mesh size.

2. Discretization of Ginzburg-Landau-BBM Equations and a Priori Estimates

Let $\Omega = [0, 2\pi]$ be a domain in real one dimensional Euclidean space $\mathbf{R}$, and J a nonnegative integer and $h = \frac{2\pi}{J}$ the space step length. The interval Ω is divided into discrete lattice $\Omega_h = \{x_1, x_2, \cdots, x_J\}$, where $x_j = jh$. The discretized function $u = (u_1, u_2, \cdots, u_J)^T$ and $u_j = u(x_j)$.

The difference operators are defined by

$$u_{jx} = \frac{1}{h}(u_{j+1} - u_j) = \nabla_h u_j, \quad u_{j\overline{x}} = \frac{1}{h}(u_j - u_{j-1}),$$

$$u_{j\hat{x}} = \frac{1}{2h}(u_{j+1} - u_{j-1}) = \frac{1}{2h}\Delta_0 u_j, \;\; u_{jx\overline{x}} = \frac{1}{h^2}(u_{j+1} - 2u_j + u_{j-1}) = \Delta_h u_j.$$

Spatially finite difference discretized version of the problem (1.1)-(1.4) can be defined by

$$\frac{d}{dt}\varepsilon_j + \nu\varepsilon_j - (\alpha_1 + i\alpha_2)\varepsilon_{jx\overline{x}} + (\beta_1 + i\beta_2)|\varepsilon_j|^2\varepsilon_j - i\delta n_j\varepsilon_j = g_{1j}, \tag{2.1}$$

$$\frac{d}{dt}n_j + D_0 f(n_j) + \gamma n_j - \alpha n_{jx\overline{x}} - n_{jx\overline{x}t} + |\varepsilon_j|^2_{\hat{x}} = g_{2j}, \tag{2.2}$$

$$\varepsilon_{j+rJ}(t) = \varepsilon_j(t), \quad n_{j+rJ}(t) = n_j(t), \tag{2.3}$$

$$\varepsilon_j(0) = \varepsilon_0(x_j), \quad n_j(0) = n_0(x_j). \tag{2.4}$$

where $D_0 f(n_j) = \frac{1}{3}n_j\Delta_0 n_j + \frac{1}{3}\Delta_0(n_j^2)$. $\nu, \gamma, \alpha, \alpha_1, \beta_1 > 0$, r is an integer, $j = 1, 2, \cdots, J$.

Denote the scalar product of two discrete complex periodic functions $u = \{u_j \mid j = 1, 2, \cdots, J\}$ and $v = \{v_j \mid j = 1, 2, \cdots, J\}$ by

$$(u, v)_h = \sum_{j=1}^{J} u_j \overline{v}_j h.$$

Here $\overline{v}_j$ denotes the complex conjugate of v_j. For the norms of the discrete function u and its difference quotients $\nabla_h^k u$ of order $k > 0$, we take the expressions

$$\|\nabla_h^k u\|_s = \left(\sum_{j=1}^{J-k} |\nabla_h^k u_j|^s h\right)^{\frac{1}{s}}, \quad 1 \leq s < \infty$$

and

$$\|\nabla_h^k u\|_\infty = \max_{j=1,\cdots,J-k} |\nabla_h^k u_j|,$$

where $\nabla_h^k = \nabla_h \cdot \nabla_h \cdots \nabla_h$ and $k \geq 0$ is any nonnegative integer and p is a real number.

Now we state some interpolation relations between the norms of several difference quotients for the discrete function u on the finite interval Ω.

Lemma 2.1. (Zhou [11]) *For any discrete function* $u = \{u_j | j = 1, 2, \cdot, J\}$, *there are*

$$\begin{aligned}
\|u\|_\infty &\le K_1 \|u\|^{\frac{1}{2}} (\|\nabla_h u\| + \|u\|)^{\frac{1}{2}},\\
\|\nabla_h u\| &\le K_2 \|u\|^{\frac{1}{2}} (\|\Delta_h u\| + \|u\|)^{\frac{1}{2}},\\
\|\nabla_h^k u\|_p &\le K_3 \|u\|^{1-\frac{k+\frac{1}{2}-\frac{1}{p}}{n}} (\|\nabla_h^n u\| + \|u\|)^{\frac{k+\frac{1}{2}-\frac{1}{p}}{n}}.
\end{aligned}$$

where K_1, K_2 *and* K_3 *are constants independent of the discrete functions* u *and the steplength* h, $2 \le p < \infty$, $0 \le k < n$.

Lemma 2.2. (Guo and Chang [4]) *Assume that the two complex discrete functions* $u = \{u_j |j = 1, \cdots, J\}$ *and* $v = \{v_j \ |j = 1, \cdots, J\}$ *satisfy the periodic conditions* $u_j = u_{j+J}$, $v_j = v_{j+J}$, *there are the identities*

$$(u_j v_j)_x = u_{j+1} v_{jx} + u_{jx} v_j, \quad (u, v_{\overline{x}}) = -(u_x, v).$$

Lemma 2.3. *Assume that* $\varepsilon_0(x) \in L_p^2(\Omega)$, $g_1(x) \in L_p^2(\Omega)$. *Then for the solution of the problem (2.1)-(2.4), there is*

$$\|\varepsilon(t)\|^2 \le \|\varepsilon_0\|^2 e^{-\nu t} + \frac{\|g_1\|^2}{\nu^2}(1 - e^{-\nu t}). \tag{2.5}$$

Proof. Multiplying (2.1) by $\overline{\varepsilon}_j$, summating the equation about j from 1 to J and taking the real part, one have

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon\|^2 + \nu\|\varepsilon\|^2 - \mathrm{Re}\sum_{j=1}^{J}(\alpha_1 + i\alpha_2)\varepsilon_{jx\overline{x}}\overline{\varepsilon}_j h + \mathrm{Re}\sum_{j=1}^{J}(\beta_1 + i\beta_2)|\varepsilon_j|^4 h = \sum_{j=1}^{J} g_{1j}\overline{\varepsilon}_j h.$$

This implies that

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon\|^2 + \frac{\nu}{2}\|\varepsilon\|^2 + \alpha_1\|\nabla_h\varepsilon\|^2 + \beta_1\|\varepsilon\|_4^4 \le \frac{1}{2\nu}\|g_1\|^2. \tag{2.6}$$

By Gronwall's inequality, we get (2.5) and

$$\overline{\lim_{t\to\infty}}\|\varepsilon(t)\|^2 \le \frac{\|g_1\|^2}{\nu^2} = E_0.$$

Lemma 2.4. *Suppose that* $\varepsilon_0(x), n_0(x) \in H_p^1(\Omega)$, $g_1(x) \in H_p^1(\Omega)$, $g_2(x) \in L_p^2(\Omega)$, $x \in \mathbf{R}$. *Then for the solution of the problem(2.1)-(2.4), there are*

$$\overline{\lim_{t\to\infty}}(\|\varepsilon(t)\|_{H_p^1}^2 + \|n(t)\|_{H_p^1}^2) \le E_1, \tag{2.7}$$

where E_1 *is a constant independent of the mesh size* h.

Proof. Multiplying (2.1) by $\overline{\varepsilon}_{jx\overline{x}}$, summating about j from 1 to J, taking the real part of equation, we have

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\|\nabla_h\varepsilon\|^2 + \nu\|\nabla_h\varepsilon\|^2 + \alpha_1\|\Delta_h\varepsilon\|^2 - \mathrm{Re}(\beta_1 + i\beta_2)\sum_{j=1}^{J}|\varepsilon_j|^2\varepsilon_j\overline{\varepsilon}_{jx\overline{x}}h\\
&\quad + \mathrm{Re}(i\delta\sum_{j=1}^{J} n_j\varepsilon_j\overline{\varepsilon}_{jx\overline{x}}h) = \sum_{j=1}^{J} g_{1jx}\overline{\varepsilon}_{jx}h.
\end{aligned} \tag{2.8}$$

Lemma 2.1 implies

$$\begin{aligned}\|u\|_6^3 &\le K_4\|u\|^{\frac{5}{2}}(\Delta_h u\| + \|u\|)^{\frac{1}{2}},\\ \|u\|_\infty &\le K_5\|u\|^{\frac{3}{4}}(\|\Delta_h u\| + \|u\|)^{\frac{1}{4}},\\ \|\nabla_h u\| &\le K_6\|u\|^{\frac{1}{2}}(\|\Delta_h u\| + \|u\|)^{\frac{1}{2}}.\end{aligned}$$

Noting that for $a,b>0, p<1, (a+b)^p \le a^p + b^p$. Thus by Young's inequality and above inequalities, we get

$$\begin{aligned}&\mathrm{Re}(\beta_1 + i\beta_2)\sum_{j=1}^{J}|\varepsilon_j|^2\varepsilon_j\overline{\varepsilon}_{jx\overline{x}}h\\ &\quad\le (\beta_1^2+\beta_2^2)^{\frac{1}{2}}\|\Delta_h\varepsilon\|\,\|\varepsilon\|_6^3\\ &\quad\le K_4(\beta_1^2+\beta_2^2)^{\frac{1}{2}}\|\Delta_h\varepsilon\|\,\|\varepsilon\|^{\frac{5}{2}}(\|\Delta_h\varepsilon\| + \|\varepsilon\|)^{\frac{1}{2}}\\ &\quad\le \tfrac{\alpha_1}{4}\|\Delta_h\varepsilon\|^2 + C_1,\\ &\mathrm{Re}(i\delta\sum_{j=1}^{J}(n_j\varepsilon_j)_x\overline{\varepsilon}_{jx}h) = \mathrm{Im}\delta\sum_{j=1}^{J}n_{jx}\varepsilon_j\overline{\varepsilon}_{jx}h\\ &\quad\le |\delta|\,\|\nabla_h\varepsilon\|\,\|\nabla_h n\|\,\|\varepsilon\|_\infty\\ &\quad\le K_7|\delta|\,\|\varepsilon\|^{\frac{5}{4}}(\|\Delta_h\varepsilon\| + \|\varepsilon\|)^{\frac{3}{4}}\|\nabla_h n\|\\ &\quad\le \tfrac{\alpha_1}{4}\|\Delta_h\varepsilon\|^2 + \tfrac{\alpha}{4}\|\nabla_h n\|^2 + C_2,\end{aligned}$$

where

$$\begin{aligned}C_1 &= \tfrac{1}{4}\big(\tfrac{\alpha_1}{6}\big)^{-3}K_4^4(\beta_1^2+\beta_2^2)^2E_0^5 + \tfrac{2}{\alpha_1}K_4^2(\beta_1^2+\beta_2^2)E_0^2,\\ C_2 &= 4\alpha^{-4}\big(\tfrac{\alpha_1}{3}\big)^{-3}K_7^8\delta^8E_0^5 + \tfrac{2}{\alpha_1}K_7^2\delta^2E_0^2.\end{aligned}$$

Thus, (2.8) follows

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h\varepsilon\|^2 + \frac{\nu}{2}\|\nabla_h\varepsilon\|^2 + \frac{\alpha_1}{2}\|\Delta_h\varepsilon\|^2 \le \frac{\alpha}{4}\|\nabla_h n\|^2 + C_3. \tag{2.9}$$

Secondly, multiplying (2.2) by n_j and summating about j from 1 to J, we get

$$\frac{1}{2}\frac{d}{dt}(\|n|^2 + \|\nabla_h n\|^2) + \gamma\|n\|^2 + \alpha\|\nabla_h n\|^2 + F_0(n_j) + \sum_{j=1}^{J}|\varepsilon_j|_{\hat{x}}^2 n_j h = \sum_{j=1}^{J}g_{2j}n_j h, \tag{2.10}$$

where

$$\begin{aligned}F_0(n_j) &= \sum_{j=1}^{J}n_jD_0f(n_j)h = \sum_{j=1}^{J}n_j\big(\frac{1}{3}n_j\Delta_0 n_j + \frac{1}{3}\Delta_0 n_j^2\big)h\\ &= \tfrac{1}{3}\sum_{j=1}^{J}[n_j^2(n_{j+1}-n_{j-1}) + n_j(n_{j+1}^2 - n_{j-1}^2)]h\\ &= 0,\\ -\sum_{j=1}^{J}|\varepsilon_j|_{\hat{x}}^2 n_j h &= -\tfrac{1}{2}\sum_{j=1}^{J}\big(|\varepsilon_j|_x^2 + |\varepsilon_j|_{\overline{x}}^2\big)n_j h \le \frac{\alpha}{2}\|\nabla_h n\|^2 + \frac{1}{2\alpha}\sum_{j=1}^{J}|\varepsilon_j|^4 h,\\ \tfrac{1}{2\alpha}\sum_{j=1}^{J}|\varepsilon_j|^4 h &= \tfrac{1}{2\alpha}\|\varepsilon\|_4^4 \le K_8\|\varepsilon\|^{\frac{7}{2}}(\|\Delta_h\varepsilon\| + \|\varepsilon\|)^{\frac{1}{2}}\\ &\le \tfrac{\alpha_1}{4}\|\Delta_h\varepsilon\|^2 + C_4.\end{aligned}$$

(2.10) gives

$$\frac{1}{2}\frac{d}{dt}(\|n\|^2+\|\nabla_h n\|^2)+\frac{\gamma}{2}\|n\|^2+\frac{\alpha}{2}\|\nabla_h n\|^2\le\frac{\alpha_1}{4}\|\Delta_h\varepsilon\|^2+C_5. \tag{2.11}$$

The summation of (2.6), (2.9) and (2.11) implies

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}(\|\varepsilon\|_{H^1}^2+\|n\|_{H^1}^2)+\tfrac{\nu}{2}\|\varepsilon\|^2+\big(\alpha_1+\tfrac{\nu}{2}\big)\|\nabla_h\varepsilon\|^2+\tfrac{\gamma}{2}\|n\|^2\\&+\tfrac{\alpha}{4}\|\nabla_h n\|^2+\tfrac{\alpha_1}{4}\|\Delta_h\varepsilon\|^2+\beta_1\|\varepsilon\|_4^4\le C_6.\end{aligned}$$

By Gronwall's inequality, (2.7) holds.

Lemma 2.5. *Suppose that $\varepsilon_0(x), n_0(x)\in H_p^2(\Omega), g_1(x)\in H_p^2(\Omega)$, $g_2(x)\in H_p^1(\Omega)$. Then for the solution of the problem (2.1)-(2.4), there are*

$$\overline{\lim_{t\to\infty}}(\|\Delta_h\varepsilon\|^2+\|\Delta_h n\|^2)\le E_2, \tag{2.12}$$

where E_2 is a constant independent of the mesh size h.

Proof. Multiplying (2.1) by $\overline{\varepsilon}_{jx\overline{x}x\overline{x}}$, summating about j from 1 to J, taking the real part, we have

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\|\Delta_h\varepsilon\|^2+\nu\|\Delta_h\varepsilon\|^2+\alpha_1\|\nabla_h^3\varepsilon\|^2-\mathrm{Re}(\beta_1+i\beta_2)\sum_{j=1}^{J}(|\varepsilon_j|^2\varepsilon_j)_x\overline{\varepsilon}_{jx\overline{x}x}h\\&-\mathrm{Re}(i\delta\sum_{j=1}^{J}(n_j\varepsilon_j)_{x\overline{x}}\overline{\varepsilon}_{jx\overline{x}}h)=\sum_{j=1}^{J}g_{1jx\overline{x}}\overline{\varepsilon}_{jx\overline{x}}h.\end{aligned} \tag{2.13}$$

Similarly

$$\begin{aligned}&\mathrm{Re}(\beta_1+i\beta_2)\sum_{j=1}^{J}(|\varepsilon_j|^2\varepsilon_j)_x\overline{\varepsilon}_{jx\overline{x}x}h\\&\quad\le 3(\beta_1^2+\beta_2^2)^{\frac{1}{2}}\|\varepsilon\|_\infty\|\nabla_h\varepsilon\|\,\|\nabla_h^3\varepsilon\|\\&\quad\le K_9\|\nabla_h^3\varepsilon\|\,\|\varepsilon\|^{\frac{7}{3}}(\|\nabla_h^3\varepsilon\|+\|\varepsilon\|)^{\frac{2}{3}}\\&\quad\le\tfrac{\alpha_1}{4}\|\nabla_h^3\varepsilon\|^2+C_7,\\&|\mathrm{Re}(i\delta\sum_{j=1}^{J}(n_j\varepsilon_j)_{x\overline{x}}\overline{\varepsilon}_{jx\overline{x}}h)|=|\mathrm{Im}\delta\sum_{j=1}^{J}(n_j\varepsilon_j)_x\overline{\varepsilon}_{jx\overline{x}x}h|\\&\quad\le|\delta|(\|n\|_\infty\|\nabla_h\varepsilon\|+\|\varepsilon\|_\infty\|\nabla_h n\|)\|\nabla_h^3\varepsilon\|\\&\quad\le\tfrac{\alpha_1}{4}\|\nabla_h^3\varepsilon\|^2+C_8.\end{aligned}$$

Thus, (2.13) gives

$$\frac{1}{2}\frac{d}{dt}\|\Delta_h\varepsilon\|^2+\frac{\nu}{2}\|\Delta_h\varepsilon\|^2+\frac{\alpha_1}{2}\|\nabla_h^3\varepsilon\|^2\le C_9. \tag{2.14}$$

On the other hand, multiplying (2.2) by $n_{jx\overline{x}}$ to get

$$\frac{1}{2}\frac{d}{dt}(\|\nabla_h n\|^2+\|\Delta_h n\|^2)+\frac{\gamma}{2}\|\nabla_h n\|^2+\frac{\alpha}{2}\|\Delta_h n\|^2\le C_{10}. \tag{2.15}$$

Combining (2.13) with (2.15), (2.12) holds.

3. Existence of the Attractors

In view of the priori estimates in section 2, one can obtain the existence of the attractors for the discrete system (2.1)-(2.4) with respect to $\|\cdot\|_{H_p^2}$-norm.

By Lemma 2.3 and 2.4, using Galerkin method, we have the following theorems.

Theorem 3.1. *Assume that* $\varepsilon_0(x), n_0(x) \in H_p^2(\Omega)$, $g_1(x) \in H_p^2(\Omega)$, $g_2(x) \in H_p^1(\Omega)$. *Then there exists a unique solution* $(\varepsilon, n) \in H_p^2 \times H_p^2$ *of the problem (2.1)-(2.4). The mapping* $(\varepsilon_0, n_0) \in H_p^2 \times H_p^2 \to (\varepsilon, n) \in H_p^2 \times H_p^2$ *defines a continuous semigroup* $S(t)$.

Theorem 3.2. *Let* $\Omega = [0, 2\pi]$ *and* $B = \{(\varepsilon, n) \in H_p^2 \times H_p^2; \|\varepsilon\|_{H_p^2}^2 + \|n\|_{H_p^2}^2 \leq 2E_2\}$ *The semigroup associates with the* 2π*-periodic initial boundary value problem (2.1)-(2.4) possesses a maximal attractor* $\mathcal{A} \subset B$, *which is bounded in* $H_p^2 \times H_p^2$. $\mathcal{A}$ *attractors the bounded set of* $H_p^2 \times H_p^2$ *and it is convex and connected.*

4. Estimates of the Upper Bounds of Hausdorff and Fractal Dimensions for the Attractors

In this section, we give the estimates of the upper of Hausdorff and fractal dimensions for the attractors. Writing the problem in the abstract form:

$$\frac{d}{dt}u_j(t) = F(u_j(t)).$$

We consider the following linear variation corresponding to the problem (2.1)-(2.4):

$$\frac{d}{dt}U_j + \nu U_j - (\alpha_1 + i\alpha_2)U_{jx\overline{x}} + (\beta_1 + i\beta_2)(2|\varepsilon|^2 U_j + \varepsilon^2\overline{U_j}) - i\delta(nU_j + \varepsilon V_j) = 0, \tag{4.1}$$

$$\frac{d}{dt}V_j + (nV_j)_{\hat{x}} + \gamma V_j - \alpha V_{jx\overline{x}} - V_{jx\overline{x}t} + 2\mathrm{Re}(\overline{\varepsilon}U_j)_{\hat{x}} = 0, \tag{4.2}$$

$$U_{j+rJ}(t) = U_j(t), \quad V_{j+rJ}(t) = V_j(t), \tag{4.3}$$

$$U_j(0) = u_0(x_j), \quad V_j(0) = v_0(x_j). \tag{4.4}$$

where (ε, n) is the solution of the problem (2.1)-(2.4) with the initial data (ε_0, n_0). It is easily proved that problem (4.1)-(4.4) has a global smooth solution as long as the solution of the problem (2.1)-(2.4) is mildly smooth, and the semigroup $S(t)$ of the problem (2.1)-(2.4) is Fréchet differentiable in $L_p^2 \times L_p^2$ with differential $L : (\varepsilon, n) \to (U, V)$, where (U, V) is the solution of the problem (4.1)-(4.4).

In fact, set $\zeta(t) = S(t)(\xi_0 + \eta_0) - S(t)\xi_0 - DS(t)\xi_0)\eta_0 = \xi_1(t) - \xi(t) - \eta(t)$, then

$$\begin{cases} \frac{d}{dt}\zeta &= F(\xi_1(t)) - F(\xi(t)) + L(\xi(t))\eta(t) \\ &= F(\xi(t) + \eta(t) + \zeta(t)) - F(\xi(t)) + L(\xi(t))\eta(t), \\ \zeta(0) &= 0. \end{cases} \tag{4.5}$$

therefore, (4.5) can be rewritten in the form

$$\frac{d}{dt}\zeta(t) + L(\xi(t)\zeta(t)) = \wedge(\xi, \eta, \zeta), \tag{4.6}$$

where

$$\wedge(\xi, \eta, \zeta) = F(\xi(t) + \eta(t) + \zeta(t)) - F(\xi(t)) + L(\xi(t))(\eta + \zeta).$$

Taking the scalar product of (4.6) with ζ, we find that

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 + (L(\xi(t))\zeta(t), \zeta(t)) \leq C(\|\zeta\|^2 + \| \wedge(\xi, \eta, \zeta)\|^2). \tag{4.7}$$

Using the theory of linear partial differential equations, estimating every terms in (4.7), we can get the L^2-estimate

$$\|\zeta(t)\| \leq C\|\eta_0\|^2.$$

It dives the differentiability of the semigroup operator $S(t)$ in $L_p^2 \times L_p^2$.

Now we introduce an energy equality of the solution $(U(t), V(t))$ of the problem (4.1)-(4.4). Set $u_j = e^{\gamma t}U_j, v_j = e^{\gamma t}V_j$, then (u_j, v_j) satisfies:

$$\begin{aligned} &\tfrac{d}{dt}u_j + (\nu-\gamma)u_j - (\alpha_1 + i\alpha_2)u_{jx\overline{x}} + (\beta_1 + i\beta_2)(2|\varepsilon|^2 u_j + \varepsilon^2\overline{u_j}) - i\delta(nu_j + \varepsilon v_j) = 0, \\ &\tfrac{d}{dt}v_j + (nu_j)_{\hat{x}} + (\gamma-\alpha)v_{jx\overline{x}} - v_{jx\overline{x}t} + 2\mathrm{Re}(\overline{\varepsilon}u_j)_{\hat{x}} = 0. \end{aligned} \tag{4.8}$$

Similar to (2.2), (2.8) and (2.10), we have

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\|u\|^2 + (\nu-\gamma)\|u\|^2 + \alpha_1\|\nabla_h u\|^2 + \mathrm{Im}\delta\sum_{j=1}^{J}(\varepsilon v_j \overline{u}_j h) \\ &+\mathrm{Re}(\beta_1 + i\beta_2)\sum_{j=1}^{J}(2|\varepsilon|^2|u_j|^2 + \overline{\varepsilon}^2\overline{u}_j^2)h = 0, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\|\nabla_h u\|^2 + (\nu-\gamma)\|\nabla_h u\|^2 + \alpha_1\|\Delta_h u\|^2 + \mathrm{Im}\delta\sum_{j=1}^{J}(nu_j\overline{u}_{jx\overline{x}} + \varepsilon v_j\overline{u}_{jx\overline{x}})h \\ &+\mathrm{Re}(\beta_1 + i\beta_2)\sum_{j=1}^{J}(2|\varepsilon|^2 u_j\overline{u}_{jx\overline{x}} + \varepsilon^2\overline{u}_j\overline{u}_{jx\overline{x}})h = 0, \end{aligned} \tag{4.10}$$

$$\frac{1}{2}\frac{d}{dt}[\|v\|^2 + \|\nabla_h v\|^2] + (\alpha-\gamma)\|\nabla_h v\|^2 + \sum_{j=1}^{J}(nv_j)_{\hat{x}}v_j h + 2\mathrm{Re}(\overline{\varepsilon}u_j)_{\hat{x}}v_j h = 0. \tag{4.11}$$

Define

$$J_\mu(u,v) = \|u\|^2 + \|\nabla_h u\|^2 + \mu\|v\|^2 + \mu\|\nabla_h v\|^2,$$

$$\begin{aligned} I_\mu(u,v) = \quad & 2[(\gamma-\nu)\|u\|^2 + (\gamma-\nu-\alpha_1)\|\nabla_h u\|^2 - \alpha_1\|\Delta_h u\|^2 - \mathrm{Im}\delta\sum_{j=1}^{J}(\varepsilon v_j\overline{u}_j h) \\ & -\mathrm{Re}(\beta_1 + i\beta_2)\sum_{j=1}^{J}(2|\varepsilon|^2|u_j|^2 + \overline{\varepsilon}^2\overline{u}_j^2)h \\ & -\mathrm{Re}(\beta_1 + i\beta_2)\sum_{j=1}^{J}(2|\varepsilon|^2 u_j\overline{u}_{jxx} + \varepsilon^2\overline{u}_j\overline{u}_{jx\overline{x}})h \\ & -\mathrm{Im}\delta\sum_{j=1}^{J}(nu_j\overline{u}_{jx\overline{x}} + \varepsilon v_j\overline{u}_{jxx})h + \mu(\gamma-\alpha)\|\nabla_h v\|^2 \\ & -\mu\sum_{j=1}^{J}(nv_j)_{\hat{x}}v_j h - 2\mu\mathrm{Re}(\overline{\varepsilon}u_j)_{\hat{x}}v_j h. \end{aligned}$$

Then (4.9)-(4.11) have

$$\frac{dJ_\mu(u,v)}{dt} = I_\mu(u,v), \tag{4.12}$$

where μ is a positive constant sufficiently large such that $J_\mu(u,v)$ induces an equivalent norm on $H_p^1 \times H_p^1$.

Let X be a bounded invariant of $S(t)$ in $H_p^1 \times H_p^1$, then

$$S(t)X = X, \ \text{ for all } \ t \geq 0.$$

Let $\xi_0 \in X$, then $S(t)\xi_0 = S(t)(\varepsilon_0, n_0) = (\varepsilon(t), n(t)) \in X$. So

$$\|X\|_\infty = \sup_{\xi_0\in X}\sup_{t\geq 0}\{\|\varepsilon(t)\|_\infty + \|\nabla_h\varepsilon(t)\|_\infty + \|n(t)\|_\infty + \|\nabla_h n(t)\|_\infty\}, \tag{4.13}$$

is finite. By definition of $J_\mu(u,v)$, we deduce that there exist $\mu>0$ and $M_1, M_2>0$ such that

$$M_2(\|u\|^2_{H^1_p}+\mu\|v\|^2_{H^1_p})\le J_\mu(u,v)\le M_1(\|u\|^2_{H^1_p}+\mu\|v\|^2_{H^1_p}), \tag{4.14}$$

for all $(u,v)\in X$, that is an equivalent norm on $H^1_p\times H^1_p$. Suppose that $\eta=(\eta_1,\eta_2)$, $\xi=(\xi_1,\xi_2)\in X$. Define

$$\Psi(\eta,\xi)=\nabla_h\eta_1\cdot\overline{\nabla_h\xi_1}+\nabla_h\eta_2\cdot\nabla_h\xi_2+\mu\eta_1\overline{\xi}_1+\mu\eta_2\overline{\xi}_2.$$

It is well known that $\Psi(\eta,\xi)$ is an $\mathbf{R}$-linear symmetric form on $H^1_p\times H^1_p$, and $\Psi(\eta,\eta)=J_\mu(\eta,\eta)$ is coercive by (4.13). It means $\Psi(\eta,\eta)^{\frac12}$ is an equivalent norm on $H^1_p\times H^1_p$.

Now let $\xi^j=(\varepsilon^j,n^j), j=1,2,\cdots,m$ be m elements in $H^1_p\times H^1_p$, $\xi^j(t)=(DS(t)\xi_0)\eta_0$ the corresponding solution of the problem (4.1)-(4.4). Let $\eta_j(t)=e^{\gamma t}\xi^j(t)$, $j=1,2,\cdots,m$. Then $\eta^j(t)=(u^j(t),v^j(t))$ satisfies (4.8) with $\eta^j(0)=(u^j_0,v^j_0)$. Note that

$$\begin{aligned}|\xi^1(t)\wedge\xi^2(t)\wedge\cdots\xi^m(t)|^2_{\wedge^m(H^1)} &= e^{-2\gamma mt}|\eta^1\wedge\eta^2\wedge\cdots\wedge\eta^m|_{\wedge^m(H^1)}\\ &= e^{-2\gamma mt}\det{}_{1\le i,j\le m}\Psi(\eta^i(t),\eta^j(t)).\end{aligned}$$

Set $H_m(t)=\det_{1\le i,j\le m}\Psi(\eta^i(t),\eta^j(t))$, we have

$$\frac{dH_m(t)}{dt}=H_m(t)\sum_{l=1}^m\max_{\substack{F\subset\mathbf{R}^m\\ dimF=l}}\min_{\substack{x\in F\\ x\ne 0}}\frac{I_\mu(\sum_{j=1}^m x_j\eta^j(t))}{J_\mu(\sum_{j=1}^m x_j\eta^j(t))}.$$

Since $\|n\|_\infty,\|\varepsilon\|_\infty$ are uniformly bounded for $(\varepsilon_0,n_0)\in H^1_p\times H^1_p$, and

$$\begin{aligned}
&(\gamma-\nu-\alpha_1)\|\nabla_h u\|^2\le(\gamma-\nu-\alpha_1)K_2^2\|u\|(\|\Delta_h u\|+\|u\|)\\
&\quad\le\tfrac{\alpha_1}{4}\|\Delta_h u\|^2+\left[\tfrac{(\gamma-\nu-\alpha_1)^2K_2^4}{\alpha_1}+(\gamma-\nu-\alpha_1)K_2^2\right]\|u\|^2,\\
&|\mathrm{Im}\delta\sum_{j=1}^J(\varepsilon v_j\overline{u}_jh)|\le|\delta|\|\varepsilon\|_\infty\|u\|\,\|v\|\le\frac12|\delta|\,\|\varepsilon\|_\infty(\|u\|^2+\|v\|^2),\\
&|\mathrm{Im}\delta\sum_{j=1}^J(nu_j\overline{u}_{jx\overline{x}}+\varepsilon v_j\overline{u}_{jx\overline{x}})h|\le|\delta|(\|n\|_\infty\|u\|+\|\varepsilon\|_\infty\|v\|)\|\Delta_h u\|\\
&\quad\le\tfrac{\alpha_1}{4}\|\Delta_h u\|^2+\tfrac{2\delta^2}{\alpha_1}(\|n\|^2_\infty\|u\|^2+\|\varepsilon\|^2_\infty\|v\|^2),\\
&|\mathrm{Re}(\beta_1+i\beta_2)\sum_{j=1}^J(2|\varepsilon|^2|u_j|^2+\overline{\varepsilon}^2\overline{u}_j^2)h|\le(\beta_1^2+\beta_2^2)^{\frac12}3\|\varepsilon\|^2_\infty\|u\|^2,\\
&|\mathrm{Re}(\beta_1+i\beta_2)\sum_{j=1}^J(2|\varepsilon|^2u_j\overline{u}_{jxx}+\varepsilon^2\overline{u}_j\overline{u}_{jxx})h\\
&\quad\le(bt_1^2+\beta_2^2)^{\frac12}3\|\varepsilon\|^2\|u\|\,\|\Delta_h u\|^2\\
&\quad\le\tfrac{\alpha_1}{4}\|\Delta_h u\|^2+\tfrac{9}{\alpha_1}(\beta_1^2+\beta_2^2)\|\varepsilon\|^4_\infty\|u\|^2,\\
&\sum_{j=1}^J(nv_j)_{\hat{x}}v_jh=\frac12\sum_{j=1}^Jn(v_{j+1}-v_{j-1}v_j)=0,\\
&|2\mathrm{Re}(\overline{\varepsilon}u_j)_{\hat{x}}v_jh\le\|\varepsilon\|_\infty\sum_{j=1}^Ju_{j\hat{x}}v_jh\le(\alpha-\gamma)\|\Delta_h v\|^2+\frac{\|\varepsilon\|^2_\infty}{4(\alpha-\gamma)}\|u\|^2.
\end{aligned}$$

Thus we get

$$\begin{aligned} I_\mu(u,v) &\le 2M_1(\|u\|^2+\|v\|^2)=2M_1(\|u\|^{\frac32}\|u\|^{\frac12}+\|v\|^{\frac32}\|v\|^{\frac12}) \\ &\le 2M_1\left(\|u\|_{H^1}^{\frac32}\|u\|^{\frac12}+\|v\|_{H^1}^{\frac32}\|v\|^{\frac12}\right), \end{aligned} \tag{4.15}$$

here

$$\begin{aligned} M_1=\max\Big\{&\tfrac12|\delta|\,\|\varepsilon\|_\infty+\tfrac{2\delta^2}{\alpha_1}\|\varepsilon\|_\infty^2\,, \\ &\gamma-\nu+(\gamma-\nu-\alpha)K_2^2+\tfrac{(\gamma-\nu-\alpha_1)^2K_2^4}{\alpha_1}+\tfrac12|\delta|\|\varepsilon\|_\infty+\tfrac{2\delta^2}{\alpha_1}\|n\|_\infty^2 \\ &+3(\beta_1^2+\beta_2^2)^{\frac12}\|\varepsilon\|_\infty^2+\tfrac{9}{\alpha_1}(\beta_1^2+\beta_2^2)\|\varepsilon\|_\infty^4+\tfrac{1}{4(\alpha_1-\gamma)}\|\varepsilon\|_\infty^2\Big\} \end{aligned}$$

From (4.15) and (4.15), we have

$$\begin{aligned} \frac{dH_m(t)}{dt} &\le \frac{2M_1}{M_2}H_m(t)\sum_{l=1}^m \max_{\substack{F\subset \mathbf{R}^m\\ \dim F=l}}\ \min_{\substack{x\in F\\ x\neq 0}} \\ &\left\{\frac{\|\sum_{j=1}^m x_ju_j\|_{H^1}^{\frac32}\|\sum_{j=1}^m x_ju_j\|^{\frac12}}{\|\sum_{j=1}^m x_ju_j\|_{H^1}^2+\|\sum_{j=1}^m x_jv_j\|_{H^1}^2}+\frac{\|\sum_{j=1}^m x_jv_j\|_{H^1}^{\frac32}\|\sum_{j=1}^m x_jv_j\|^{\frac12}}{\|\sum_{j=1}^m x_ju_j\|_{H^1}^2+\|\sum_{j=1}^m x_jv_j\|_{H^1}^2}\right\} \\ &\le \frac{2M_1}{M_2}H_m(t)\sum_{l=1}^m \max_{\substack{F\subset \mathbf{R}^m\\ \dim F=l}}\ \min_{\substack{x\in F\\ x\neq 0}}\left\{\left[\frac{\|\sum_{j=1}^m x_ju_j\|}{\|\sum_{j=1}^m x_ju_j\|_{H^1}}\right]^{\frac12}+\left[\frac{\|\sum_{j=1}^m x_jv_j\|}{\|\sum_{j=1}^m x_jv_j\|_{H^1}}\right]^{\frac12}\right\} \\ &\le \frac{4M_1}{M_2}H_m(t)\sum_{l=1}^m\frac{1}{\lambda_l^{\frac14}}, \end{aligned}$$

where we have used the Max-Min Theorem and λ_l is the l'th eigenvalue of $-\Delta$.

Note that $\phi_k\in L^2(\Omega), k\in N$ be an orthogonal basis which the eigenvectors of the difference operator $-\Delta_h$ and $\lambda_k=\frac{J^2}{\pi^2}\sin^2(\frac{k\pi}{J})$, $k=1,2,\cdots,\left[\frac J2\right]$ are the corresponding double eigenvalues of the $\{\phi_k\}$ and $\frac{\sin x}{x}>\frac{2}{\pi}$ $(0<x<\frac{\pi}{2})$, we have that: for $0<s<\frac12$

$$\begin{aligned} \sum_{k=1}^m\lambda_k^{-s} &= \sum_{k=1}^m\big(\frac{J}{\pi}\big)^{-2s}\sin^{-2s}\big(\frac{k\pi}{J}\big) \\ &\le \sum_{k=1}^m\big(\frac{2}{\pi}\big)^{-2s}k^{-2s} \\ &\le \big(\tfrac{2}{\pi}\big)^{-2s}\cdot\tfrac{1}{1-2s}\cdot\tfrac{1}{m^{2s-1}}. \end{aligned}$$

Then

$$\frac{dH_m(t)}{dt}\le\frac{4M_1}{M_2}H_m(t)\sum_{l=1}^m\lambda_l^{-\frac14}\le 4Cm^{\frac12}H_m(t),$$

and

$$H_m(t)\le e^{4Cm^{\frac12}t}H_m(0).$$

Then we have

Theorem 4.1. *The universal attractor $\mathcal{A}$ defined in Theorem 3.2 has finite Hausdorff and fractal dimensions in $H_p^1\times H_p^1$.*

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