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CONSTRUCTION OF A PRECONDITIONER FOR DOMAIN DECOMPOSITION METHODS WITH POLYNOMIAL LAGRANGIAN MULTIPLIERS*1)

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Abstract

In this paper we consider domain decomposition methods with polynomial Lagrangian multipliers to two-dimentional elliptic problems, and construct a kind of simple preconditioners for the corresponding interface equation. It will be shown that condition number of the resulting preconditioned interface matrix is almost optimal (namely, it has only logarithmic growth with dimension of the local interface space).

Key words: Domain Decomposition, Non-matching grids, Lagrangian multipliers, Preconditioner, Condition number.

1. Introduction

In recent years the non-overlapping domain decomposition methods (DDMs) with non-matching grids have attracted particular attention of computional experts and engineers (see [1]-[9]). This kind of DDM allows non-coincidence of nodal points at common edges (or common faces) of two neighbouring subdomains. Thus it can be applied to solving the problems of changing meshes (for example, the multi-body contact problems in solid mechanics and the relative motion problems in oil exploration), and can be applied to designing the optimal meshes, namely, one can choose different mesh-sizes and different orders of approximate polynomials in different subdomains according to different properties of solutions and different requirements of practical problems.

There are three kinds of important algorithms to deal with the interface non-conformity generated by the non-matching grids, namely, the mortar element method (see [1]-[3], [8]-[10]), the Lagrangian multipliers method (see [4], [5], [10]-[14]) and the augmented Lagrangian method (see [9]). For the mortar element method, the interface variable is chosen as a proper approximation of the trace of numerical solution on the interface, thus it is a direct extention of the usual non-overlapping DDM. For the Lagrangian multipliers method, the interface variable (namely, the Lagrangian multiplier) is chosen as a proper approximation of the normal derivative on the interface, which transforms the minimization problem with restriction (namely, weak continuity of the trace on the interface) into the corresponding saddle-point problem without restriction, thus it is the dual algorithm of the mortar element method. The augmented Lagrangian method can be understood as a mixed algorithm generated by combining the mortar element method with the Lagrangian multiplier method.

The DDM with Lagrangian multiplier (DDMLM) has obvious advantages over the mortar element DDM: (a) the interface variable associated with DDMLM need not be continuous at the cross-points (for the case of two-dimension) or on the cross-edges (for the case of three-dimension), so the corresponding interface equation can be formed easily; (b) the construction of the interface subspace associated with the DDMLM is flexible, thus the non-matching grids do not bring about any difficulty; (c) the DDMLM may reduce the size of the interface problem.

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In [5], we introduced and analysed a kind of DDMLM in which the space of the Lagrangian multipliers consists of polynomials of the certain degree n (in [11] and [12], this method was mentioned too). This method has advantages in comparison with another kind of DDMLM in which special partitioning of the interface is introduced and the space of the Lagrangian multiplier is chosen as the corresponding finite element space (refer to [1], [9], [10] and [12]-[14]): (i) the numerical integrations defined on the interface can be calculated conveniently; (ii) the size of the interface problem can be reduced greatly when the exact solution has good smoothness on the interface. However, for this kind of DDMLM condition number of the interface matrices is highly sensitive to the number n.

It is well known that, for non-overlapping DDMs, construction of interface preconditioners is a core problem. From the advantage (a) mentioned above, we know that construction of interface preconditioner associated with DDMLM is essentially different from the case of the usual non-overlapping DD method (how to construct coarse subspace ?).

In this paper we advanced a new idea (refer to [19] and [20]) in which the coarse subspace consists of piecewise contants. Based on this, we construct a kind of preconditioner for the DDMLM to two-dimensional elliptic problems, and show that condition number of the correspondding preconditioned interface matrix is almost optimal. The preconditioner proposed in here has obvious advantages: (i) it is independent of the cross-points, thus the computational procedure is easy to design (in comparison with the preconditioners constructed in [15] and [16]); (ii) it is independent of the trace space, thus the problems of changing meshes do not bring about any difficulty (note that all the preconditioners introduced in [4], [13] and [14] depend on the usual Scur complement); (iii) the local solvers are defined on the common edges of two neighbouring subdomains, thus it results in cheap calculation (in comparison with the preconditioners discussed in [8], [13], [14], [17] and [18]);

The idea advanced in this paper is also fit for three-dimensional elliptic problems (see [19]).

2. The DDMLM

For ease of notation, we consider the following model problems:

$$\begin{cases} -\Delta u + \eta u = f, & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset R^2$ is a polygonal domain, and η is a positive number which is bounded above.

The domain Ω is decomposed into N polygons Ω_i. We first make the following assumptions:
H_1: all subdomain Ω_i are of size d in the sense that there exists constants c_0 and c_1 independent of d such that each Ω_i contains (resp.is contained in) a circle of radius $c_0 d$ (resp.$c_1 d$);
H_2 :For $i \neq j$, we require that if two (open) edges $F' \subset \partial\Omega_i$ and $F'' \subset \partial\Omega_j$ share a common point, then $\overline{F}' = \overline{F}'' = \overline{\Omega}_i \bigcap \overline{\Omega}_j = F_{ij}$. Let F=$\bigcup F_{ij}$ denotes the entire interface;
H_3: each subdomain Ω_i consists of quasi-uniform triangular or quadrilateral elements with diamenter h_i.

For a natural number n, and $h = \max_{1\leq i\leq N} h_i$, we assume that
$H_4 : \lim_{h\to 0}(n^2 h/d) = 0$.

Since N is in general large, namely, d is small, thus we can assume that
$H_5 : \eta {\leq} C d^{-2}$.

Now, we define the approximation spaces as follows:

Let $S_{h_i}(\Omega_i)$ be the space of continuous piecewise m_i degree polynomials defined on Ω_i. Set

$$S_h(\Omega) = \{\varphi : \varphi\mid_{\partial\Omega} = 0, \varphi\mid_{\Omega_i} \in S_h(\Omega_i), i = 1, \cdots, N\}$$

$$S_n(F_{ij}) = \{\lambda_{ij} : \lambda_{ij}\ is\ a\ polynomial\ of\ degree {\leq} n\ on\ F_{ij}\}$$

$$S_n(\partial\Omega_i) = \{\lambda_i : \lambda_i\mid_{F_{ij}} \in S_n(F_{ij}), F_{ij} \subset \partial\Omega_i\}$$

$$S_n(F) = \{\lambda : \lambda\mid_{F_{ij}} \in S_n(F_{ij})\ for\ all\ F_{ij}\}$$

$$S_{h\times n} = S_h(\Omega) \times S_n(F)$$

Remark 1. The boundary nodes of the triangulation of Ω_i and Ω_j need not coincide on the common edges (namely, we have not imposed any matching conditions for the grids at subdomains interfaces, refer to [1]-[9]), thus $S_h(\Omega)$ need not be in $C(\Omega)$.Besides, the functions in $S_n(F)$ may be discontinuous on the cross-points.

Remark 2. The interface space $S_n(F)$ is independent of the nodes of the space $S_h(\Omega)$, thus non-matching grids will not bring about any difficulty in numerical integrations (on the interface).

Set $a_i(u,v) = (\nabla u, \nabla v)_{\Omega_i} + \eta(u,v)_{\Omega_i}$ and $< \cdot, \cdot >_{\partial\Omega_i} = \sum\limits_{F_{ij}\subset\partial\Omega_i} < \cdot, \cdot >_{F_{ij}}$ with $< \lambda, \mu >_{F_{ij}} = \int_{F_{ij}} \lambda\mu ds$. The discrete problem for (2.1) is: to find $(u_h, \lambda)\in S_{h\times n}$, such that

$$\begin{cases} \sum\limits_{i=1}^{N} \{a_i(u_h, v) - < \lambda, v >_{\partial\Omega_i}\} = \sum\limits_{i=1}^{N} (f,v)_{\Omega_i}, \ \forall v \in S_h(\Omega), \\ \sum\limits_{i=1}^{N} < u_h, \mu >_{\partial\Omega_i} = 0, \qquad \forall \mu \in S_n(F), \end{cases} \tag{2.2}$$

where μ (or λ) has opposite sign on the two sides of F_{ij}.

In the following discussion, C denotes a positive constant which is independent of h, n, d, η.

Set $\mathbf{X} = \prod\limits_i H^1(\Omega_i)$, $\mathbf{M} = H^{-\frac{1}{2}}(F)$ and define the norms:

$$\|v\|_{\mathbf{X}} = (\sum_{i=1}^{N} \|v\|^2_{1,\Omega_i})^{\frac{1}{2}}, \|\mu\|_{\mathbf{M}} = (\sum_{i=1}^{N} |\mu|^2_{-\frac{1}{2},\partial\Omega_i})^{\frac{1}{2}},$$

where

$$\|v\|^2_{1,\Omega_i} = (\nabla v, \nabla v)_{\Omega_i} + (v,v)_{\Omega_i}, |\mu|_{-\frac{1}{2},\partial\Omega_i} = \sup_{v\in H^{\frac{1}{2}}(\partial\Omega_i)} \frac{| < \mu, v >_{\partial\Omega_i} |}{\|v\|_{\frac{1}{2},\partial\Omega_i}}$$

with

$$\|v\|_{\frac{1}{2},\partial\Omega_i} = (|v|^2_{\frac{1}{2},\partial\Omega_i} + d^{-1}\|v\|^2_{0,\partial\Omega_i})^{\frac{1}{2}}$$

Here, LBB condition is satisfied under the assumption H_4 (see Theorem 1 of [5]). The assumption H_4 means that the dimension of the local interface $S_n(F_{ij})$ is much less than the number of degrees of freedom on the boundary F_{ij} from the approximation spaces $S_h(\Omega_i)$ and $S_h(\Omega_j)$. In [7], the case of $n = 0$ is not considered. In fact, the number n in the inverse estimate given by Lemma 3 of [5] should be replaced by $n+1$, thus this Theorem is true even if $n = 0$). On the other hand, from the generalized Poincaré inequality (see [7]), we know that the following ellipticity condition is fulfilled (even if η=0, refer to [1])

$$\sum_{i}^{N} a_i(v_h, v_h) \geq C\|v_h\|^2_{\mathbf{X}}, \quad \forall\ v_h \in \overline{S}_h(\Omega),$$

where $\overline{S}_h(\Omega) = \{v : v\in S_h(\Omega), \sum\limits_{i=1}^{N} < v, \mu >_{\partial\Omega_i} = 0, \ \ \forall \mu \in S_n(F)\}$.

Therefore, by the general theory of mixed finite element, we obtain the following result (refer to [5], from the proof given in [5] we can see that the smoothness assumptions of Theorem 2 given in [5] can be weaken, because λ is defined only on the interface):

Theorem 1. *Let* m=$\min m_i$. *Assume that* $u\in H^{m+1}(\Omega_i)$ *and* $u\ |_{\partial\Omega_i\backslash\partial\Omega}\ \in H^{k+\frac{1}{2}}(\partial\Omega_i\backslash\partial\Omega)$ $(i = 1, \cdots, N)$ *for some* $k\geq 2m$. *Then*

$$\|u_h - u\|_{\mathbf{X}} + \|\lambda - \frac{\partial u}{\partial n}\|_{\mathbf{M}} \leq Ch^m \sum_{i=1}^{N} (|u|_{m+1,\Omega_i} + d^{\frac{1}{2}}|u|_{k+\frac{1}{2},\partial\Omega_i\backslash\partial\Omega})$$

Remark 3. The assumption H_4 implies that $n^{-1} \geq (h/d)^{\frac{1}{2}}$ (when $h \to 0$), so, in order to obtain this optimal error estimate, we must assume in this Theorem that " $u\ |_{\partial\Omega_i\backslash\partial\Omega}\ \in H^{k+\frac{1}{2}}(\partial\Omega_i\backslash\partial\Omega)$ $(i = 1, \cdots, N)$ for some $k\geq 2m$", which means that the function u must be smooth enough in the "internal" subdomain Ω_i. This assumption can be satisfied in most circumstance. When the nature number k is great, n can be chosen as a small nature number (namely, the size of the interface problem can be reduced). This is just the merit of the well-known $p-version$ method.Like the $p-version$ method, this method is accurate and stable provided that the exact solution u has "good" smoothness in the "internal" subdomains, and n is not large.

Remark 4. Theorem 1 is also true for the case of η=0. Let $\{\phi_j^i\}_{j=1}^{N_i}$ and $\{\psi_k\}_{k=1}^{M}$ denotes respectively a set of basis functions on $S_h(\Omega_i)$ and $S_n(F)$, where ψ_k has opposite sign on the two sides of F_{ij}. The coordinate vectors of $u_h\mid_{\Omega_i}$ and λ in these bases are denoted by U_i and χ respectively. Set

$$a_{jk}^i = a_i(\phi_j^i, \phi_k^i), j, k = 1, \cdots, N_i$$

and

$$b_{rl}^i =< \phi_r^i, \psi_l >_{\partial\Omega_i}, r = 1, \cdots, N_i; l = 1, \cdots, M,$$

the forming matrices are denoted by A_i and B_i respectively;set

$$c_j^i = (f, \phi_j^i)_{\Omega_i}, j = 1, \cdots, N_i,$$

the forming vector is denoted by f_i. Then U_i and χ can be computed by (refer to [5], [12])

$$(\sum_{i=1}^{N} B_i^T A_i^{-1} B_i)\chi = -\sum_{i=1}^{N} B_i^T A_i^{-1} f_i \tag{2.3}$$

and

$$U_i = A_i^{-1}(f_i + B_i\chi), i = 1, \cdots, N$$

The following result can be proved as in [5].

Theorem 2. *Let* $\overline{S} = \sum_{i=1}^{N} B_i^T A_i^{-1} B_i$ *(which is a symmetric and positive definite matrix). Then the condition number of* $\overline{S}$ *can be estimated by*

$$cond(\overline{S}) \leq C(n^2 d^{-2} \eta^{-1}) \qquad (\ \eta \leq C d^{-2}\ ) \tag{2.4}$$

Remark 5. The inequality (2.4) indicates that the condition number of the interface matrix $\overline{S}$ may be great. Thus, it is important to construct an efficient preconditioner to $\overline{S}$.

3. Main Result

Let $\{\mu_r^{(ij)}\}_{r=1}^{n+1}$ be a set of basis functions of $S_n(F_{ij})$ $(i < j)$.Set

$$b_{kr}^{(ij)} =< \phi_k^i, \mu_r^{(ij)} >_{F_{ij}},\ r = 1, \cdots, n+1; k = 1, \cdots, N_i$$

and

$$b_{kr}^{(ji)} =< \phi_k^j, \mu_r^{(ij)} >_{F_{ij}},\ r = 1, \cdots, n+1;\ k = 1, \cdots, N_j,$$

the forming matrices are denoted respectively by B_{ij} *and* B_{ji}. Set

$$\overline{S}_{ij} = B_{ij}^T A_i^{-1} B_{ij} + B_{ji}^T A_j^{-1} B_{ji},$$

Let the set of the edges $\{F_{ij} : F_{ij} \subset \partial\Omega_i \ for\ some\ i\ satisfying\ \partial\Omega_i \bigcap \partial\Omega = \phi\}$ be written as $\{\Gamma_{0r}\}_{r=1}^{N_0}$, and $D_{0r} : S_n(\Gamma_{0r}) \rightarrow S_n(F)$ denotes the "zero extension" operator, namely,

$$D_{0r}\mu = \begin{cases} \mu, & on\ \Gamma_{0r}, \\ 0, & on\ F\backslash\Gamma_{0r}. \end{cases} \qquad \mu \in S_n(\Gamma_{0r})$$

Set

$$b_{kr}^{(0i)} =< \phi_k^i, D_{0r} 1 >_{\partial\Omega_i},\ k = 1, \cdots, N_i; r = 1, \cdots, N_0,$$

and let B_{0i} denotes the matrix whose elements are $b_{kr}^{(0i)}$.Set

$$\overline{S}_0 = \sum_{i=1}^{N} B_{0i}^T A_i^{-1} B_{0i}$$

It can be verified directly that the matrices $\overline{S}_0$ and $\overline{S}_{ij}$ are symmetric and positive definite (refer to [19], note that the assumption H_4). We define the preconditioner of the interface matrix $\overline{S}$ by

$$\overline{M} = I_0 \overline{S}_0^{-1} I_0^T + \sum_{i<j} \overline{D}_{ij} \overline{S}_{ij}^{-1} \overline{D}_{ij}^T$$

Here $\overline{D}_{ij}$ denotes the 0,1 sign matrix associated with the "zero extension" operator D_{ij} : $S_n(F_{ij}) \rightarrow S_n(F)$, and I_0 denotes the 0,1 sign matrix defined by

$$(D_{01}1, \cdots, D_{0N_0}1) = (\psi_1, \cdots, \psi_M) I_0.$$

Remark 6. If there is no internal subdomain, then $\overline{M} = \sum_{i<j} \overline{D}_{ij} \overline{S}_{ij}^{-1} \overline{D}_{ij}^T$. However, this case is not of interest.

Theorem 3. *Under the assumptions given in Section 2, we have*

$$cond(\overline{M}\ \overline{S}) \leq C(1+\log^2 n).$$

Remark 7. Because $D_{0r}1$ is a "natural" basis function of $S_n(F)$, the matrix $\overline{S}_0$ is a submatrix of $\overline{S}$ (in fact, $\overline{S}_0$ is just the interface matrix $\overline{S}$ when n=0). Besides, $\overline{S}_{ij}$ is just the diagonal subblock of $\overline{S}$. Moreover, $\overline{S}_0$ is a lower order and sparse matrix; the order of the "local" matrix $\overline{S}_{ij}$, which equals to n+1, is in general very low. Therefore, action of $\overline{S}_0^{-1}$ and $\overline{S}_{ij}^{-1}$ can be implemented easily. On the other hand, vertices do not play any role in the definition of $\overline{M}$. These mean that the preconditioners presented in this paper have obvious advantages over the other preconditioners: they not only have simple structure but also have cheap computation.

4. Proof of Theorem 3

In order to prove Theorem 3, we have to consider the operator form of $\overline{M}$.

Let $R_i : S_n(\partial\Omega_i)\to S_h(\Omega_i)$ and $R_{ij} : S_n(F_{ij})\to S_h(\Omega_i)$ denotes the operators defined respectively by

$$a_i(R_i\lambda_i, v) = <\lambda_i, v>_{\partial\Omega_i}, \lambda_i\in S_n(\partial\Omega_i), \forall v\in S_h(\Omega_i)$$

and

$$a_i(R_{ij}\lambda_{ij}, v) = <\lambda_{ij}, v>_{F_{ij}}, \lambda_{ij}\in S_n(F_{ij}), \forall v\in S_h(\Omega_i).$$

The operators $S_i : S_n(F)\to S_n(F)$ and $S_{ij} : S_n(F_{ij})\to S_n(F_{ij})$ are defined respectively by

$$<S_i\lambda, \mu>_{\partial\Omega_i} = a_i(R_i\lambda, R_i\mu), \lambda\in S_n(F), \forall\mu\in S_n(F),$$

and

$$<S_{ij}\lambda_{ij}, \mu_{ij}>_{F_{ij}} = a_i(R_{ij}\lambda_{ij}, R_{ij}\mu_{ij}) + a_j(R_{ji}\lambda_{ij}, R_{ji}\mu_{ij}),$$
$$\lambda_{ij}\in S_n(F_{ij}), \forall\mu_{ij}\in S_n(F_{ij}),$$

Set $S = \sum_{i=1}^{N} S_i$, then the operator form of the interface equation (2.3) can be written as

$$S\lambda = g, \quad \lambda, g\in S_n(F)$$

We define a coarse subspace of $S_n(F)$ by

$$V_0 = span\{D_{ij}1 : F_{ij} \subset \partial\Omega_i \ for\ some\ i\ satisfying\ \partial\Omega_i \cap \partial\Omega = \phi\}.$$

The coarse solver $S_0 : V_0\to V_0$ and the projection operator $Q_0 : S_n(F)\to V_0$ are defined respectively by

$$<S_0\lambda_0, \mu_0> = <S\lambda_0, \mu_0>, \lambda_0\in V_0,\ \forall\mu_0\in V_0,$$

and

$$<Q_0\lambda, \mu_0> = <\lambda, \mu_0>, \lambda\in S_n(F),\ \forall\mu_0\in V_0,$$

where $<\cdot,\cdot> = \sum_{i<j} <\cdot,\cdot>_{F_{ij}}$.

It can be verified directly that the operator forms (associated the given bases) of the preconditioner $\overline{M}$ is

$$M = S_0^{-1}Q_0 + \sum_{i<j} D_{ij}S_{ij}^{-1}D_{ij}^T$$

where $D_{ij}^T : S_n(F)\to S_n(F_{ij})$ is the "natural" restriction operator (namely, the adjoint operator of D_{ij}).

The following result can be proved in the usual way (refer to [23] and [24])

Lemma 1. *If the following conditions are satisfied:*

(1) *for any $\lambda\in S_n(F)$, there exists a decomposition $\lambda = \lambda_0 + \sum_{i<j} D_{ij}\lambda_{ij}$ with $\lambda_0\in V_0$ and $\lambda_{ij}\in S_n(F_{ij})$, such that*

$$<S_0\lambda_0, l\lambda_0> + \sum_{i<j} <S_{ij}\lambda_{ij}, \lambda_{ij}>_{F_{ij}} \leq \alpha_1 <S\lambda, \lambda>; \tag{4.1}$$

(2) *for any $\lambda_{ij}\in S_n(F_{ij})$, we have*

$$<S(\sum_{i<j} D_{ij}\lambda_{ij}), \sum_{i<j} D_{ij}\lambda_{ij}> \leq \alpha_2 \sum_{i<j} <S_{ij}\lambda_{ij}, \lambda_{ij}>_{F_{ij}}, \tag{4.2}$$

then

$$\alpha_1^{-1} < \lambda, S\lambda > \leq < MS\lambda, S\lambda > \leq \alpha_2 < \lambda, S\lambda >, \ \forall \ \lambda \in S_n(F), \tag{4.3}$$

namely,

$$cond(MS) \leq \alpha_1\alpha_2 \tag{4.4}$$

Lemma 2. *Assume that* $v \in H^{\frac{1}{2}}(F_{ij})$*. Then*

$$\|v - \gamma_{F_{ij}}(v)\|_{\frac{1}{2},F_{ij}} \leq C|v|_{\frac{1}{2},F_{ij}}, \tag{4.5}$$

where $\gamma_{F_{ij}}(v)$ *is the integration average of* v *on* F_{ij}*.*

Proof. Without loss of generality, we assume $F_{ij} = \{(x,0) : 0 \leq x \leq d\}$. Set $\widetilde{\Omega} = [0,d]^2$, and let $\overline{v} \in H^{\frac{1}{2}}(\partial\widetilde{\Omega})$ be the function defined by

$$\overline{v}(x,y) = \begin{cases} v(x,0), & if \ 0\leq x\leq d \ and \ y=0, \\ v(d-x,0), & if \ 0\leq x\leq d \ and \ y=d, \\ v(y,0), & if \ 0\leq y\leq d \ and \ x=0, \\ v(d-y,0), & if \ 0\leq y\leq d \ and \ x=d, \end{cases}$$

namely, the space curve $z = \overline{v}(x,y)$ $((x,y) \in \partial\widetilde{\Omega})$ is symmetric with respect to the planes x=y and x+y=d. It can be verified directly by the definition of $\overline{v}$ that

$$|\overline{v}|_{\frac{1}{2},\partial\widetilde{\Omega}} \leq C|v|_{\frac{1}{2},F_{ij}} \tag{4.6}$$

Let $\widetilde{v} \in H^1(\widetilde{\Omega})$ denotes the harmonic extension of $\overline{v}$. Thus

$$|\widetilde{v}|_{1,\widetilde{\Omega}} \leq C|\overline{v}|_{\frac{1}{2},\partial\widetilde{\Omega}}$$

Hence, by the trace Theorem and the Friedrichs' inequality

$$d^{-2}\|\widetilde{v} - \gamma_{F_{ij}}(v)\|^2_{0,\widetilde{\Omega}} \leq C|\widetilde{v}|^2_{1,\widetilde{\Omega}},$$

(Here, the factor "d^{-2}" guarantee that the constant "C" is independent of d)
we have

$$\begin{aligned} \|v - \gamma_{F_{ij}}(v)\|_{\frac{1}{2},F_{ij}} \leq & \ C\|\overline{v} - \gamma_{F_{ij}}(v)\|_{\frac{1}{2},\partial\widetilde{\Omega}} \\ \leq & \ C(|\widetilde{v} - \gamma_{F_{ij}}(v)|^2_{1,\widetilde{\Omega}} + d^{-2}\|\widetilde{v} - \gamma_{F_{ij}}(v)\|^2_{0,\widetilde{\Omega}})^{\frac{1}{2}} \\ \leq & \ C|\widetilde{v}|_{1,\widetilde{\Omega}} \leq C|\overline{v}|_{\frac{1}{2},\partial\widetilde{\Omega}} \\ & (since \ \ |\widetilde{v} - \gamma_{F_{ij}}(v)|^2_{1,\widetilde{\Omega}} = |\widetilde{v}|^2_{1,\widetilde{\Omega}}) \end{aligned}$$

By (4.6),this leads to (4.5).

Set $F' = \{\Gamma \in \{F_{ij}\} : there \ is \ a \ \partial\Omega_i \ satisfying \ \partial\Omega_i \bigcap \partial\Omega = \phi, \ such \ that \ \Gamma \subset \partial\Omega_i\}$. For $F_{ij} \in F'$,set $V_{ij}^0 = span\{1\} \subset S_n(F_{ij})$ (It is obvious that dim V_{ij}^0=1). Let $P_{ij}^0 : S_n(F_{ij}) \rightarrow V_{ij}^0$ be the orthogonal projection with respect to the inner product $< S_{ij}\cdot, \cdot >_{F_{ij}}$.For $\lambda_{ij} \in S_n(F_{ij})$, set $\lambda_{ij}^0 = P_{ij}^0\lambda_{ij}, \overline{\lambda}_{ij} = (I - P_{ij}^0)\lambda_{ij} = \lambda_{ij} - \lambda_{ij}^0$.

Set $\|\cdot\|^*_{1,\Omega_i} = (a_i(\cdot,\cdot))^{\frac{1}{2}}$.

Lemma 3. *Assume that* $\lambda_{ij} \in S_n(F_{ij})$*. Then*

$$< S_{ij}\overline{\lambda}_{ij}, \overline{\lambda}_{ij} >_{F_{ij}} \leq C|\lambda_{ij}|^2_{-\frac{1}{2},F_{ij}}, \ if \ F_{ij} \in F', \tag{4.7}$$

$$< S_{ij}\lambda_{ij}, \lambda_{ij} >_{F_{ij}} \leq C|\lambda_{ij}|^2_{-\frac{1}{2},F_{ij}}, \ if \ F_{ij} \notin F' \tag{4.8}$$

Proof. We consider only (4.7) ((4.8) can be proved by analogy). From the definition of $\overline{\lambda}_{ij}$,we have

$$< S_{ij}\overline{\lambda}_{ij}, \lambda_{ij}^0 >_{F_{ij}} = 0,$$

thus

$$\begin{aligned} & < S_{ij}\overline{\lambda}_{ij}, \overline{\lambda}_{ij} >_{F_{ij}} = < S_{ij}\overline{\lambda}_{ij}, \lambda_{ij} >_{F_{ij}} \\ & = a_i(R_{ij}\overline{\lambda}_{ij}, R_{ij}\lambda_{ij}) + a_j(R_{ji}\overline{\lambda}_{ij}, R_{ji}\lambda_{ij}) \\ & = < \lambda_{ij}, R_{ij}\overline{\lambda}_{ij} >_{F_{ij}} + < \lambda_{ij}, R_{ji}\overline{\lambda}_{ij} >_{F_{ij}} \\ & \leq |\lambda_{ij}|_{-\frac{1}{2},F_{ij}} \cdot \|R_{ij}\overline{\lambda}_{ij} + R_{ji}\overline{\lambda}_{ij}\|_{\frac{1}{2},F_{ij}} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \gamma_{F_{ij}}(R_{ij}\overline{\lambda}_{ij} + R_{ji}\overline{\lambda}_{ij}) = & \ |F_{ij}|^{-1} < 1, R_{ij}\overline{\lambda}_{ij} + R_{ji}\overline{\lambda}_{ij} >_{F_{ij}} \\ = & \ |F_{ij}|^{-1}(a_i(R_{ij}1, R_{ij}\overline{\lambda}_{ij}) + a_j(R_{ji}1, R_{ji}\overline{\lambda}_{ij})) \\ = & \ |F_{ij}|^{-1} < S_{ij}\overline{\lambda}_{ij}, 1 >_{F_{ij}} = 0. \end{aligned} \tag{4.10}$$

Using (4.5), together with (4.10) and the trace Theorem, we obtain

$$\begin{aligned}\|R_{ij}\overline{\lambda}_{ij}+R_{ji}\overline{\lambda}_{ij}\|_{\frac{1}{2},F_{ij}}\leq\ & C|R_{ij}\overline{\lambda}_{ij}+R_{ji}\overline{\lambda}_{ij}|_{\frac{1}{2},F_{ij}}\\ \leq\ & C(|R_{ij}\overline{\lambda}_{ij}|_{\frac{1}{2},\partial\Omega_i}+|R_{ji}\overline{\lambda}_{ij}|_{\frac{1}{2},\partial\Omega_i})\\ \leq\ & C(|R_{ij}\overline{\lambda}_{ij}|_{1,\Omega_i}+|R_{ji}\overline{\lambda}_{ij}|_{1,\Omega_i})\\ \leq\ & C(\|R_{ij}\overline{\lambda}_{ij}\|^*_{1,\Omega_i}+\|R_{ji}\overline{\lambda}_{ij}\|^*_{1,\Omega_i})\\ \leq\ & C<S_{ij}\overline{\lambda}_{ij},\overline{\lambda}_{ij}>^{\frac{1}{2}},\end{aligned}$$

which combining with (4.10) gives the desired result.

Lemma 4.[5] *Let $\lambda\in S_n(\partial\Omega_i)$. Then*

$$|\lambda|_{0,\partial\Omega_i}\leq Cnd^{-\frac{1}{2}}|\lambda|_{-\frac{1}{2},\partial\Omega_i}\quad(n\geq 1)\tag{4.11}$$

Lemma 5. *Assume that $\lambda\in S_n(F)$. Then*

$$|\lambda|_{-\frac{1}{2},\partial\Omega_i}\leq C<S_i\lambda,\lambda>,\ (\eta\leq Cd^{-2})\tag{4.12}$$

Proof. For $v\in H^{\frac{1}{2}}(\partial\Omega_i)$, let $\varphi_v\in H^1(\Omega_i)$ and $w_v\in H^1(\Omega_i)$ denote, respectively, the weak solution of problems:$-\triangle\varphi_v+d^{-2}\varphi_v=0$ (in Ω_i), $\varphi_v\ |_{\partial\Omega_i}=v$ and $-\triangle w_v+\eta w_v=0$ (in Ω_i),$w_v\ |_{\partial\Omega_i}=v$. It can be verified directly that

$$\|w_v\|^*_{1,\Omega_i}=\inf_{\psi-w_v\in H^1_0(\Omega_i)}\|\psi\|^*_{1,\Omega_i},$$

thus

$$\|w_v\|^*_{1,\Omega_i}\leq\|\varphi_v\|^*_{1,\Omega_i}\leq C(|\varphi_v|^2_{1,\Omega_i}+d^{-2}\|\varphi_v\|^2_{0,\Omega_i})^{\frac{1}{2}}\leq C\|v\|_{\frac{1}{2},\partial\Omega_i},\ (since\ \eta\leq Cd^{-2}).$$

Hence,from the definitions of $|\cdot|_{-\frac{1}{2},\partial\Omega_i}$, we have

$$|\lambda|_{-\frac{1}{2},\partial\Omega_i}\leq C\|u_\lambda\|^*_{1,\Omega_i}=(<u_\lambda,\lambda>_{\partial\Omega_i})^{\frac{1}{2}},\ (if\ \eta\leq Cd^{-2})$$

where u_λ is the solution of problem:$-\triangle u_\lambda+\eta u_\lambda=0$ (in Ω_i) and $\frac{\partial u_\lambda}{\partial n}\ |_{\partial\Omega_i}=\lambda$.

Let $u_{h\lambda}\in S_{h_i}(\Omega_i)$ denotes the usual L^2 projection of u_λ.Furthermore, using H_4 and (4.13), we can deduce (refer to [5])

$$\|u_{h\lambda}\|^*_{1,\Omega_i}\cdot|\lambda|_{-\frac{1}{2},\partial\Omega_i}\leq C<u_{h\lambda},\lambda>_{\partial\Omega_i},\ (\eta\leq Cd^{-2})$$

This leads to (4.12).

Lemma 6. *Let $\lambda_i\in S_n(\partial\Omega_i)$ and $\lambda_{ij}=\lambda_i\ |_{F_{ij}}$.Then*

$$|\lambda_{ij}|_{-\frac{1}{2},F_{ij}}\leq C(1+\log n)|\lambda_i|_{-\frac{1}{2},\partial\Omega_i}\quad(n\geq 1)\tag{4.13}$$

Proof. Let $\partial\Omega_i$ be divided into quasi-uniform units with diameter $\overline{h}=dn^{-2}$, such that the vertices are in the knots. Let $S_{\overline{h}}(\partial\Omega_i)$ and $S_{\overline{h}}(F_{ij})$ denotes respectively the space which consists of continuous and piecewise linear polynomials defined on $\partial\Omega_i$ and on F_{ij}.For $v\in H^{\frac{1}{2}}(F_{ij})$, let $v_h\in S_{\overline{h}}(F_{ij})$ be the L^2 projection of v. Then

$$\|v_h-v\|_{0,F_{ij}}\leq C\overline{h}^{\frac{1}{2}}\|v\|_{\frac{1}{2},F_{ij}},$$

by (4.11) this leads to

$$\begin{aligned}|<\lambda_{ij},v-v_h>_{F_{ij}}|\leq\ & \|\lambda_{ij}\|_{0,F_{ij}}\cdot\|v-v_h\|_{0,F_{ij}}\\ \leq\ & C\overline{h}^{\frac{1}{2}}\|v\|_{\frac{1}{2},F_{ij}}\cdot\|\lambda_i\|_{0,\partial\Omega_i}\\ \leq\ & C\|v\|_{\frac{1}{2},F_{ij}}\cdot\|\lambda_i\|_{-\frac{1}{2},\partial\Omega_i}\end{aligned}\tag{4.14}$$

Let s_1 and s_4 be the two end-points of F_{ij}, and s_2 and s_3 denotes respectively the interior nodes nearby s_1 and s_4. Set $e=[s_1,s_2]\cup[s_3,s_4]$.

We define $v_{1h}\in S_{\overline{h}}(F_{ij})$ and $v_{2h}\in S_{\overline{h}}(\partial\Omega_i)$ by

$$v_{1h}=\begin{cases}v_h(s), & if\ s=s_1\ or\ s=s_4,\\ linear function, & if s\in e,\\ 0, & if\ s\in[s_2,s_3]\end{cases};v_{2h}=\begin{cases}v_h-v_{1h}, & on\ F_{ij},\\ 0, & on\ \partial\Omega_i\backslash F_{ij}.\end{cases}$$

Since (refer to [24] and its references)

$$\|v_h\|_{0,\infty,F_{ij}}\leq C(1+\log^{\frac{1}{2}}(d/\overline{h}))\|v_h\|_{\frac{1}{2},F_{ij}},$$

from (4.13), we have

$$\begin{aligned}|<\lambda_{ij},v_{1h}>_{F_{ij}}|\leq\ & \|\lambda_{ij}\|_{0,F_{ij}}\cdot\|v_{1h}\|_{0,F_{ij}}\\ \leq\ & \|\lambda_i\|_{0,\partial\Omega_i}\cdot\|v_{1h}\|_{0,e}\\ \leq\ & Cnd^{-\frac{1}{2}}|\lambda_i|_{-\frac{1}{2},\partial\Omega_i}\cdot\overline{h}^{\frac{1}{2}}\|v_h\|_{0,\infty,F_{ij}}\\ \leq\ & C(1+\log^{\frac{1}{2}}n)|\lambda_i|_{-\frac{1}{2},\partial\Omega_i}\cdot\|v_h\|_{\frac{1}{2},F_{ij}}.\end{aligned}\tag{4.15}$$

It can be verified directly that (refer to [15] and [24])

$$\|v_{2h}\|_{\frac{1}{2},\partial\Omega_i}\leq C(1+\log(d/\overline{h}))\|v_h\|_{\frac{1}{2},F_{ij}}.$$

Thus

$$\begin{aligned} |<\lambda_i, v_{2h}>_{\partial\Omega_i}| &\leq |\lambda_i|_{-\frac{1}{2},\partial\Omega_i}\cdot\|v_{2h}\|_{\frac{1}{2},\partial\Omega_i} \\ &\leq C(1+\log n)|\lambda_i|_{-\frac{1}{2},\partial\Omega_i}\cdot\|v_h\|_{\frac{1}{2},F_{ij}}, \end{aligned} \tag{4.16}$$

On the other hand, we have

$$\|v_h\|_{\frac{1}{2},F_{ij}}\leq C\|v\|_{\frac{1}{2},F_{ij}}$$

Hence, from (4.14), (4.15) and (4.16), we obtain

$$\begin{aligned} |<\lambda_{ij}, v>_{F_{ij}}| &\leq |<\lambda_{ij}, v_h>_{F_{ij}}| + |<\lambda_{ij}, v-v_h>_{F_{ij}}| \\ &\leq |<\lambda_{ij}, v_{1h}>_{F_{ij}}| + |<\lambda_i, v_{2h}>_{\partial\Omega_i}| \\ &+ C\|v\|_{\frac{1}{2},F_{ij}}\|\lambda_i\|_{-\frac{1}{2},\partial\Omega_i} \\ &\leq C(1+\log n)|\lambda_i|_{-\frac{1}{2},\partial\Omega_i}\cdot\|v\|_{\frac{1}{2},F_{ij}}, \end{aligned}$$

which leads to (4.13).

Proof of Theorem 3. Set $\Lambda_{ij}=\{(r,k): F_{rk}\subset\partial\Omega_i\bigcup\partial\Omega_j, r<k\}$ For any $\lambda_{ij}\in S_n(F_{ij})$, we have

$$\begin{aligned} <S(\sum_{i<j}D_{ij}\lambda_{ij}), \sum_{i<j}D_{ij}\lambda_{ij}> &= \sum_{i<j}<SD_{ij}\lambda_{ij}), \sum_{i<j}D_{ij}\lambda_{ij}> \\ &= \sum_{i<j}\sum_{(r,k)\in\Lambda_{ij}}<SD_{ij}\lambda_{ij}, D_{rk}\lambda_{rk}> \end{aligned} \tag{4.17}$$

Since

$$\begin{aligned} <SD_{ij}\lambda_{ij}, D_{rk}\lambda_{rk}> &= <S_iD_{ij}\lambda_{ij}, D_{rk}\lambda_{rk}>_{\partial\Omega_i} + <S_jD_{ij}\lambda_{ij}, D_{rk}\lambda_{rk}>_{\partial\Omega_j} \\ &\leq (<S_iD_{ij}\lambda_{ij}, D_{ij}\lambda_{ij}>_{\partial\Omega_i} + <S_jD_{ij}\lambda_{ij}, D_{ij}\lambda_{ij}>_{\partial\Omega_j})^{\frac{1}{2}} \\ &\cdot (<S_iD_{rk}\lambda_{rk}, D_{rk}\lambda_{rk}>_{\partial\Omega_i} + <S_jD_{rk}\lambda_{rk}, D_{rk}\lambda_{rk}>_{\partial\Omega_j})^{\frac{1}{2}} \\ &\leq (<S_{ij}\lambda_{ij}, \lambda_{ij}>_{F_{ij}})^{\frac{1}{2}}\cdot(<S_{rk}\lambda_{rk}, \lambda_{rk}>_{F_{rk}})^{\frac{1}{2}} \\ &\leq \frac{1}{2}(<S_{ij}\lambda_{ij}, \lambda_{ij}>_{F_{ij}} + <S_{rk}\lambda_{rk}, \lambda_{rk}>_{F_{rk}}), \end{aligned}$$

by (4.17), we obtain

$$<S(\sum_{i<j}D_{ij}\lambda_{ij}), \sum_{i<j}D_{ij}\lambda_{ij}> \leq C\sum_{i<j}<S_{ij}\lambda_{ij}), \lambda_{ij}>_{F_{ij}} \tag{4.18}$$

For any $\lambda\in S_n(F)$,set $\lambda_{ij}=\lambda\,|_{F_{ij}}$. Let $\lambda_0=\sum\limits_{i<j}D_{ij}P_{ij}^0\lambda_{ij}$, $\overline{\lambda}_{ij}=(I-P_{ij}^0)\lambda_{ij}$,and set

$$\lambda_{ij}^*=\begin{cases}\overline{\lambda}_{ij}, & if\ F_{ij}\in F', \\ \lambda_{ij}, & if\ F_{ij}\notin F'.\end{cases}$$

It is clear that $\lambda=\lambda_0+\sum\limits_{i<j}D_{ij}\lambda_{ij}^*$ with $\lambda_0\in V_0$ and $\lambda_{ij}^*\in S_n(F_{ij})$. From (4.7), (4.8) and (4.13), we have

$$<S_{ij}\lambda_{ij}^*, \lambda_{ij}^*>_{F_{ij}} \leq C(1+\log^2 n)(|\lambda|^2_{-\frac{1}{2},\partial\Omega_i}+|\lambda|^2_{-\frac{1}{2},\partial\Omega_j}). \tag{4.19}$$

Since $\eta\leq Cd^{-2}$, thus (4.12) and (4.19) infer that

$$\sum_{i<j}<S_{ij}\lambda_{ij}^*, \lambda_{ij}^*>_{F_{ij}} \leq C(1+\log^2 n)<S\lambda, \lambda> \tag{4.20}$$

Furthermore (note (4.18))

$$\begin{aligned} <S_0\lambda_0, \lambda_0> &= <S(\lambda-\sum_{i<j}D_{ij}\lambda_{ij}^*), \lambda-\sum_{i<j}D_{ij}\lambda_{ij}^*> \\ &\leq C(<S\lambda, \lambda> + <S(\sum_{i<j}D_{ij}\lambda_{ij}^*), \sum_{i<j}D_{ij}\lambda_{ij}^*>) \\ &\leq C(<S\lambda, \lambda> + \sum_{i<j}<S_{ij}\lambda_{ij}^*), \lambda_{ij}^*>_{F_{ij}}) \\ &\leq C(1+\log^2 n)<S\lambda, \lambda>, \ (\eta\leq Cd^{-2}). \end{aligned} \tag{4.21}$$

Using Lemma 1, together with (4.18), (4.20) and (4.21),we deduce to Theorem 3.

5. The Case of η=0

In this section,we consider Poisson equations:

$$\begin{cases} -\Delta u=f, & in\ \Omega, \\ u=0, & on\ \partial\Omega, \end{cases} \tag{5.1}$$

For these equations the interface equation can not be built directly like (3), because the corresponding local stiffness matrix A_i is only semi-positive definite. Now, we introduce a "approximation-extrapolation" framework (this idea can also be found in [21]). This algorithm can preserve the merits of the preconditioner introduced in Section 3. Moreover, it increases slightly cost of calculation only from parallel point of view.

For a given natural number r, and a "small" positive number η,let $(u_{hj}, \lambda_{nj}) \in S_{h\times n}$ be the solutions of the following "approximated" problems ($j = 0, \cdots, r-1$):

$$\begin{cases} \sum_{i=1}^{N} \{a_{ij}(u_{hj}, v) - < \lambda_{nj}, v >_{\partial\Omega_i}\} = \sum_{i=1}^{N} (f, v)_{\Omega_i}, \ \forall v \in S_h(\Omega), & (5.2) \\ \sum_{i=1}^{N} < u_{hj}, \mu >_{\partial\Omega_i} = 0, \qquad \forall \mu \in S_n(F), & (5.3) \end{cases}$$

where $a_{ij}(u_{hj}, v) = (\nabla u_{hj}, \nabla v)_{\Omega_i} + (\eta/2^j)(u_{hj}, v)_{\Omega_i}$ $(0 \le j \le r-1)$.

Set

$$u_{hj}^{(0)} = u_{hj}, \lambda_{nj}^{(0)} = \lambda_{nj}, j = 0, \cdots, r-1,$$

$$u_{hj}^{(l)} = \frac{2^l u_{hj+1}^{(l-1)} - u_{hj}^{(l-1)}}{2^l - 1}, \lambda_{nj}^{(l)} = \frac{2^l \lambda_{nj+1}^{(l-1)} - \lambda_{nj}^{(l-1)}}{2^l - 1},$$

$$j = 0, \cdots, r-1-l; l = 1, \cdots, r-1 (if\ r \ge 2)$$

Theorem 4. *Under the assumptions of Theorem 1, we have*

$$\begin{aligned} & \|u_{h0}^{(r-1)} - u\|_{\mathbf{X}} + \|\lambda_{n0}^{(r-1)} - \tfrac{\partial u}{\partial n}\|_{\mathbf{M}} \\ & \le C(\eta^r + h^m) \sum_{i=1}^{N} (|u|_{m+1,\Omega_i} + \|u\|_{0,\Omega_i} + d^{\frac{1}{2}} |u|_{k+\frac{1}{2}, \partial\Omega_i \backslash \partial\Omega}), \end{aligned} \tag{5.4}$$

where u is the exact solution of (5.1).

Proof. Let $(\overline{u}_h, \overline{\lambda}) \in S_{h\times n}$ be the solution of problem:

$$\begin{cases} \sum_{i=1}^{N} \{(\nabla \overline{u}_h, \nabla v)_{\Omega_i} - < \overline{\lambda}, v >_{\partial\Omega_i}\} = \sum_{i=1}^{N} (f, v)_{\Omega_i}, \quad \forall v \in S_h(\Omega), & (5.5) \\ \sum_{i=1}^{N} < \overline{u}_h, \mu >_{\partial\Omega_i} = 0, \qquad \forall \mu \in S_n(F). & (5.6) \end{cases}$$

Since (see Remark 3)

$$\|\overline{u}_h - u\|_{\mathbf{X}} + \|\overline{\lambda} - \frac{\partial u}{\partial n}\|_{\mathbf{M}} \le C h^m \sum_{i=1}^{N} (|u|_{m+1,\Omega_i} + d^{\frac{1}{2}} |u|_{k+\frac{1}{2}, \partial\Omega_i \backslash \partial\Omega}),$$

therefore it suffices to prove

$$\begin{aligned} & \|u_{h0}^{(r-1)} - \overline{u}_h\|_{\mathbf{X}} + \|\lambda_{n0}^{(r-1)} - \overline{\lambda}\|_{\mathbf{M}} \\ & \le C\eta^r \sum_{i=1}^{N} (|u|_{m+1,\Omega_i} + \|u\|_{0,\Omega_i} + d^{\frac{1}{2}} |u|_{k+\frac{1}{2}, \partial\Omega_i \backslash \partial\Omega}) \end{aligned} \tag{5.7}$$

Subtracting (5.5) from (5.2) with j=0 and j=1 leads respectively to (set $v = \overline{u}_h$ in (5.2) and (5.5))

$$\sum_{i=1}^{N} \{(\nabla(u_{h0} - \overline{u}_h), \nabla \overline{u}_h)_{\Omega_i} + \eta(u_{h0}, \overline{u}_h)_{\Omega_i}\} = 0 \tag{5.8}$$

and

$$\sum_{i=1}^{N} \{(\nabla(u_{h1} - \overline{u}_h), \nabla \overline{u}_h)_{\Omega_i} + (\eta/2)(u_{h1}, \overline{u}_h)_{\Omega_i}\} = 0 \tag{5.9}$$

Subtracting (5.8) from $2 \times (5.9)$, we obtain

$$\sum_{i=1}^{N} \{(\nabla(2u_{h1} - u_{h0} - \overline{u}_h), \nabla \overline{u}_h)_{\Omega_i} + \eta(u_{h1} - u_{h0}, \overline{u}_h)_{\Omega_i}\} = 0 \tag{5.10}$$

Repeating the above process,we have (set $v = 2u_{h1} - u_{h0}$ in (5.2) and (5.5))

$$\sum_{i=1}^{N} \{(\nabla(2u_{h1} - u_{h0} - \overline{u}_h), \nabla(2u_{h1} - u_{h0}))_{\Omega_i} + \eta(u_{h1} - u_{h0}, 2u_{h1} - u_{h0})_{\Omega_i}\} = 0,$$

which subtracting (5.10) yields that

$$\sum_{i=1}^{N}\{|u_{h0}^{(1)}-\overline{u}_h|_{1,\Omega_i}+\eta(u_{h1}-u_{h0},u_{h0}^{(1)}-\overline{u}_h)_{\Omega_i}\}=0,$$

namely,

$$\sum_{i=1}^{N}a_{i0}(u_{h0}^{(1)}-\overline{u}_h,u_{h0}^{(1)}-\overline{u}_h)=\eta\sum_{i=1}^{N}(u_{h1}-\overline{u}_h,u_{h0}^{(1)}-\overline{u}_h)_{\Omega_i}.$$

Thus, by the generalized Poincare' inequality (see [7]) and the Cauchy-Schwarz inequality,we obtain

$$\|u_{h0}^{(1)}-\overline{u}_h\|_{\mathbf{X}}\leq C\eta\|u_{h1}-\overline{u}_h\|_{\mathbf{X}} \tag{5.11}$$

Similarly,we have

$$\|u_{h1}-\overline{u}_h\|_{\mathbf{X}}\leq C\eta(\sum_{i=1}^{N}\|\overline{u}_h\|_{0,\Omega_i}^2)^{\frac{1}{2}},$$

which,together with (5.11), yields that

$$\|u_{h0}^{(1)}-\overline{u}_h\|_{\mathbf{X}}\leq C\eta^2\sum_{i=1}^{N}(|u|_{m+1,\Omega_i}+\|u\|_{0,\Omega_i}+d^{\frac{1}{2}}|u|_{k+\frac{1}{2},\partial\Omega_i\backslash\partial\Omega}). \tag{5.12}$$

On the other hand, from the LBB condition (see Thereom 1 in [5]), we know that there is a function $v_h\in S_h(\Omega)$ such that

$$\|\lambda_{n0}^{(1)}-\overline{\lambda}\|_{\mathbf{M}}\cdot\|v_h\|_{\mathbf{X}}\leq C\sum_{i=1}^{N}<\lambda_{n0}^{(1)}-\overline{\lambda},v_h>_{\partial\Omega_i},$$

by (5.2), (5.5), (5.12) and the Cauchy-Schwarz inequality, this leads to

$$\|\lambda_{n0}^{(1)}-\overline{\lambda}\|_{\mathbf{M}}\leq C\eta^2\sum_{i=1}^{N}(|u|_{m+1,\Omega_i}+\|u\|_{0,\Omega_i}+d^{\frac{1}{2}}|u|_{k+\frac{1}{2},\partial\Omega_i\backslash\partial\Omega}).$$

The rest may be deduced by analogy.

Remark 8. The solutions (u_{hj},λ_{nj}) $(j=0,\cdots,r-1)$ can be obtained in the same way as (u_h,λ) (note that the local stiffness matrix A_{ij} associated with the local bilinear form $a_{ij}(\cdot,\cdot)$ has the same size as A_i), which may be computed parallelly for different j. Moreover, the corresponding preconditioned interface matrix has the same condition number as $\overline{M}\ \overline{S}$.

Remark 9. Theorem 4 indicates that an ideal approximate solution of (5.1) can be obtained provided that $\eta^r\leq h^m$.The positive number η can not be very small, otherwise, the condition number of the local stiffness matrix A_{ij} may be great. It is clear that, when h^m is not very small,extrapolation will unnecessary (namely, set $\eta=h^m$). In most circumstance, "extrapolation" need to be done for one time at most.

6. Numerical Examples

At first,we consider example

$$\begin{cases}-\Delta u+\eta u=f, & in\ \Omega=[0,4]^2,\\ u=g, & on\ \partial\Omega,\end{cases} \tag{6.1}$$

where f and g are given functions such that its exact solution $u=\sin(x+y)$.

The domain Ω is decomposed into 4×4 or 5×5 squares with same size, and each square is divided alternately (number the squares in the usual way) into 20×20 or 18×18 small squares with the same size (the grids is non-matching). We consider the Q_1 finite element approximation space.

In order to show effect of our preconditioner, we calculate condition numbers of the matrices $\overline{S}$ and $\overline{M}\ \overline{S}$, and iteration times of solving the interface equation by conjugate gradient (CG) and preconditioned conjugate gradient (PCG) methods. The effect is confirmed by Table 1 and Table 2, here, the iteration domination error is 10^{-6}. The resulting solution error is less than 2.107E-3.

Table 1
4×4 subdomains,d=1

n	$\eta=1$				$\eta=0.1$			
	condition numbers		iteration times		condition numbers		iteration times	
	$\overline{S}$	$\overline{M}\ \overline{S}$	CG	PCG	$\overline{S}$	$\overline{M}\ \overline{S}$	CG	PCG
3	354	16	48	13	3318	21	74	15
4	930	27	65	14	8714	36	100	17
5	2401	35	91	16	22387	49	157	18
6	6785	90	114	17	63712	120	182	19

Table 2
5×5 subdomains,d=0.8

n	$\eta=1$				$\eta=0.1$			
	condition numbers		iteration times		condition numbers		iteration times	
	$\overline{S}$	$\overline{M}\ \overline{S}$	CG	PCG	$\overline{S}$	$\overline{M}\ \overline{S}$	CG	PCG
3	602	19	76	19	3790	21	123	22
4	1574	32	102	21	9901	36	170	23
5	4052	40	136	22	2362	49	221	25
6	11388	107	172	24	72163	125	289	26

Table 1 and Table 2 indicate that the preconditioner advanced in this paper are very effective. However, the condition numbers $cond(\overline{M}\ \overline{S})$ in Table 1 and Table 2 has not polylogarithmic bound proven in Theorem 3. This phenomenon does not contradict our theoretical result, because Theorem 3 is valid only under the assumption H_4.

In applications, we are interested only in the effect of our preconditioner for the same n, h and d (or N), so the assumption H_4 is not absolutely necessary (provided that n^2h/d is not large, refer to Table 1 and Table 2).

Now we consider Poisson equation:

$$\begin{cases} -\Delta u = f_0, & in\ \Omega_0 = [0,1]^2, \\ u = g_0, & on\ \partial\Omega_0, \end{cases} \tag{6.2}$$

where f_0 and g_0 are given functions such that its exact solution $u = \sin(x+y)$.

The "approximation" equation is (6.1) with $f = f_0, g = g_0$ and $\Omega = \Omega_0$. The theoretical results stated in §5 are confirmed by Table 3. For convenience' sake, we use the L^2 norm instead of the H^1 norm. Note that from two $r-1$ level "neighbouring" extrapolation results, we can only obtain one r level extrapolation result.

Table 3
4×4 subdomains,d=0.25,h=0.05,n=4,η=0.4

j	$\|u_{hj}-u\|_{\mathbf{X}}$	$\|u_{hj}^{(1)}-u\|_{\mathbf{X}}$	$\|u_{hj}^{(2)}-u\|_{\mathbf{X}}$
0	3.34D-1	4.82D-2	6.13D-4
1	1.91D-1	1.22D-2	7.66D-5
2	1.02D-1	3.06D-3	\
3	5.03D-2	\	\

All numerical results are obtained by using Liang's finite element program generator (see [22]).

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