Journal of Computational Mathematics, Vol.19, No.3, 2001, 327–336.

THE NUMERICAL METHODS FOR SOLVING EULER SYSTEM OF EQUATIONS IN REPRODUCING KERNEL SPACE $H^2(R)$*1)

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Abstract

A new method is presented by means of the theory of reproducing kernel space and finite difference method, to calculate Euler system of equations in this paper. The results show that the method has many advantages, such as higher precision, better stability, less amount of calculation than any other methods and the reproducing kernel function has good local properties and its derived function is wavelet function.

Key words: Euler system of equations, Reproducing kernel method, Finite difference method, Wavelet function.

1. Introduction

In recent years, more and more people are interested in solving Euler system of equations. They presented various methods to simulate the flow of the complicated fluid field. It is well known that Euler system of equations has described many practical engineering problems, such as spherically symmetric flow, the flow inside a pipe, whose sectional area changed slowly, the radius of curvature is large, sectional area is small and so on. And it not only describes the incompressible ideal fluid one-dimension unsteady flow but also is the foundation for solving Navier-Stokes system of equations, which describes the viscous flow, hence it lies at the heart of fluid mechanics, that is why, solving Euler system of equations possesses important significance. But the various methods given for solving Euler system of equations so far are only confined to classical ones, such as finite difference and finite element methods, their effects are not very well. This paper combines the reproducing kernel with the finite difference method and gives the approximate solution to Euler system of equations. Numerical experiment results indicate that the effect is very well. Because the reproducing kernel function possesses the good local properties, such as the odd order vanishing moment, symmetry and regularity, its dilation has fast attenuation etc., its derived function is a wavelet function which possesses even order vanishing moment and anti-symmetry, regularity and its dilation has fast attenuation. So reproducing kernel function possesses its operation superiority and the description superiority of wavelet function, that is why the superiority of this method prevails over others. Firstly, this method can command simply, conveniently and easily. Secondly, it can apply extensively, especially solving complicated non-linear partial difference equations, which are solved hardly. Thirdly, the method possesses the advantages of higher precision and better stability compared with the others. In addition, we can construct two-dimension reproducing kernel space by tensor product form and extend it to two-dimension Euler system of equations.

* Received July 26, 1998.
1)Project 19971020 supported by NSFC and Project (HIT 2000.01) supported by the Scientific Research Foundation of Harbin Institute of Technology.

2. Reproducing Kernel Space $H^2(R)$

2.1 Definition of reproducing kernel space $H^2(R)$

Definition. *$H^2(R) = \{u(x)|u(x)$ and $u'(x)$ are absolutely continuous function in R, u, u' and $u'' \in L^2(R)\}$. Inner product on $H^2(R)$ is defined as follows:*

$$< u, v >= \int_R (uv + 2u'v' + u''v'')dx.$$

By reference [1], we know $H^2(R)$ is a reproducing kernel space and its reproducing kernel is:

$$k_2(x-\xi) = \frac{1}{4}(1 + |x-\xi|)e^{-|x-\xi|},$$

namely

$$\forall u \in H^2(R), \quad < u(\xi), K_2(x-\xi) >= u(x).$$

Theorem. *Let $\{x_i\}_{i=1}^n$ denote a system where x_i are pairwise different nodes in R, and let $\phi_i(x) = K_2(x - x_i)$, then $\{\phi_i(x)\}_{i=1}^n$ is a linearly independent function system in $H^2(R)$, we obtain an orthonormal system $\{\phi_i^*(x)\}_{i=1}^n$ by Schmidt orthogonalization, where $\phi_i^*(x) = \sum\limits_{k=1}^{i} \alpha_{ki}\phi_k(x)$, when $\{x_i\}_{i=1}^\infty$ is dense in R, $\{\phi_i^*\}_{i=1}^\infty$ is an orthonormal basis in $H^2(R)$, namely*

$$\lim_{n\to\infty}(H_n u)(x) = u(x), \text{ where } u(x) \in H^2(R), \ (H_n u)(x) = \sum_{i=1}^n (u, \phi_i^*)\phi_i^*.$$

Proof. Let

$$C_1\phi_1 + C_2\phi_2 + \cdots + C_n\phi_n = 0 \tag{2.1}$$

We use proof by contradiction. Assume $\{\phi_i(x)\}_{i=1}^n$ is linearly independent, we apply Fourier transform to both sides of the (2.1), then we get

$$C_1 e^{-ix_1\omega} + C_2 e^{-ix_2\omega} + \cdots C_n e^{-ix_n\omega} = 0. \tag{2.2}$$

We may as well let $C_1 \neq 0$, obtain

$$1 = -(\frac{C_2}{C_1}e^{i(x_1-x_2)\omega} + \cdots + \frac{C_n}{C_1}e^{i(x_1-x_n)\omega}) \tag{2.3}$$

By (2.3), we know there is one that doesn't equal zero in $C_1, \cdots, C_n$, setting $C_2 \neq 0$, then $\frac{C_2}{C_1} \neq 0$. Let $x_1 - x_2 = y_1, \cdots, x_1 - x_n = y_n$, $\frac{C_2}{C_1} = a_2$, $\frac{C_n}{C_1} = a_n$, and by deriving (2.3), we get

$$0 = -(a_2 y_2 e^{-iy_2\omega} + \cdots + a_n y_n e^{-iy_n\omega}). \tag{2.4}$$

Obviously $y_2, \cdots, y_n \neq 0$, let $a_2 y_2 = b_2, \cdots, a_n y_n = b_n$, then (2.4) is converted into (2.5)

$$0 = -(b_2 e^{-iy_2\omega} + \cdots + b_n e^{-iy_n\omega}). \tag{2.5}$$

Since $b_2 \neq 0$, at least there is one of them doesn't equal zero in $b_3, \cdots, b_n$, setting $b_3 \neq 0$, and so on and so forth, we get $c_n e^{-iy_n\omega} = 0$ $(c_n \neq 0)$ which leads to an absurdity. So $\{\phi_i\}_{i=1}^n$ are linearly independent.

By means of Schmidt orthogonalization of $\{\phi_i(x)\}_{i=1}^n$, we get

$$\phi_i^* = \sum_{k=1}^{i} a_{ki}\phi_k(x) \quad (i = 1, 2, \cdots, n)$$

let $< \phi_i^*, u >= 0$, $i = 1, 2, \cdots$, namely

$$\sum_{k=1}^{i} a_{ki} u(x_k) = 0$$

it follows that $u(x_k) = 0$, $k = 1, 2, \cdots$

By the continuity of $u(x)$ and the density of $\{x_i\}_{i=1}^\infty$ in R we know $\{\phi_i^*\}_{i=1}^\infty$ is an orthonormal basis on $H^2(R)$. So from $(H_n u)(x) = \sum_{i=1}^{n}(u, \phi_i^*)\phi_i^*$, we get

$$(H_n u)(x) \to u(x), \quad n \to \infty$$

2.2 The properties of reproducing kernel function

Property 1. *Let $K_2(x) = K_2(x - 0)$, then $K_2(x) = \frac{1}{4}(1 + |x|)e^{-|x|}$ and $K_2'(x)$ is a wavelet function.*

Proof. $\int_R K_2'(x)dx = 0$ is obvious, and $\hat{K_2'}(\omega) = \frac{i\omega}{(1+\omega^2)^2}$, $\hat{K_2'}(0) = 0$. So $K_2'(x)$ is a wavelet function.

Property 2. *$K_2(x)$ and $K_2'(x)$ have odd order and even order vanishing moments respectively.*

Proof. In fact by representations of $K_2(x)$ and $K_2'(x)$, it is easy to proof that

$$\int_R x^{2n+1} K_2(x)dx = \int_R x^{2n} K_2'(x)dx = 0,$$

where n is a nonnegative integer.

Property 3. *$K_2(x)$ has symmetry and $K_2'(x)$ has anti-symmetry.*

Proof. The result is obvious by the representations of $K_2(x)$ and $K_2'(x)$.

Property 4. *$K_2(x)$ and $K_2'(x)$ have regularity and their dilatations have fast attenuation .*

Proof. In fact when n (n is a positive integer) is large enough, it is obvious that

$$K_2(x) = \frac{1}{4}(1 + |x|)e^{-|x|} \leq C_n(1 + |x|)^{-n} \quad (C_n \text{ is a positive coefficient})$$

$$K_2'(x) = \frac{1}{4}|x|e^{-|x|} \leq D_n(1 + |x|)^{-n} \quad (D_n \text{ is a positive coefficient}).$$

Property 5. *$K_2'(x)$has neighboring region orthogonality, namely*

$$\textit{when } |b - a| = 1, \ < K_2'(x - a), K_2'(x - b) >_{H^2(R)} = 0.$$

Proof.

$$\begin{aligned}
& I(a,b) \\
= & < K_2'(x-a), K_2'(x-b) >_{H^2(R)} \\
= & < K_2'(x-a), K_2'(x-b) >_{L^2(R)} +2 < K_2''(x-a), K_2''(x-b) >_{L^2(R)} \\
& + < K_2'''(x-a), K_2'''(x-b) >_{L^2(R)} \\
= & \frac{1}{2\pi} < i\omega e^{-ia\omega}\hat{K}_2(\omega), i\omega e^{-ib\omega}\hat{K}_2(\omega) >_{L^2(R)} \\
& + \frac{1}{\pi} < (i\omega)^2 e^{-ia\omega}\hat{K}_2(\omega), (i\omega)^2 e^{-ib\omega}\hat{K}_2(\omega) >_{L^2(R)} + \\
& + \frac{1}{2\pi} < (i\omega)^3 e^{-ia\omega}\hat{K}_2(\omega), (i\omega)^3 e^{-ib\omega}\hat{K}_2(\omega) >_{L^2(R)} \\
= & \frac{1}{2\pi}\int_R \frac{\omega^2}{(1+\omega^2)^2} e^{i(b-a)\omega} d\omega \\
= & K_1(b-a) - K_2(b-a) \\
= & \frac{1}{4}(1-|b-a|)e^{-|b-a|} \quad (K_1(x) = \frac{1}{2}e^{-|x|}),
\end{aligned}$$

thus, when $|b-a| = 1$, $I(a,b) = 0$.

3. The Case for Single Equation

3.1 Construction of reproducing kernel finite difference method

Considering discontinuous solution to model equation as follows, namely nonlinear convective equation, since it is the typical model equation of unsteady fluid mechanics equations

$$u_t + uu_x = 0 \quad (t \geq 0) \tag{3.1}$$

First, we solve the $u(x,t)$ with $x \in [0,1]$, we assume $u(x,t) = 0$ with $x \notin [0,1]$. Then we have $u \in H^2(R)$.

By introducing a time step Δt and a time discretization to (3.1), we obtain

$$u^{n+1} + \Delta t u^n u_x^{n+1} = u^n \tag{3.2}$$

where $u^n = u(x, t_n) = u(x, n\Delta t)$, $u_x^{n+1} = \frac{\partial u^{n+1}}{\partial x}$, Δt is a time step.

Setting $u^{n+1} = \sum\limits_{i=1}^{N} b_i\phi_i^*$, where $b_i =< u^{n+1}, \phi_i^* >_{H^2(R)}$, $\phi_i^* = \sum\limits_{k=1}^{i} a_{ki}\phi_k$, $\phi_k = K_2(x - x_k)$. a_{ki} is a coefficient of Schmidt's orthogonalization. In $\{x_k\}_{k=1}^{\infty}$, x_k are pairwise different knots and dense in R, applying theorem, we know $\{\phi_i^*\}$ is the orthonormal basis on $H^2(R)$.

Since $u(x,t)$ is zero on $x \notin [0,1]$, we may as well set $\{x_k\}_{k=1}^{\infty}$ is dense on $[0,1]$, where x_k are pairwise different knots. For

$$b_i =< u^{n+1}, \sum_{k=1}^{i} a_{ki}\phi_k >_{H^2(R)} = \sum_{k=1}^{i} a_{ki}u^{n+1}(x_k) = \sum_{k=1}^{i} a_{ki}u_k^{n+1}.$$

Substituting $u^{n+1} = \sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i} a_{ki}u_k^{n+1}\phi_i^*$ into equation (3.2) we get

$$u^{n+1} + \Delta t u^n \sum_{i=1}^{N}\sum_{k=1}^{i} a_{ki}u_k^{n+1}(\phi_i^*)' = u^n. \tag{3.3}$$

Calculating the inner product of $\phi_l = K_2(x - x_l)$ and (3.3), we get

$$u_l^{n+1} + \Delta t u_l^n \sum_{i=1}^{N} \sum_{k=1}^{i} a_{ki} u_k^{n+1} (\phi_{li}^*)' = u_l^n. \ \ (l = 1, 2, \cdots, N)$$

namely

$$\begin{cases} \sum\limits_{i=1}^{N} \sum\limits_{k=i}^{N} (\Delta t u_1^n a_{ik} (\phi_{1k}^*)' + \delta_{1i}) u_i^{n+1} = u_1^n \\ \sum\limits_{i=1}^{N} \sum\limits_{k=i}^{N} (\Delta t u_2^n a_{ik} (\phi_{2k}^*)' + \delta_{2i}) u_i^{n+1} = u_2^n \\ \cdots \\ \sum\limits_{i=1}^{N} \sum\limits_{k=i}^{N} (\Delta t u_N^n a_{ik} (\phi_{Nk}^*)' + \delta_{Ni}) u_i^{n+1} = u_N^n \end{cases} \tag{3.4}$$

Let

$$A = \begin{pmatrix} 1 + \sum\limits_{k=1}^{N} \Delta t u_1^n a_{1k} (\phi_{1k}^*)' & \sum\limits_{k=2}^{N} \Delta t u_1^n a_{2k} (\phi_{1k}^*)' & \cdots & \Delta t u_1^n a_{NN} (\phi_{1N}^*)' \\ \sum\limits_{k=1}^{N} \Delta t u_2^n a_{1k} (\phi_{2k}^*)' & 1 + \sum\limits_{k=2}^{N} \Delta t u_2^n a_{2k} (\phi_{2k}^*)' & \cdots & \Delta t u_2^n a_{NN} (\phi_{2N}^*)' \\ \vdots & \vdots & \cdots & \vdots \\ \sum\limits_{k=1}^{N} \Delta t u_N^n a_{1k} (\phi_{Nk}^*)' & \sum\limits_{k=2}^{N} \Delta t u_N^n a_{2k} (\phi_{Nk}^*)' & \cdots & 1 + \Delta t u_N^n a_{NN} (\phi_{NN}^*)' \end{pmatrix},$$

$B = (u_1^{n+1}, u_2^{n+1}, \cdots, u_N^{n+1})^T$, $C = (u_1^n, u_2^n \cdots, u_N^n)^T$ then $AB = C$, so $B = A^{-1}C$. Making use of the initial condition, we can get u_k^n $(k = 1, 2, \cdots, N; n = 1, 2, \cdots)$.

3.2 Error analysis

(3.2) follows that

$$u^{n+1} + \Delta t u^n u_x^{n+1} - u^n = 0. \tag{3.5}$$

Taking Taylor expansion of u^{n+1} follows that $u^{n+1} = u^n + \Delta t u_t^n + O(\Delta t^2)$. Substituting into (3.5) then we have

$$\begin{aligned} R &= u^n + \Delta t u_t^n + O(\Delta t^2) + \Delta t u^n (u_x^n + \Delta t u_{tx}^n) - u^n \\ &= \Delta t (u_t^n + u^n u_x^n) + (\Delta t)^2) u^n u_{tx}^n + O(\Delta t^2) \\ &= (\Delta t)^2 u^n u_{tx}^n + O(\Delta t^2). \end{aligned}$$

So truncation error $R = O(\Delta t^2)$, when $\Delta t \to 0$, $\|R\| \to 0$, it has two order precision with respect to time. Because $u^{n+1} = \sum\limits_{i=1}^{N} b_i \phi_i^*$ is the project of u^{n+1} on span$\{\phi_i^*\}_{i=1}^N$, when $\{x_k\}_{k=1}^{\infty}$ is dense in $[0, 1]$, we have

$$\|u^{n+1} - P_N u^{n+1}\| \to 0 \ \ (N \to \infty)$$

Namely u^{n+1} is convergent with respect to x.

Thus approximate solution using reproducing kernel finite difference method has higher precision, at the same time that it is sensitive to the singularity, so that the approximate solution can approximate perfectly to exact solution at the neighbor of the discontinuous point.

4. The Case of System of Equations

One-dimension unsteady compressible Euler system of equations is: (4.1)

$$\frac{\partial U}{\partial t}+A\frac{\partial U}{\partial x}=0 \tag{4.1}$$

here $U=\begin{pmatrix}\rho\\ u\\ p\end{pmatrix}$, $A=\begin{pmatrix}u & \rho & 0\\ 0 & u & \rho^{-1}\\ 0 & \rho c^2 & u\end{pmatrix}$, $c=\sqrt{\frac{kp}{\rho}}$, turning Euler system of equations into

$$\begin{cases}\rho_t+\rho u_x+u\rho_x=0\\ u_t+uu_x+\rho^{-1}p_x=0\\ p_t+kpu_x+up_x=0\end{cases} \tag{4.2}$$

We can get its difference system of equations with respect to t

$$\begin{cases}\rho^{n+1}+\Delta t\rho^n u_x^{n+1}+\Delta t u^n\rho_x^{n+1}=\rho^n\\ u^{n+1}+\Delta t u^n u_x^{n+1}+\Delta t(\rho^n)^{-1}p_x^{n+1}=u^n\\ p^{n+1}+k\Delta t p^n u_x^{n+1}+\Delta t u^n p_x^{n+1}=p^n\end{cases}$$

where Δt is the time step.

Firstly, we evaluate ρ, u and p on $[-1,1]$. We may assume ρ, u and p equal zero when $x\notin[-1,1]$, it follows that $\rho,u,p\in H^2(R)$.

Let $\rho^{n+1}=\sum\limits_{i=1}^{N}b_i\phi_i^*$, $u^{n+1}=\sum\limits_{i=1}^{N}c_i\phi_i^*$ and $p^{n+1}=\sum\limits_{i=1}^{N}d_i\phi_i^*$, where $b_i=<\rho^{n+1},\phi_i^*>_{H^2(R)}=\sum\limits_{k=1}^{i}a_{ki}\rho_k^{n+1}$, $c_i=<u^{n+1},\phi_i^*>_{H^2(R)}=\sum\limits_{k=1}^{i}a_{ki}u_k^{n+1}$, $d_i=<p^{n+1},\phi_i^*>_{H^2(R)}=\sum\limits_{k=1}^{i}a_{ki}p_k^{n+1}$, $\phi_i^*=\sum\limits_{k=1}^{i}a_{ki}\phi_k$, $\phi_k=K_2(x-x_k)$.

We may set that in $\{x_k\}_{k=1}^{\infty}$, x_k are dense and pairwise different knots in $[-1,1]$.

Substituting them into difference system of equations, we obtain

$$\begin{cases}\rho^{n+1}+\Delta t\rho^n\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}u_k^{n+1}(\phi_i^*)'+\Delta t u^n\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}\rho_k^{n+1}(\phi_i^*)'=\rho^n\\ u^{n+1}+\Delta t u^n\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}u_k^{n+1}(\phi_i^*)'+\Delta t(\rho^n)^{-1}\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}p_k^{n+1}(\phi_i^*)'=u^n\\ p^{n+1}+k\Delta t p^n\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}u_k^{n+1}(\phi_i^*)'+\Delta t u^n\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}p_k^{n+1}(\phi_i^*)'=p^n\end{cases}$$

Calculating inner product of $\phi_l=K_2(x-x_l)$ $(l=1,2,\cdots,N)$ and above system, it follows that

$$\begin{cases}\rho_l^{n+1}+\Delta t\rho_l^n\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}u_k^{n+1}(\phi_{li}^*)'+\Delta t u_l^n\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}\rho_k^{n+1}(\phi_{li}^*)'=\rho_l^n\\ u_l^{n+1}+\Delta t u_l^n\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}u_k^{n+1}(\phi_{li}^*)'+\Delta t(\rho_l^n)^{-1}\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}p_k^{n+1}(\phi_{li}^*)'=u_l^n\\ p_l^{n+1}+k\Delta t p_l^n\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}u_k^{n+1}(\phi_{li}^*)'+\Delta t u_l^n\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{i}a_{ki}p_k^{n+1}(\phi_{li}^*)'=p_l^n\end{cases}$$

Taking $k, l = 1, 2, \cdots, N$, we obtain

$$\begin{cases}
\sum_{i=1}^{N}\sum_{k=i}^{N}(\Delta t u_1^n a_{ik}(\phi_{1k}^*)' + \delta_{1i})\rho_i^{n+1} + \sum_{i=1}^{N}\sum_{k=i}^{N}\Delta t \rho_1^n a_{ik}(\phi_{1k}^*)' u_i^{n+1} + \sum 0 \cdot p_i^{n+1} = \rho_1^n \\
\cdots \\
\sum_{i=1}^{N}\sum_{k=i}^{N}(\Delta t u_N^n a_{ik}(\phi_{Nk}^*)' + \delta_{Ni})\rho_i^{n+1} + \sum_{i=1}^{N}\sum_{k=i}^{N}\Delta t \rho_N^n a_{ik}(\phi_{Nk}^*)' u_i^{n+1} + \sum 0 \cdot p_i^{n+1} = \rho_N^n \\
\sum_{i=1}^{N}\sum_{k=i}^{N}(\Delta t u_1^n a_{ik}(\phi_{1k}^*)' + \delta_{1i})u_i^{n+1} + \sum_{i=1}^{N}\sum_{k=i}^{N}\Delta t (\rho_1^n)^{-1} a_{ik}(\phi_{1k}^*)' p_i^{n+1} + \sum 0 \cdot \rho_i^{n+1} = u_1^n \\
\cdots \\
\sum_{i=1}^{N}\sum_{k=i}^{N}(\Delta t u_N^n a_{ik}(\phi_{Nk}^*)' + \delta_{Ni})u_i^{n+1} + \sum_{i=1}^{N}\sum_{k=i}^{N}\Delta t (\rho_N^n)^{-1} a_{ik}(\phi_{Nk}^*)' p_i^{n+1} + \sum 0 \cdot \rho_i^{n+1} = u_N^n \\
\sum_{i=1}^{N}\sum_{k=i}^{N}(\Delta t u_1^n a_{ik}(\phi_{1k}^*)' + \delta_{1i})p_i^{n+1} + \sum_{i=1}^{N}\sum_{k=i}^{N}k\Delta t p_1^n a_{ik}(\phi_{1k}^*)' u_i^{n+1} + \sum 0 \cdot \rho_i^{n+1} = p_1^n \\
\cdots \\
\sum_{i=1}^{N}\sum_{k=i}^{N}(\Delta t u_N^n a_{ik}(\phi_{Nk}^*)' + \delta_{Ni})p_i^{n+1} + \sum_{i=1}^{N}\sum_{k=i}^{N}k\Delta t p_N^n a_{ik}(\phi_{Ni}^*)' u_i^{n+1} + \sum 0 \cdot \rho_i^{n+1} = p_N^n
\end{cases}$$

Let $A =$

$$\begin{pmatrix}
1+\sum_{k=1}^{N}\Delta t u_1^n a_{1k}(\varphi_{1k}^*)' & \cdots & \Delta t u_1^n a_{NN}(\varphi_{1N}^*)' & \sum_{k=1}^{N}\Delta t \rho_1^n a_{1k}(\varphi_{1k}^*)' & \cdots & \Delta t \rho_1^n a_{NN}(\varphi_{1N}^*)' & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sum_{k=1}^{N}\Delta t u_N^n a_{1k}(\varphi_{Nk}^*)' & \cdots & 1+\Delta t u_N^n a_{NN}(\varphi_{NN}^*)' & \sum_{k=1}^{N}\Delta t \rho_N^n a_{1k}(\varphi_{Nk}^*)' & \cdots & \Delta t \rho_N^n a_{NN}(\varphi_{NN}^*)' & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1+\sum_{k=1}^{N}\Delta t u_1^n a_{1k}(\varphi_{1k}^*)' & \cdots & \Delta t u_1^n a_{NN}(\varphi_{1N}^*)' & \sum_{k=1}^{N}\Delta t a_{1k}(\varphi_{1k}^*)'/\rho_1^n & \cdots & \Delta t a_{NN}(\varphi_{1N}^*)'/\rho_1^n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \sum_{k=1}^{N}\Delta t u_N^n a_{1k}(\varphi_{Nk}^*)' & \cdots & 1+\Delta t u_N^n a_{NN}(\varphi_{NN}^*)' & \sum_{k=1}^{N}\Delta t a_{1k}(\varphi_{Nk}^*)'/\rho_N^n & \cdots & \Delta t a_{NN}(\varphi_{NN}^*)'/\rho_N^n \\
0 & \cdots & 0 & \sum_{k=1}^{N}k\Delta t a_{1k} p_1^n(\varphi_{1k}^*)' & \cdots & k\Delta t p_1^n a_{NN}(\varphi_{1N}^*)' & 1+\sum_{k=1}^{N}\Delta t u_1^n a_{1k}(\varphi_{1k}^*)' & \cdots & \Delta t u_1^n a_{NN}(\varphi_{1N}^*)' \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \sum_{k=1}^{N}k\Delta t p_N^n a_{1k}(\varphi_{Nk}^*)' & \cdots & k\Delta t p_N^n a_{NN}(\varphi_{NN}^*)' & \sum_{k=1}^{N}\Delta t u_N^n a_{1k}(\varphi_{Nk}^*)' & \cdots & 1+\Delta t u_N^n a_{NN}(\varphi_{NN}^*)'
\end{pmatrix}$$

$$B = (\rho_1^{n+1}, \rho_2^{n+1}, \cdots, \rho_N^{n+1}, u_1^{n+1}, u_2^{n+1}, \cdots, u_N^{n+1}, p_1^{n+1}, p_2^{n+1}, \cdots, p_N^{n+1})^T$$

$$C = (\rho_1^n, \rho_2^n, \cdots, \rho_N^n, u_1^n, u_2^n, \cdots, u_N^n, p_1^n, p_2^n, \cdots, p_N^n)^T.$$

Then $AB = C$, so $B = A^{-1}C$.

5. Numerical Results

Example 1. Model equation

$$u_t + uu_x = 0 \quad (t \geq 0, x \in [0, 1])$$

given initial condition

$$u(x,0)=\begin{cases} x & \frac{1}{2}\le x\le 1 \\ -x & 0\le x\le \frac{1}{2} \\ 0 & \text{otherwise}\end{cases} \quad u(0,t)=0$$

then its exact solution is

$$u(x,t)=\begin{cases} \frac{x}{1+t} & \frac{1}{2}\le x\le 1 \\ -\frac{x}{1+t} & 0\le x\le \frac{1}{2} \\ 0 & \text{otherwise.}\end{cases}$$

We make use of this paper's method to solve the above equation. Following figure 1 and 2 gives the numerical results, taking $\Delta x=0.1$ (the step of x)and $\Delta t=0.1$ (the step of t) (compared with Lax-wendroff scheme).

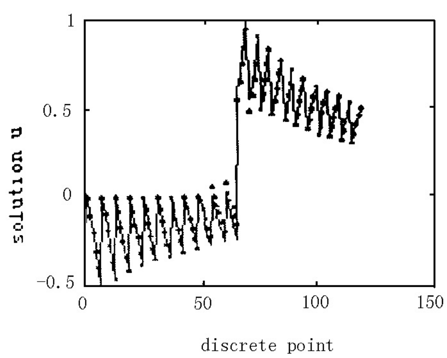

Figure 1.Reproducing kernel solution (point) and exact solution(solid line)

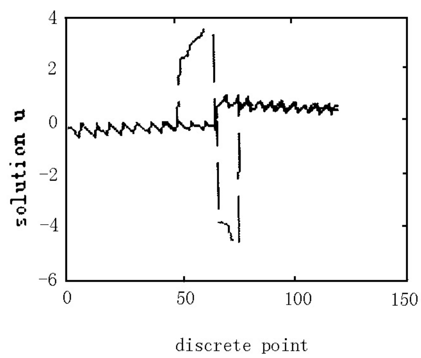

Figure 2. Difference solution (dotted line) and exact solution(solid line)

Above results are consistent with preceding analysis, this paper's method has better behavior on a neighborhood of the discontinuous points.

Example 2. Considering the exact solution of one-dimension unsteady Euler system of equations's initial value problem, that is Riemann Problem's discontinuous solution.

Let the initial values be constant distributions respectively at $t=0$, namely, when $x<0$, the distribution is ρ_1, u_1, p_1; when $x>0$, the distribution is ρ_2, u_2, p_2. And it satisfies following discontinuous conditions on discontinuous line.

$$\begin{cases} \rho_1(u_1-Z)=\rho_2(u_2-Z) \\ \rho_1 u_1(u_1-Z)+p_1=\rho_2 u_2(u_2-Z)+p_2 \\ \rho_1 E_1(u_1-Z)+p_1 u_1=\rho_2 E_2(u_2-Z)+p_2 u_2 \end{cases}$$

where Z is the velocity of discontinuous line $x=x(t)$, E is the energy per unit volume ,hence Euler system of equations (4.1) has exact solution.

We make use of this paper's method to solve it. Figure 3-8 gives the computing error curves (compared with difference scheme). From these figures we can find that this paper's algorithm discussed is stable for the system.

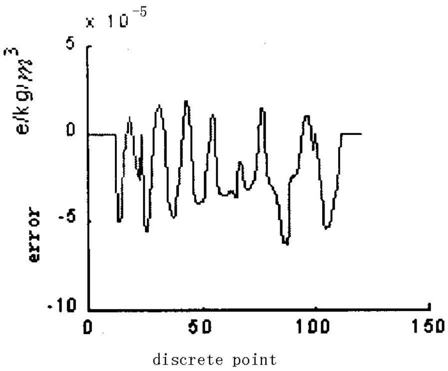

Fig3. when $\Delta x = 0.1, \Delta t = 0.1$, the error of ρ in reproducing kernel method

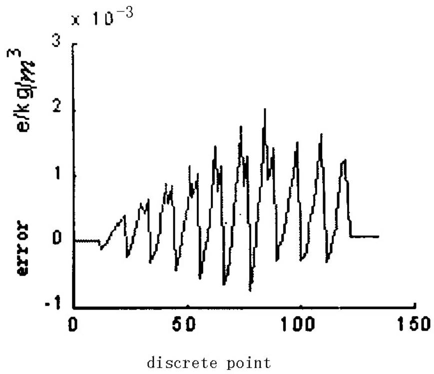

Fig 4. when $\Delta x = 0.1, \Delta t = 0.1$, the error of ρ in difference method

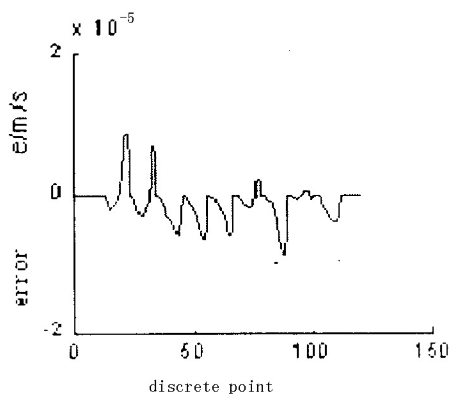

Fig5. When $\Delta x = 0.1, \Delta t = 0.1$, the error of u in reproducing kernel method

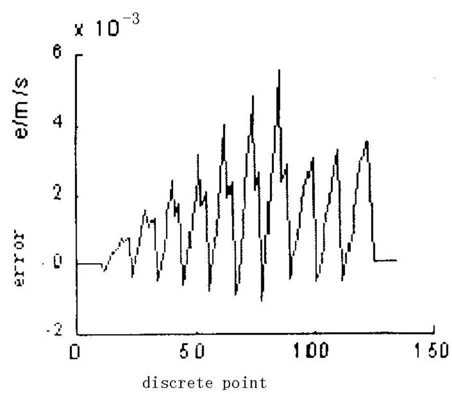

Fig 6. When $\Delta x = 0.1, \Delta t = 0.1$, the error of u in difference method

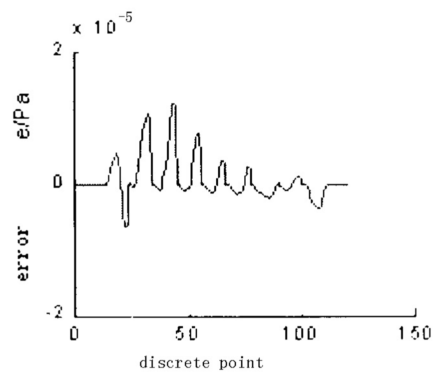

Fig 7. When $\Delta x = 0.1, \Delta t = 0.1$, ,the error of P in reproducing kernel method

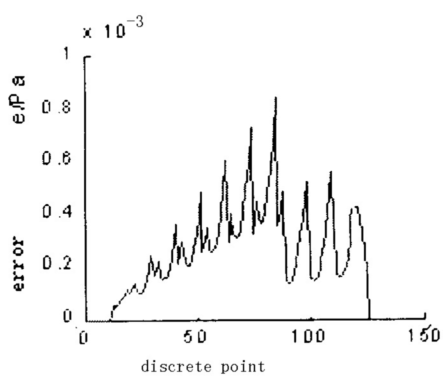

Fig8. When $\Delta x = 0.1, \Delta t = 0.1$,the error of P in difference method

All those numerical results above have shown that reproducing kernel finite difference method given in this paper can capture efficiently shock wave, can approximate the true solution exactly around the discontinuous points, and its unphysical oscillation is implicit, it possesses satisfied resolution ratio, simple scheme, small amount of calculation, so as to be extended and applied easily.

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