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PARTITION PROPERTY OF DOMAIN DECOMPOSITION WITHOUT ELLIPTICITY[*1)]

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Abstract

Partition property plays a central role in domain decomposition methods. Existing theory essentially assumes certain ellipticity. We prove the partition property for problems without ellipticity which are of practical importance. Example applications include implicit schemes applied to degenerate parabolic partial differential equations arising from superconductors, superfluids and liquid crystals. With this partition property, Schwarz algorithms can be applied to general non-elliptic problems with an h-independent optimal convergence rate. Application to the time-dependent Ginzburg-Landau model of superconductivity is illustrated and numerical results are presented.

Key words: Partition property, Domain decomposition, Non-ellipticity, Degenerate parabolic problems, Time-dependent Ginzburg-Landau model, Superconductivity, Preconditioning, Schwarz algorithms.

1. Introduction

Domain decomposition methods have undergone great development in the past decade and there has been rich mathematical theory for model problems. It is worthy to examine how these methods can be effectively applied to practical problems where the model problem analysis does not trivially apply. It is far from claiming that there remains not much new for domain decomposition theoretically and practically. In terms of domain partition, domain decomposition methods are classified as two types: overlapping and non-overlapping. Due to P. L. Lions [7], a general framework based on projection operators has been established for the convergence analysis of overlapping methods, where a so-called *partition property* play a central role. Following this approach, various overlapping methods such as the multiplicative and additive Schwarz algorithms have been studied for the model elliptic and parabolic problems as well as their extension to nonlinear, non-selfadjoint, and indefinite problems. For each case, the essential task is to verify the validity of the partition property associated with the particular problem. Although mathematically the fundamental analysis framework remains more or less unchanged, these works are of practical importance because they make the application of domain decomposition broader and broader, and more and more practical. There is vast literature available

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in this field, and a complete survey and references are recently found in [5], from which we cite only a few when necessary.

In this paper, we are interested in problems without ellipticity, such as degenerate parabolic PDEs which are not toy problems, nor trivial extension of the parabolic model problem for which the overlapping methods have already been well understood. The study is motivated by superconductor simulation using the three-dimensional time-dependent Ginzburg-Landau model (TDGL), which involves solution to a degenerate parabolic system of coupled nonlinear PDEs [8] [10]. An implicit scheme is used due to the stability consideration. To simulate the superconducting vortex dynamics over a long time period and the vortex structures at the equilibrium state, a fairly large step size Δt has to be used. On the other hand, a small spacing h is required for adequate resolution. This implies that the condition number of the discrete system cannot be improved by decreasing Δt. To simulate a larger sample with more vortices, one must use many more spatial grid points, which in turn implies larger condition number and much more intensive computation which takes considerable time even on IBM SP2 and Intel PARAGON parallel supercomputers. Apparently, preconditioning is necessary and crucial for reducing the computational time of iterative solvers in order to perform practical simulations. In fact, this type of degenerate parabolic problems also occur in other important applications such as superfluids and liquid crystals. We apply overlapping methods to speed up the iteration and prove the optimal convergence rate for these methods. As for other problems mentioned earlier, the key task is to verify that the partition property also holds for this type of problems under the weak assumption. We show that these methods are indeed numerically effective.

The remainder of the paper is organized as follows. We outline P. L. Lions' general framework and describe the partition property in Section 2. In Section 3, the partition property for problems without ellipticity is proved. The convergence theory developed in Section 3 is applied to the Ginzburg-Landau model in Section 4 and numerical results are presented.

2. Partition Property

Starting with a non-overlapping quasi-uniform partition

$$\overline{\Omega} = \bigcup_{i=1}^{I} \overline{\Omega}_i^0 \tag{2.1}$$

of coarse mesh size H, we define an overlapping domain decomposition by extending each subdomain Ω_i^0 to Ω_i such that

$$dist\left(\partial\Omega_i \bigcap \Omega, \partial\Omega_i^0 \bigcap \Omega\right) \geq \beta H, \quad i = 1, \cdots, I, \tag{2.2}$$

where β measures the overlapping width and that for any point $x \in \Omega$, there exist at most N_c subdomains Ω_i containing x. Let V be a finite element space corresponding to a quasi-uniform fine mesh of size h, and assume that the boundaries of the overlapping subdomains align with the fine mesh lines. Associated with the domain decomposition, we define subspaces $V_i \subset V (1 \leq i \leq I)$ as

$$V_i = \{v \in V;\ v = 0 \text{ in } \Omega/\Omega_i\}.$$

One can verify that $V = \sum_{i=1}^{I} V_i$. In addition, one often needs to use V_0 which denotes the finite element space corresponding to the coarse mesh (2.1).

Associated with a symmetric, positive definite bilinear form $a(\cdot,\cdot)$ in V, a partition property of domain decomposition is defined as follows.

Partition Property. There exists a constant $C_P \geq 1$ such that for any $v \in V$, the partition $v = \sum_{i=I_0}^{I} v_i$, where $v_i \in V_i$, satisfies

$$\sum_{i=I_0}^{I} a(v_i, v_i) \leq C_P a(v, v), \tag{2.3}$$

where $I_0 = 0$ or 1, depending on whether V_0 is included.

Let $\mathbf{P}_i$: V -> V_i be the orthogonal projection operator with respect to $a(\cdot\,,\cdot)$. According to P. L. Lions [7], the error propagation operator of the classical Schwarz alternating iterative procedure can be expressed in terms of $\{\mathbf{P}_i\}$ as

$$\mathbf{E}_I = (\mathbf{I} - \mathbf{P}_I)\,(\mathbf{I} - \mathbf{P}_{I-1}) \cdots (\mathbf{I} - \mathbf{P}_1)\,. \tag{2.4}$$

For this reason, the Schwarz alternating method is also referred to as the multiplicative Schwarz algorithm. There is also an additive Schwarz preconditioner for the operator $\mathbf{A}$ induced from $a(\cdot,\cdot)$. The preconditioned operator can be written as

$$\tilde{\mathbf{A}} = \sum_{i=I_0}^{I} \mathbf{P}_i. \tag{2.5}$$

Thus, the convergence of the Schwarz algorithms merely relies on the operators $\{\mathbf{P}_i\}$.

To analyze the additive Schwarz preconditioner, for example, an upper bound for the spectra of $\tilde{\mathbf{A}}$ is easily obtained by a standard functional analysis argument using Cauchy's inequality, the definition of the projection, and the assumption on domain decomposition.

Lemma 2.1. [[5]]

$$\lambda_{\max}\left(\sum_{i=I_0}^{I} \boldsymbol{P}_i\right) \leq N_m, \tag{2.6}$$

where $N_m = N_c$ *for* $I_0 = 1$*, and* $N_m = N_c + 1$ *for* $I_0 = 0$.

Recall the Rayleigh quotient

$$\lambda_{\min}\left(\sum_{i=I_0}^{I} \mathbf{P}_i\right) = \min_{u\neq 0} \sum_{i=I_0}^{I} a(\mathbf{P}_i u, u) \Big/ a(u,u).$$

A lower bound for the spectra of $\tilde{\mathbf{A}}$ then immediately follows from the partition property and Cauchy's inequality.

Lemma 2.2.

$$\lambda_{\min}\left(\sum_{i=I_0}^{I} \boldsymbol{P}_i\right) \geq \frac{1}{C_P}. \tag{2.7}$$

Combining Lemma 2.1 and Lemma 2.2 yields a bound for the condition number $cond(\tilde{\mathbf{A}})$, which characterizes the convergence rates of many popular iterative methods with the additive Schwarz preconditioner.

Theorem 2.3. *The condition number cond* $(\tilde{\mathbf{A}})$ *of the additive Schwarz preconditioned operator is bounded by* $N_m C_P$.

With the partition constant C_P, the convergence rate of the multiplicative Schwarz algorithm can also be bounded from a general convergence theory [11] (see the fundamental theorem II and Lemma 4.7 therein).

Theorem 2.4.

$$\|\mathbf{E}_I\|_a \le 1 - \frac{1}{(1+N_C)^2 C_P}. \tag{2.8}$$

Under the above framework, the convergence analysis of overlapping methods is essentially reduced to establishing the partition property for the problem dependent bilinear form $a(\cdot,\cdot)$, as in most of the well-known papers [3, 4, 5, 6, 7, 11]. If $a(\cdot,\cdot)$ induces an H^1-equivalent norm, namely there exist positive constants C_1 and C_2 independent of h such that

$$C_1 \|u\|_1^2 \le a(u,u) \le C_2 \|u\|_1^2, \quad \forall u \in V, \tag{2.9}$$

then the partition property holds with $C_P = O(H^{-2})$ for $I_0 = 1$. The factor $O(H^{-2})$ can be removed from the bound for $I_0 = 0$, i. e., if the coarse mesh model is included. A more detailed analysis can also show the dependence of the bound on the overlapping parameter β. In fact, the proof of this result was carried out in terms of H^1-norm, which was problem independent, and then passed to the energy norm under assumption (2.9). Therefore, it appears that (2.9) is an essential requirement for the partition property. Assumption (2.9) is naturally satisfied for second order elliptic problems and for implicit schemes applied to the associated parabolic problems. In the latter case, the convergence is also shown to be uniform without the coarse mesh as soon as $\Delta t \le CH^2$. Although the left hand side of (2.9) is relaxed in [4] to Garding's inequality

$$e(u,u) - C\|u\|_0^2 \le a(u,u) \quad \forall u \in H_0^1(\Omega), \tag{2.10}$$

where $e(\cdot,\cdot)$ is elliptic, the ellipticity is still an essential requirement. In practice, there are applications which result in $a(\cdot,\cdot)$ such that $a(u,u)$ can only be lower bounded by L_2-norm $\|u\|_0^2$. We show in the next section that the partition property still holds under this very weak assumption.

3. Partition Property without Ellipticity

Let $W \subset H^1(\Omega)$ be a subspace, where $\Omega \in R^d$ is a bounded domain. Consider a bilinear form $a(\cdot,\cdot)$ satisfying

$$\begin{cases} a(u,v) = a(v,u), & \forall u,v \in W; \\ |a(u,v)| \le C\|u\|_1 \|v\|_1, & \forall u,v \in W; \\ a(u,u) \ge C\|u\|_0, & \forall u \in W. \end{cases} \tag{3.1}$$

Theorem 3.1. *Let $V \subset W$ be a finite element subspace. Under the assumption of (3.1), for any $v \in V$, there exist $v_i \in V_i$ such that $v = \sum_{i=1}^{I} v_i$ and*

$$\sum_{i=1}^{I} a(v_i,v_i) \le CN_c\left(1+\beta^{-2}H^{-2}\right) a(v,v), \tag{3.2}$$

where C is a generic constant independent of h, H and I.

Proof. Recall that in the analysis with ellipticity, one derived the partition property by actually working on H^1-norm and then passing to the energy norm due to the equivalence between the two norms. It does not appear trivial to extend this result to the non-elliptic case. Instead, our approach is to work on the bilinear form directly without going through H^1-norm.

As usual, let $\{\theta_i\}_1^I$ be a smooth partition of unity defined on Ω satisfying

$$\sum_{i=1}^{I}\theta_i = 1; \quad 0 \le \theta_i \le 1; \quad \text{supp } \theta_i \subset \Omega_i \bigcup \partial\Omega; \quad |\nabla\theta_i|_\infty \le C\beta^{-1}H^{-1}. \tag{3.3}$$

A decomposition $v = \sum_{i=1}^{I} v_i$ is constructed by

$$v_i = I_h(\theta_i v), \quad i = 1, 2, \cdots, I, \tag{3.4}$$

where I_h is the nodal interpolant on V.

We start directly with the following inequality in the a-form:

$$a_e(v_i, v_i) \le 2a_e(\theta_{ie} I_h v, \theta_{ie} I_h v) + 2a_e\left(I_h\left((\theta_i - \theta_{ie})v\right), I_h\left((\theta_i - \theta_{ie})v\right)\right), \tag{3.5}$$

where $a_e(u, v)$ is the restriction of $a(u, v)$ on element e and θ_{ie} is the average of θ_i on e. From (3.3) we have

$$|\theta_i - \theta_{ie}|_\infty \le Ch\,|\nabla\theta_i|_{L_\infty(e)} \le Ch\beta^{-1}H^{-1}. \tag{3.6}$$

Using the inverse estimate $\|v_h\|_1 \le Ch^{-1}\|v_h\|_0$ and the fact that $\|I_h(fv_h)\|_0 \le |f|_\infty \|v_h\|_0$, we have the estimate for the second term on the right hand side of (3.5)

$$\begin{aligned}
&a_e\left(I_h\left((\theta_i - \theta_{ie})\,v\right), I_h\left((\theta_i - \theta_{ie})\,v\right)\right) \\
&\le C\left\|I_h\left((\theta_i - \theta_{i\theta})\,v\right)\right\|_{H^1(e)}^2 \\
&\le Ch^{-2}\left\|I_h\left((\theta_i - \theta_{i\theta})\,v\right)\right\|_{L_2(e)}^2 \\
&\le C\beta^{-2}H^{-2}\|v\|_{L_2(e)}^2 .
\end{aligned} \tag{3.7}$$

For the first term on the right hand side of (3.5), we have

$$a_e\left(\theta_{ie} I_h v, \theta_{ie} I_h v\right) = \theta_{ie}^2 a_e(v, v) \le a_e(v, v), \tag{3.8}$$

where we use the facts that θ_{ie} is a constant; $I_h v = v$ for $v \in V$; and $\theta_{ie}^2 \le 1$. We note that (3.8) is the key for us to successfully obtain the factor $a(v, v)$ in the upper bound. If only getting $\|v\|_1^2$ in the bound as in the analysis with ellipticity, we cannot convert it to $a(v, v)$. Combining (3.7), (3.8) and summing over e leads to

$$\begin{aligned}
\textstyle\sum_{i=1}^{I} a(v_i, v_i) &\le 2N_c\left[a(v, v) + C\beta^{-2}H^{-2}\|v\|_0^2\right] \\
&\le CN_c\left[1 + \beta^{-2}H^{-2}\right]a(v, v),
\end{aligned} \tag{3.9}$$

where assumption (3.1) is applied for the last inequality, which completes the proof.

We remark that with the inclusion of the coarse mesh, the estimate for the partition constant C_P cannot be improved in our case. To see this, recall that in the analysis with ellipticity the $O(H^{-2})$ factor is removed by taking the advantage that

$$H^{-2}\left\|v-v_0\right\|_0^2 \leq CH^{-2}\cdot H^2\left\|v\right\|_1^2 \leq Ca(v,v). \tag{3.10}$$

We observe that the ellipticity of $a(\cdot\,,\cdot)$ is essential in (3.10) because the factor $\|v\|_1$ is essential for obtaining the first inequality from the approximation theory. Fortunately, in our applications we are more concerned with the scalability with respect to h, not to H because the number of processors, or equivalently H, is fixed in general. In fact, we are interested in larger and larger samples. So H is increasing, not decreasing.

As an application of this general theory, let us consider a degenerate parabolic problem: seek $u \in W$ such that

$$\begin{cases} (\partial_t u, v) + b(u,v) = (f,v), & \forall v \in W, \\ u(x,0) = u_0 \in W, \end{cases} \tag{3.11}$$

where the bilinear form $b(\cdot\,,\cdot)$ satisfies

$$\begin{cases} b(u,v) = b(v,u), & \forall u,v \in W; \\ |b(u,v)| \leq C\left\|u\right\|_1\left\|v\right\|_1, & \forall u,v \in W; \\ b(u,u) \geq 0, & \forall u \in W. \end{cases} \tag{3.12}$$

We call (3.11) a degenerate parabolic problem because the bilinear form $b(u,v)$ is only non-negative, not elliptic. Thus this problem is essentially different from, even not a trivial extension from that considered in [3]. Applying finite elements in space and implicit finite differences in time leads to, at every time step, a discrete problem: seek $u_h \in V \subset W$ such that

$$a_\tau(u_h, v_h) = g_\tau(v_h), \quad \forall v_h \in V, \tag{3.13}$$

where

$$a_\tau(u,v) = (u,v) + \tau b(u,v) \tag{3.14}$$

with $\tau = \Delta t$.

It is easy to verify that $a_\tau(\cdot\,,\cdot)$ satisfies (3.1). Thus (3.13) is uniquely solvable, and Theorem 3.1 can be applied with $a_\tau(\cdot\,,\cdot)$ as the bilinear form. We also note that the last inequality in assumption (3.12) can be extended to

$$b(u,u) \geq -\alpha\left\|u\right\|_0^2, \quad \forall u \in W, \tag{3.15}$$

where $\alpha > 0$ is a positive constant. In this case, we introduce

$$b^*(u,v) = b(u,v) + \alpha(u,v),$$

so that $b^*(\cdot\,,\cdot)$ satisfies (3.12). Then, corresponding to (3.14), we have

$$a_\tau(u,v) = (u,v) + \frac{\tau}{1-\alpha\tau} b^*(u,v), \quad for\ \tau < 1/\alpha. \tag{3.16}$$

We can also derive the partition property with Δt involved in the upper bound.

Theorem 3.2. *Under assumption (3.12), for any $v \in V$, there exist $v_i \in V_i$ such that $v = \sum_{i=1}^{I} v_i$ and*

$$\sum_{i=1}^{I} a_\tau(v_i, v_i) \le CN_c \left(1 + \tau\beta^{-2}H^{-2}\right) a_\tau(v, v), \tag{3.17}$$

where C is a generic constant independent of h, H and I.

Proof. Applying the argument in Theorem 3.1 we obtain

$$\sum_{i=1}^{I} b(v_i, v_i) \le 2N_c \left[b(v, v) + C\beta^{-2}H^{-2} \|v\|_0^2\right]. \tag{3.18}$$

For the inner-product term, we have

$$\sum_{i=1}^{I} (v_i, v_i) = \sum_{i=1}^{I} \|I_h(\theta_i v)\|_0^2 \le CN_c \|v\|_0^2. \tag{3.19}$$

Therefore,

$$\begin{aligned} &\textstyle\sum_{i=1}^{I} a_\tau(v_i, v_i) \\ &= \textstyle\sum_{i=1}^{I} \left[(v_i, v_i) + \tau b(v_i, v_i)\right] \\ &\le CN_c(v, v) + 2\tau N_c \left[b(v, v) + C\beta^{-2}H^{-2}(v, v)\right] \\ &\le N_c \left[C\left(1 + \tau\beta^{-2}H^{-2}\right)(v, v) + 2\tau b(v, v)\right] \\ &\le CN_c \left(1 + \tau\beta^{-2}H^{-2}\right) a_\tau(v, v), \end{aligned} \tag{3.20}$$

which completes the proof.

4. Application to the Ginzburg-Landau Model

Let us now consider a practical degenerate parabolic problem in the time-dependent Ginzburg-Landau model which is used extensively in superconductor simulation. The governing equations are

$$\begin{cases} \frac{\partial \psi}{\partial t} + (\frac{i}{\kappa}\nabla + \mathbf{A})^2\psi - \psi + |\psi|^2\psi = 0, \\ \frac{\partial \mathbf{A}}{\partial t} + \mathrm{curlcurl}\mathbf{A} + \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + |\psi|^2\mathbf{A} = \mathrm{curl}\mathbf{H}, \end{cases} \quad \mathrm{in}\Omega \times (0, T], \tag{4.1}$$

with boundary conditions

$$\begin{cases} \frac{\partial \psi}{\partial n} = 0, \\ \mathrm{curl}\mathbf{A} \times \mathbf{n} = \mathbf{H} \times \mathbf{n}, \quad \mathrm{on}\partial\Omega \times (0, T], \\ \mathbf{A} \cdot \mathbf{n} = 0, \end{cases} \tag{4.2}$$

and initial conditions

$$\begin{cases} \psi(\mathbf{x},0)=\psi^0(\mathbf{x})\text{with}|\psi^0|\le 1; \\ \\ \mathbf{A}(\mathbf{x},0)=\mathbf{A}^0(\mathbf{x})\text{withdiv}\mathbf{A}^0=0, \mathbf{A}^0\cdot\mathbf{n}=0, \end{cases} \quad \text{in}\Omega, \tag{4.3}$$

where $\mathbf{n}$ is the unit outer normal vector, and others are conventional notations. In this model, ψ is a complex variable and $\mathbf{A}=(\mathbf{A_x},\mathbf{A_y},\mathbf{A_z})$ is a vector.

We apply Crank-Nicolson-Galerkin decoupling methods in [8, 10]. A typical step in these algorithms is to solve two decoupled linear systems

$$\begin{cases} (\psi,\eta)+\tau a_\psi(\psi,\eta)=f_\psi(\eta), \qquad \forall\eta\in V; \\ \\ (\mathbf{A},\mathbf{v})+\tau a_{\mathbf{A}}(\mathbf{A},\mathbf{v})=f_{\mathbf{A}}(\mathbf{v}), \quad \forall\mathbf{v}\in\mathbf{V}, \end{cases} \tag{4.4}$$

with

$$\begin{cases} a_\psi(\psi,\eta)=\left(\left(\frac{i}{\kappa}\nabla+\tilde{\mathbf{A}}\right)\psi,\left(\frac{i}{\kappa}\nabla+\tilde{\mathbf{A}}\right)\eta\right)+((\alpha-1)\psi,\eta)\,; \\ \\ a_{\mathbf{A}}(\mathbf{A},\mathbf{v})=(curl\mathbf{A},curl\mathbf{v})+(\alpha\mathbf{A},\mathbf{v}), \end{cases} \tag{4.5}$$

where $0\le\alpha\le1$ and $\tilde{\mathbf{A}}$ are known from previous time steps.

For the first equation of (4.4), $a_\psi(\cdot,\cdot)$ is not elliptic in general. However, we have

$$\begin{aligned} a_\psi(v,v) &= \tfrac{1}{\kappa^2}|v|_1^2+\tfrac{i}{\kappa}\left((\nabla v v^*-v\nabla v^*),\tilde{\mathbf{A}}\right)+\left(\tilde{\mathbf{A}}v,\tilde{\mathbf{A}}v\right)+((\alpha-1)v,v) \\ &\ge \tfrac{1}{\kappa^2}|v|_1^2-C\|\nabla v\|_{L^2}\|v\|_{L^4}\left\|\tilde{\mathbf{A}}\right\|_{L^4}-\|v\|_0^2 \\ &\ge \tfrac{1}{\kappa^2}|v|_1^2-C|v|_1\|v\|_{L^4}-\|v\|_0^2 \\ &\ge \tfrac{1}{\kappa^2}|v|_1^2-|v|_1\left(\tfrac{1}{4\kappa^2}|v|_1+C\|v\|_0\right)-\|v\|_0^2 \\ &\ge \tfrac{1}{\kappa^2}|v|_1^2-\left(\tfrac{1}{4\kappa^2}|v|_1^2+\tfrac{1}{4\kappa^2}|v|_1^2+C\|v\|_0^2\right)-\|v\|_0^2 \\ &= \tfrac{1}{2\kappa^2}|v|_1^2-C\|v\|_0^2\,, \end{aligned}$$

where we used $\left\|\tilde{\mathbf{A}}\right\|_{L^4}\le C$ [10] for the second inequality and an interpolation inequality estimate

$$\|v\|_{L_4}\le\|v\|_{L_2}^{\frac{1}{2}}\|v\|_{L_6}^{\frac{1}{2}}\le\beta\|v\|_{L_6}+\frac{1}{4\beta}\|v\|_{L_2}\le\beta\|\nabla v\|_0+\frac{C}{\beta}\|v\|_0\,,\quad\forall\beta,$$

for the third inequality. Therefore, if τ is sufficiently small,

$$\begin{aligned} a_\tau(v,v) &\ge \tfrac{\tau}{\kappa^2}|v|_1^2+(1-C\tau)\|v\|_0^2 \\ &\ge \min\left(\tfrac{\tau}{\kappa^2},1-C\tau\right)\|v\|_1^2\,. \end{aligned}$$

Thus $a_\tau(\cdot,\cdot)$ is elliptic so that the convergence analysis with ellipticity can be applied to this case. However, we note that on one hand, τ should be quite small because the generic constant C above would be very large; and on the other hand, the partition parameter $C_P=$

$O\left(1/\min\left(\frac{\tau}{\kappa^2}, 1 - C\tau\right)\right)$ should be very large in practice because κ would also be large in a physically interesting regime. These difficulties are not present in our analysis by viewing it as a degenerate parabolic problem. For this particular case, we can also apply the GMRES-Schwarz algorithm for indefinite elliptic problems [4] because

$$a_\tau(v,v) \geq \frac{\tau}{\kappa^2} \|v\|_1^2 + (1 - C\tau - \frac{\tau}{\kappa^2}) \|v\|_0^2 .$$

However, CG method is generally more efficiently than GMRES.

For the second equation of (4.4), we note that $\left(\|curl\mathbf{A}\|_0^2 + \|div\mathbf{A}\|_0^2\right)^{\frac{1}{2}}$ defines a norm in $\{\mathbf{A} \in \mathbf{H}^1(\Omega) | \mathbf{A}\cdot n = 0\}$. But $\mathbf{A}$ is not divergence-free in general for the TDGL model. Therefore, even with the inclusion of a term $\|\mathbf{A}\|_0^2$ in $a_\tau(\mathbf{A},\mathbf{A})$, one cannot obtain an H^1-equivalent norm as in the first equation because $\|\mathbf{A}\|_0$ cannot control $\|div\mathbf{A}\|_0$. Yet, this is simply another particular example of the degenerate parabolic problem. Therefore, the convergence theory developed in the previous section can be applied to it.

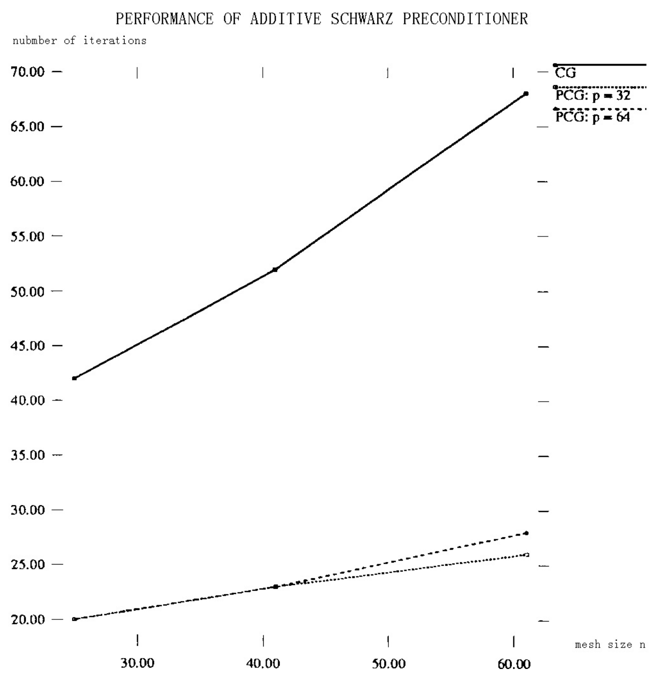

Fig 4.1. Number of iterations versus mesh size n for an $n \times n \times n$ grid for solving a typical magnetic potential equation with additive Schwarz preconditioning, where p denotes the number of processors.

Fig. 4.1 shows the number of iterations versus the grid size for both CG and preconditioned CG methods with the additive Schwarz preconditioner in solving a typical magnetic potential equation. The overlapping width is h and a tolerance 10^{-10} is used in the stopping criterion. p denotes the number of processors, or equivalently the number of subdomains. At every preconditioning step, a local problem needs to be solved for each subdomain. We again apply the CG iteration in the local solver and a tolerance 10^{-6} is used in the stopping criterion for this inner loop. So the preconditioner is inexact in this sense. Double precision is used in the computation. It is seen that the convergence rate of the PCG iteration is almost independent of the spatial resolution h and moderately varies with the number of subdomains. The performance is consistent for all time levels. Similar performance is observed for solving the order parameter equations. Readers are referred to [9] for more numerical results and practical considerations of stationary preconditioners.

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