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THE CONVERGENCE ON A FAMILY OF ITERATIONS WITH CUBIC ORDER*[1)]

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Abstract

In this paper, we establish a convergent theorem for a family of iterations with cubic order by using general convergence hypotheses. A sharp error estimate is also given explicitly.

Key words: Convergence, Majorizing method, γ-Condition, Error estimate.

1. Introduction

Let E and F be real or complex Banach spaces and $f : D \subseteq E \to F$ be a nonlinear twice differentiable operator. For solving the equation

$$f(x) = 0, \tag{1.1}$$

consider a one-parametered family of iterations

$$x_{n+1} = x_n - \left[I + \frac{1}{2} H_f(x_n)(I - \lambda H_f(x_n))^{-1} \right] f'(x_n)^{-1} f(x_n),\ n = 0, 1, \cdots, \tag{1.2}$$

where $H_f(x) = f'(x)^{-1} f''(x) f'(x)^{-1} f(x)$ and $0 \le \lambda \le 1$. This family is cubically convergent (see [3, 7]) and includes, as particular cases, Chebyshev method ($\lambda = 0$, see [5, 9]), Halley method ($\lambda = \frac{1}{2}$, see [1, 4, 6, 10]) and super-Halley method ($\lambda = 1$, see [7]).

In [3], I.K. Argyros et al analyze the convergence of (1.2) by using the quadratic majorizing function. But we found that there are some mistakes in the analysis. Especially, when the norm of the operator is estimated by the majorizing function, the second term of the last expression on page 270 in [3] is wrong . The reason for this is that the second term in the identity (4) in [3] is not always positive for all λ. In addition, the authors of [3] also give a convergence condition, but is not uniform for the parameter λ. In fact, just as we analyze below, for the iteration (1.2), it is unable to give a uniform condition independent on the parameter λ by the quadratic majorizing function. Therefore, other majorizing functions are required. In [7], M.A. Hernandez et al use cubic polynomial as a majorizing function and establish a convergence theorem. But the disadvantage of using the cubic majorizing function is that for the iteration (1.2),the error estimate can't be represented by an explicit form. Besides, the convergence condition is also too complicated.

To overcome these disadvantages above, we present more general or even weaker convergence hypotheses and establish a convergence theorem with a very simple convergence condition. Meanwhile, a sharp error estimate is given explicitly.

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2. Majorizing Function

Let β, γ be positive numbers and $h : [0, (1-\frac{1}{\sqrt{2}})\frac{1}{\gamma}] \to R$ be a real function defined by

$$h(t) = \beta - t + \frac{\gamma\, t^2}{1-\gamma\, t}\ . \tag{2.1}$$

Write $\alpha = \beta \cdot \gamma$. Then when $\alpha \le 3 - 2\sqrt{2}$, h has two real roots

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{1+\alpha \mp \sqrt{(1+\alpha)^2 - 8\alpha}}{4\gamma}.$$

They satisfy the inequality

$$\beta \le r_1 \le (1+\frac{1}{\sqrt{2}})\beta \le (1-\frac{1}{\sqrt{2}})\frac{1}{\gamma} \le r_2 \le \frac{1}{2\gamma}. \tag{2.2}$$

For $x_0 \in D$, suppose $f'(x_0)^{-1}$ exists. First, we give the following γ-condition ([9])

Definition 1. *Let h be defined by* (2.1). *Then h is said to satisfy the γ-condition about f at x_0 if*

$$||f'(x_0)^{-1} f(x_0)|| \le \beta, \quad ||f'(x_0)^{-1} f''(x_0)|| \le 2\gamma, \tag{2.3}$$

$$||f'(x_0)^{-1} f'''(x)|| \le \frac{6\gamma^2}{(1-\gamma||x-x_0||)^3}, \ \ \forall\, x \in D,\ \ ||x - x_0|| < (1-\frac{1}{\sqrt{2}})\frac{1}{\gamma}. \tag{2.4}$$

If h satisfies the γ-condition about f, then h is also said to be a majorizing function of f.

Lemma 1. *Assume h satisfies the γ-condition about f at x_0. If $||x - x_0|| < (1-\frac{1}{\sqrt{2}})\frac{1}{\gamma}$, then we have*

(a) $||f'(x_0)^{-1} f''(x)|| \le h''(||x-x_0||)$;

(b) $f'(x)^{-1}$ *exists and*

$$||f'(x)^{-1} f'(x_0)|| \le -\frac{1}{h'(||x-x_0||)}.$$

Proof. By the definition, we have

$$\begin{aligned} ||f'(x_0)^{-1} f''(x)|| &\le ||f'(x_0)^{-1} f''(x_0)|| + ||\int_0^1 f'(x_0)^{-1} f'''(x_0 + s(x-x_0))|| \cdot ||x-x_0|| ds \\ &\le h''(0) + \int_0^1 h'''(s||x-x_0||)||x-x_0||ds = h''(||x-x_0||). \end{aligned}$$

Hence (a) follows.

On the other hand, by Taylor formula,

$$f'(x_0)^{-1} f'(x) = I + f'(x_0)^{-1} f''(x_0)(x-x_0) + \int_0^1 f'(x_0)^{-1} f'''(x_0 + s(x-x_0))(1-s)(x-x_0)^2 ds.$$

Thus, by (2.3), (2.4) and the nonpositivity of h' , we have

$$\begin{aligned} ||f'(x_0)^{-1} f'(x) - I|| \quad &\le \quad \int_0^1 ||f'(x_0)^{-1} f'''(x_0 + s(x-x_0))|| \cdot ||x-x_0||^2 (1-s) ds \\ &\quad + ||f'(x_0)^{-1} f''(x_0)|| \cdot ||x-x_0|| \\ &\le \quad \int_0^1 h'''(s||x-x_0||)||x-x_0||^2(1-s)ds + h''(0)||x-x_0|| \\ &\le \quad h'(||x-x_0||) - h'(0) = 1 + h'(||x-x_0||) < 1 \end{aligned} \tag{2.5}$$

By Banach lemma, (b) also follows.

3. Convergence of the Majorizing Sequence

Let G_f be an iterative function defined by

$$G_f(x) = x - \left[I + \frac{1}{2}H_f(x)(I - \lambda H_f(x))^{-1}\right] f'(x)^{-1} f(x). \tag{3.1}$$

If h is a majorizing function, then the sequence $\{t_n\}_{n=0}^{n=\infty}$ defined by $t_{n+1} = G_h(t_n),\ t_0 = 0$,is called a majorizing sequence of the iteration (1.2).

Lemma 2. *Let h be defined by* (2.1) *and $\alpha \le 3-2\sqrt{2}$. Then $0 \le H_h(t) \le 2\alpha$ for $t \in [0, r_1]$. Moreover, if $t_{n+1} = G_h(t_n),\ t_0 = 0,\ n = 0, 1, \cdots$, then the sequence $\{t_n\}$ is monotonically increasing and tending to r_1.*

Proof. Let $g(t) = \gamma\, h(t/\gamma)$ and $s_1 = \gamma\, r_1,\ s_2 = \gamma\, r_2$. Then we have $H_h(t) = H_g(\gamma t)$. Write

$$g(t) = \frac{2\, ab}{(1-t)}, \tag{3.2}$$

where $a = s_1 - t$ and $b = s_2 - t$. Thus, $g'(t) = -2\left((a+b)(1-t) - ab\right)/(1-t)^2$ and $g''(t) = 2/(1-t)^2$. Also write

$$\varphi(t) := (a+b)(1-t) - ab = a(1-s_2) + b(1-t) = b(1-s_1) + a(1-t). \tag{3.3}$$

Then

$$H_g(t) = \frac{ab}{\varphi(t)^2}.$$

Since

$$H_g'(t) = -\frac{a^2(1-s_2) + b^2(1-s_1)}{\varphi(t)^3} < 0, \quad 0 \le t \le s_1,$$

so $H_g(t)$ is decreasing monotonically. Therefore, $0 \le H_g(t) \le H_g(0) = 2\alpha$ as $0 \le t \le s_1$.

On the other hand, we can verify that

$$G_h'(t) = \frac{H_h(t)^2}{2(1-\lambda H_h(t))^2}[3(1-\lambda) + \lambda(2\lambda - 1)H_h(t) - H_{h'}(t)]. \tag{3.4}$$

Since $H_{h'}(t)$ is negative for $0 \le t \le r_1$ and $0 \le H_h(t) < 2\alpha$, we have $G_h'(t) > 0$. In addition, , $h(t)/h'(t) \le 0$ for $0 \le t < r_1$. Thus we can use mathematical induction to prove that if $t_n < r_1$ holds for some n, then $t_n \le t_{n+1} = G_h(t_n) < G_h(r_1) = r_1$. This proves the lemma.

Lemma 3. *Suppose $s' = G_g(s),\ a = s_1 - s$, $b = s_2 - s,\ a' = s_1 - s'$ and $b' = s_2 - s'$. Then*

$$\frac{a'}{b'} = \frac{1-s_2}{1-s_1} \cdot \frac{\varphi(s)(1-s) + (\frac{1}{2} - \lambda)b}{\varphi(s)(1-s) + (\frac{1}{2} - \lambda)a} \cdot \left(\frac{a}{b}\right)^3. \tag{3.5}$$

In particular, when $\frac{1}{2} \le \lambda \le 1$,

$$\frac{a'}{b'} \le \frac{1-s_2}{1-s_1} \cdot \left(\frac{a}{b}\right)^3, \tag{3.6}$$

and when $0 \le \lambda \le \frac{1}{2}$,

$$\frac{a'}{b'} \le 2(1-\lambda)\frac{1-s_2}{1-s_1} \cdot \left(\frac{a}{b}\right)^3. \tag{3.7}$$

Proof. By the iterative formula (1.2) and the computational results in the previous proof, we have

$$\begin{aligned} a' &= a - \frac{1+(\frac{1}{2}-\lambda)\dfrac{ab}{\varphi(s)^2}}{1-\lambda\dfrac{ab}{\varphi(s)^2}} \cdot \frac{ab(1-s)}{\varphi(s)} \\ &= a - \frac{\varphi(s)^2 ab(1-s) + (\frac{1}{2}-\lambda)a^2b^2(1-s)}{\varphi(s)^3 - \lambda ab\varphi(s)}. \\ &= a \cdot \frac{\varphi(s)^3 - \frac{1}{2}ab\varphi(s) - \varphi(s)^2 b(1-s) - (\lambda-\frac{1}{2})ab\varphi(s) + (\lambda-\frac{1}{2})ab^2\varphi(s)}{\varphi(s)^3 - \lambda ab\varphi(s)} \end{aligned} \tag{3.8}$$

Noticing that $\frac{1}{2} = (1-s_1)(1-s_2)$ and the identity (3.3), we have

$$\begin{aligned} \varphi(s)^3 - \tfrac{1}{2}ab\varphi(s) - \varphi(s)^2 b(1-s) &= \varphi(s)\left(\varphi(s)^2 - \tfrac{1}{2}ab - \varphi(s)b(1-s)\right) \\ &= \varphi(s)\left(a^2(1-s_2)^2 + ab(1-s_2)(1-s) - \tfrac{1}{2}ab\right) \\ &= \varphi(s)a^2(1-s)(1-s_2) \end{aligned} \tag{3.9}$$

and

$$(\lambda - \frac{1}{2})ab\varphi(s) - (\lambda - \frac{1}{2})ab^2\varphi(s) = (\lambda - \frac{1}{2})a^2 b(1-s_2). \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$a' = (1-s_2)a^3 \frac{\varphi(s)(1-s) + (\frac{1}{2}-\lambda)b}{\varphi(s)^3 - \lambda ab\varphi(s)}\ . \tag{3.11}$$

Similarly, we have

$$b' = (1-s_1)b^3 \frac{\varphi(s)(1-s) + (\frac{1}{2}-\lambda)a}{\varphi(s)^3 - \lambda ab\varphi(s)}\ . \tag{3.12}$$

Therefore, (3.5) follows from (3.11) and (3.12). When $\frac{1}{2} \le \lambda \le 1$, it is obvious that $u(s) := \dfrac{\varphi(s)(1-s) + (\frac{1}{2}-\lambda)b}{\varphi(s)(1-s) + (\frac{1}{2}-\lambda)a} \le 1$ since $b \ge a$. Hence (3.6) follows. On the other hand, when $0 \le \lambda \le \frac{1}{2}$,

$$u'(s) = \frac{[2(1-s)^2 + (\frac{1}{2}-\lambda)](\frac{1}{2}-\lambda)(s_2-s_1)}{(\varphi(s)(1-s) + (\frac{1}{2}-\lambda)a)^2} \ge 0.$$

Hence $u(s)$ is increasing monotonically in $[0, s_1]$. So $u(s) \le u(s_1)$. Since $u(s_1) = 1 + \dfrac{(\frac{1}{2}-\lambda)}{(1-s_1)^2}$ and $s_1 \le (1 - \frac{\sqrt{2}}{2})$ as $\alpha \le 3 - 2\sqrt{2}$, we have $u(s) \le 2(1-\lambda)$. Hence (3.7) follows. This completes the proof.

Theorem 3.1. *Let* h *be defined by* (2.1) *and* $t_{n+1} = G_h(t_n)$, $t_0 = 0$, $n = 0, 1, \cdots$. *Then when* $\alpha \le 3 - 2\sqrt{2}$, *we have, for* $\frac{1}{2} \le \lambda \le 1$,

$$r_1 - t_n \le \frac{\eta^{3^n}}{\sqrt{\theta} - \eta^{3^n}}(r_2 - r_1), \tag{3.13}$$

and for $0 \le \lambda \le \frac{1}{2}$,

$$r_1 - t_n \le \frac{\xi^{3^n}}{\sqrt{2(1-\lambda)\theta} - \xi^{3^n}}(r_2 - r_1), \tag{3.14}$$

where

$$\eta = \sqrt{\theta} \cdot \frac{r_1}{r_2}, \quad \xi = \sqrt{2(1-\lambda)\theta} \cdot \frac{r_1}{r_2}, \quad \theta = \frac{1-\gamma r_2}{1-\gamma r_1}.$$

Proof. Let $a_n = r_1 - t_n$ and $b_n = r_2 - t_n$. Notice $g(\gamma t) = \gamma h(t)$, $G_g(\gamma t) = \gamma G_h(t)$ and $s_i = \gamma r_i$, $i = 1, 2$. Hence, by (3.6) in Lemma 3, we have for $\frac{1}{2} \le \lambda \le 1$,

$$\frac{a_n}{b_n} \le \theta \left(\frac{a_{n-1}}{b_{n-1}}\right)^3 \le \cdots \le \theta^{1+3+\cdots+3^{n-1}} \left(\frac{a_0}{b_0}\right)^{3^n} = \frac{1}{\sqrt{\theta}} \left(\sqrt{\theta}\,\frac{r_1}{r_2}\right)^{3^n}$$

Solving this inequality for a_n, (3.13) is obtained. By (3.7) , (3.14) can also be obtained by the similar way. Thus the lemma follows.

4. Main Convergence Theorem

For $0 \le \lambda \le 1$, let

$$\begin{aligned} H_f(x_n) &= f'(x_n)^{-1} f''(x_n) f'(x_n)^{-1} f(x_n), \\ x_{n+1} &= x_n - \left[I + \tfrac{1}{2} H_f(x_n)(I - \lambda H_f(x_n))^{-1}\right] f'(x_n)^{-1} f(x_n), \\ &\qquad n = 0, 1, \cdots. \end{aligned} \tag{4.1}$$

Before the main theorem is given, we at first prove some lemmas.

Lemma 4.

$$\begin{aligned} f(x_{n+1}) &= \frac{1}{8} f''(x_n)\, z_n^2 + \frac{1-\lambda}{2} f''(x_n) f'(x_n)^{-1} f(x_n)\, z_n \\ &\quad + \int_{x_n}^{x_{n+1}} \left(\int_{x_n}^{x} f'''(y)dy\right)(x_{n+1} - x)dx, \end{aligned} \tag{4.2}$$

where

$$z_n = H_f(x_n)(1 - \lambda H_f(x_n))^{-1} f'(x_n)^{-1} f(x_n). \tag{4.3}$$

Proof. By Taylor formula and iterative relation (4.1), we have

$$\begin{aligned} f(x_{n+1}) &= f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{1}{2} f''(x_n)(x_{n+1} - x_n)^2 \\ &\quad + \int_{x_n}^{x_{n+1}} \left(\int_{x_n}^{x} f'''(y)dy\right)(x_{n+1} - x)dx \\ &= -f''(x_n) f'(x_n)^{-1} f(x_n)(I - \lambda H_f(x_n))^{-1} f'(x_n)^{-1} f(x_n) \\ &\quad + \frac{1}{2} f''(x_n)(f'(x_n)^{-1} f(x_n))^2 + \frac{1}{8} f''(x_n)\, z_n^2 \\ &\quad + \frac{1}{2} f''(x_n) f'(x_n)^{-1} f(x_n)\, z_n + \int_{x_n}^{x_{n+1}} \left(\int_{x_n}^{x} f'''(y)dy\right)(x_{n+1} - x)dx \end{aligned} \tag{4.4}$$

Since $(I - \lambda H_f(x_n))^{-1} = I + \lambda H_f(x_n)(I - \lambda H_f(x_n))^{-1}$, the lemma follows.

Lemma 5. *Assume that $f'(x_0)^{-1}$ exists and h satisfies the γ-condition about f at x_0. Then for $\lambda \in [0, 1]$ and $n \ge 0$, the sequence $\{x_n\}$ produced by* (4.1) *is well defined and converges to a zero x^* of f. Moreover, we have*

(a) $f'(x_n)^{-1}$ *exists;*
(b) $||f'(x_0)^{-1} f''(x_n)|| \le -h''(t_n)/h'(t_0)$;
(c) $||f'(x_n)^{-1} f'(x_0)|| \le -h'(t_0)/h'(t_n)$;
(d) $||f'(x_0)^{-1} f(x_n)|| \le -h(t_n)/h'(t_0)$;
(e) $(I - \lambda H_f(x_n))^{-1}$ *exists and* $||(I - \lambda H_f(x_n))^{-1}|| \le \dfrac{1}{1 - \lambda H_h(t_n)}$.

Proof. (a)-(e) are obviously true for $n = 0$. Now we inductively assume they hold until some n. Thus, by (4.1) and assumptions of induction hypotheses, we have

$$||x_{n+1} - x_n|| \le t_{n+1} - t_n \tag{4.5}$$

Hence, $||x_{n+1} - x_0|| \le t_{n+1} < r_1$. By Lemma 1, (a)-(c), (e) are valid for $n+1$. By (4.2) and (4.5), (d) is also valid for $n+1$.

From (4.5) we can see that the sequence $\{x_n\}$ is convergent to a limit x^*. Taking $n \to \infty$ in (4.1), we deduce $f(x^*) = 0$. Also from (4.5) we have

$$||x^* - x_n|| \le r_1 - t_n, \;\; n = 0, 1, \cdots. \tag{4.6}$$

This completes the lemma.

Lemma 6. *Under the assumptions of Lemma 5, we have , for* $0 \le \lambda < 1/2$

$$||x^* - x_{n+1}|| \le (r_1 - t_{n+1}) \left(\frac{||x^* - x_n||}{r_1 - t_n} \right)^2 \tag{4.7}$$

and for $1/2 \le \lambda \le 1$,

$$||x^* - x_{n+1}|| \le (r_1 - t_{n+1}) \left(\frac{||x^* - x_n||}{r_1 - t_n} \right)^3 . \tag{4.8}$$

Proof. By the same ways of [7, 11] and noting that

$$I + \frac{1}{2} H_f(x_n) Q_f(x_n) = Q_f(x_n)(I + (\frac{1}{2} - \lambda) H_f(x_n)),$$

where $Q_f(x_n) = (I - \lambda H_f(x_n))^{-1}$, we have

$$\begin{aligned} x^* - x_{n+1} &= x^* - x_n + Q_f(x_n) \left[I - \lambda H_f(x_n) + \tfrac{1}{2} H_f(x_n) \right] f'(x_n)^{-1} f(x_n) \\ &= -Q_f(x_n) f'(x_n)^{-1} \int_{x_n}^{x^*} \left(f''(x) - f''(x_n) \right) (x^* - x) dx \\ &\quad +(\tfrac{1}{2} - \lambda) Q_f(x_n) H_f(x_n) \left(x^* - x_n + f'(x_n)^{-1} f(x_n) \right) \\ &\quad -\tfrac{1}{2} Q_f(x_n) f'(x_n)^{-1} f''(x_n) \left(f'(x_n)^{-1} f(x_n)(x^* - x_n) + (x^* - x_n)^2 \right) \\ &= -Q_f(x_n) f'(x_n)^{-1} \int_{x_n}^{x^*} \left(f''(x) - f''(x_n) \right) (x^* - x) dx \\ &\quad -(\tfrac{1}{2} - \lambda) Q_f(x_n) H_f(x_n) f'(x_n)^{-1} \int_{x_n}^{x^*} f''(x)(x^* - x) dx \\ &\quad +\tfrac{1}{2} Q_f(x_n) f'(x_n)^{-1} f''(x_n) \cdot \left(f'(x_n)^{-1} \int_{x_n}^{x^*} f''(x)(x^* - x) dx \right) (x^* - x_n). \end{aligned} \tag{4.9}$$

If $1/2 \le \lambda \le 1$, noticing that $\frac{1}{2} = (\lambda - \frac{1}{2}) + (1 - \lambda)$ in (4.9), we obtain

$$\begin{aligned} x^* - x_{n+1} &= -Q_f(x_n) f'(x_n)^{-1} \int_{x_n}^{x^*} \left(f''(x) - f''(x_n) \right) (x^* - x) dx \\ &\quad -(\lambda - \tfrac{1}{2}) Q_f(x_n) f'(x_n)^{-1} f''(x_n) \left(f'(x_n)^{-1} \int_{x_n}^{x^*} f''(x)(x^* - x) dx \right)^2 \\ &\quad +(1 - \lambda) Q_f(x_n) f'(x_n)^{-1} f''(x_n) \cdot \\ &\quad \left(f'(x_n)^{-1} \int_{x_n}^{x^*} f''(x)(x^* - x) dx \right) (x^* - x_n). \end{aligned} \tag{4.10}$$

By (2.4),

$$
\begin{aligned}
&\left\| f'(x_0)^{-1} \int_{x_n}^{x^*} \left(f''(x) - f''(x_n)\right)(x^* - x)dx \right\| \\
&= \left\| \int_0^1 \int_0^1 f'(x_0)^{-1} f'''(x_n + \tau s(x^* - x_n)) s(1-s) d\tau ds (x^* - x_n)^3 \right\| \\
&\le \int_0^1 \int_0^1 h'''(||x_n - x_0|| + s\tau ||x^* - x_n||) s(1-s) d\tau ds ||x^* - x_n||^3 \\
&\le \int_0^1 \int_0^1 h'''(t_n + s\tau(r_1 - t_n))(r_1 - t_n)^3 s(1-s) d\tau ds \left(\frac{||x^* - x_n||}{r_1 - t_n}\right)^3 \\
&= \int_{t_n}^{r_1} \left(h''(s) - h''(t_n)\right)(r_1 - s) ds \left(\frac{||x^* - x_n||}{r_1 - t_n}\right)^3 .
\end{aligned} \tag{4.11}
$$

Similarly,

$$
\left\| f'(x_0)^{-1} \int_{x_n}^{x^*} f''(x)(x^* - x)dx \right\| \le \int_{t_n}^{r_1} h''(s)(r_1 - s) ds \left(\frac{||x^* - x_n||}{r_1 - t_n}\right)^2 . \tag{4.12}
$$

By combining (4.9)-(4.12), we have, for $0 \le \lambda \le 1/2$,

$$
\begin{aligned}
||x^* - x_{n+1}|| &\le \|Q_f(x_n)\| \left\| f'(x_n)^{-1} \int_{x_n}^{x^*} \left(f''(x) - f''(x_n)\right)(x^* - x)dx \right\| \\
&\quad -(\lambda - \tfrac{1}{2}) \|Q_f(x_n)\| \cdot \|H_f(x_n)\| \left\| f'(x_n)^{-1} \int_{x_n}^{x^*} f''(x)(x^* - x)dx \right\| \\
&\quad + \tfrac{1}{2} \|Q_f(x_n)\| \cdot \|f'(x_n)^{-1} f''(x_n)\| \cdot \left\| \left(f'(x_n)^{-1} \int_{x_n}^{x^*} f''(x)(x^* - x)dx \right)(x^* - x_n) \right\| .
\end{aligned} \tag{4.13}
$$

By Lemma 5 and (4.5), we obtain

$$
\begin{aligned}
||x^* - x_{n+1}|| &\le -Q_h(t_n) h'(t_n)^{-1} \int_{t_n}^{r_1} \left(h''(s) - h''(t_n)\right)(r_1 - s) ds \left(\frac{||x^* - x_n||}{r_1 - t_n}\right)^3 \\
&\quad + (\lambda - \tfrac{1}{2}) Q_h(t_n) H_h(t_n) h'(t_n)^{-1} \int_{t_n}^{r_1} h''(s)(t_1 - s) ds \left(\frac{||x^* - x_n||}{r_1 - t_n}\right)^2 \\
&\quad + \tfrac{1}{2} Q_h(t_n) h'(t_n)^{-1} h''(t_n) \cdot \left(h'(t_n)^{-1} \int_{t_n}^{r_1} h''(s)(r_1 - s) ds \right)(r_1 - t_n) \left(\frac{||x^* - x_n||}{r_1 - t_n}\right)^3 \\
&\le (r_1 - t_{n+1}) \left(\frac{||x^* - x_n||}{r_1 - t_n}\right)^2 .
\end{aligned} \tag{4.14}
$$

Similarly, for $1/2 \le \lambda \le 1$, (4.8) can also be obtained from (4.10). This finishes the proof.

Finally, by Lemmas 5 and 6, we directly have

Theorem 4.1. *Suppose $x_0 \in D$ such that $f'(x_0)^{-1}$ exists. Let h satisfy the γ-condition about f at x_0 and $\alpha \le 3 - 2\sqrt{2}$. Then the iteration (4.1) with the initial value x_0 is well defined for all $0 \le \lambda \le 1$ and sequence $\{x_n\}_{n=0}^{\infty}$ converges to a zero x^* of f. Moreover, it holds, for $0 \le \lambda \le 1/2$*

$$
||x^* - x_{n+1}|| \le (r_1 - t_{n+1}) \left(\frac{||x^* - x_n||}{r_1 - t_n}\right)^2 ,
$$

and for $1/2 \le \lambda \le 1$,

$$
||x^* - x_{n+1}|| \le (r_1 - t_{n+1}) \left(\frac{||x^* - x_n||}{r_1 - t_n}\right)^3 ,
$$

where $t_{n+1} = G_h(t_n),\ t_0 = 0.$

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