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TWO-SCALE FEM FOR ELLIPTIC MIXED BOUNDARY VALUE PROBLEMS WITH SMALL PERIODIC COEFFICIENTS*1)

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Abstract

In this paper, a dual approximate expression of the exact solution for mixed boundary value problems of second order elliptic PDE with small periodic coefficients is proposed. Meanwhile the error estimate of the dual approximate solution is discussed. Finally, a high-low order coupled two-scale finite element method is given, and its approximate error is analysed.

Key words: Two-scale FEM, Mixed boundary value, Small periodic coefficients.

1. Introduction

Composite materials have been widely used in high technology engineering as well as ordinary industrial products since they have many elegant qualities, such as high strength, high stiffness, high temperature resistance, corrosion resistance, and fatigue resistance. Most of the composite materials have small periodic configurations. Thus the static analysis of the structures of composite materials usually leads to the boundary value problems of elliptic partial differential equations with small periodic coefficients. Solving these problems by classical finite element methods is difficult because it usually requires very fine meshes and this lead to tremendous amount of computer memory and CPU time.

In order to solve this kind problem, many authors have proposed some useful methods, such as homogenization method, upscaling method, multiscale method, and so on (see [3] [5] [6] [7] [9] [10] [11] [12], and references therein). The two-scale method couples macroscopic scale and microscopic scale together, it not only reflects global mechanical and physical properties of structure, but also the effect of micro-configuration of composite material. Using this method, we can solve elliptic boundary value problems with small periodic coefficients by solving a homogenization problem with coarse meshes in whole domain and a periodic problem with fine meshes only in one small periodic subdomain.

However, most of authors only discussed pure Dirichlet boundary value problems. For mixed boundary value problems there are some difficulties. The main difficulty is that one can not obtain an asymptotic expression of the exact solution. It is well-known that mixed boundary value problems are very important in composite materials and applied mechanics. So we should study solving methods for the mixed boundary value problems arising from composite materials and the structures with small periodic configurations.

In this paper, we consider the mixed boundary value problem of second order elliptic equations with small periodic coefficients. Instead of asymptotic expression, we propose a dual

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approximate expression of the exact solution. The error of dual approximate solution is given. In order to couple macroscopic and microscopic analysis, we give a high-low order coupled two-scale finite element method, i.e., using high order finite elements with coarse grid size to approximate macroscopic problem in whole domain, while using low order finite elements with fine grid size to approximate microscopic problem in one small periodic subdomain. Based on this idea, the error of finite element approximate solution is obtained.

The remainder of this paper is outlined as follows: In §2 we introduce the problem and some notations. In §3 a dual approximate expression of the exact solution and its error are presented. In §4 a high-low order coupled finite element method is proposed and its error estimate is discussed.

In this paper, C (with or without subscripts) will denote a generic positive constant with possibly different values in different contexts. For any domain D, we use Sobolev space $W_p^m(D)$ with Sobolev norm $\|\cdot\|_{W_p^m(D)}$ and seminorm $|\cdot|_{W_p^m(D)}$ (see [1]). If $D=\Omega$, we omit D. Moreover if $D=\Omega$ and $p=2$, we denote the usual L^2 inner product by $(\cdot,\cdot)$, the Sobolev norm by $\|\cdot\|_m$ and seminorm by $|\cdot|_m$. Also we use Einstein summation notation, i.e., repeated index indicates to sum.

2. Preliminaries

Assume bounded domain $\Omega\subset\mathcal{R}^2$ to consist of entirely basic configurations, i.e., $\bar\Omega=\sum\limits_{z\in I^\epsilon}\epsilon(z+\bar Q)$, where and hereafter ϵ is a small positive number,

$$I^\epsilon=\{z\in\mathcal{Z}^2 \mid \epsilon(z+Q)\subset\Omega\},$$
$$Q=\{y \mid 0<y_i<1, i=1,2\}.$$

Consider the following problem

$$\begin{cases} L^\epsilon u^\epsilon\equiv-\nabla\cdot(a^\epsilon\nabla u^\epsilon)=f, & \text{in} \quad \Omega,\\ u^\epsilon=0, & \text{on} \quad \Gamma_1,\\ a^\epsilon\nabla u^\epsilon\cdot n=g, & \text{on} \quad \Gamma_2, \end{cases} \tag{2.1}$$

where $a^\epsilon=(a_{ij}^\epsilon(x))$ is a bounded symmetric positive definite matrix of the functions with small period ϵ, f and g are sufficiently smooth functions, $n=(n_1,n_2)$ is a unit outer normal vector, $\Gamma_1=\partial\Omega\backslash\Gamma_2\neq\emptyset$, Γ_2 is defined by

$$\Gamma_2=\bigcup_{z\in I_2^\epsilon}(\partial\epsilon(z+Q)\cap\partial\Omega), \quad I_2^\epsilon\subset I^\epsilon.$$

Let $y=\frac{x}{\epsilon}$ and $a=(a_{ij}(y))=(a_{ij}^\epsilon(x))$, then a_{ij} is a periodic function with period 1. Assume $a_{ij}(y)$ to be one order smooth. First we introduce periodic function $N_k(y)$ which is the solution of the following equation

$$\begin{cases} -\frac{\partial}{\partial y_i}(a_{ij}\frac{\partial N_k}{\partial y_j})=\frac{\partial}{\partial y_i}a_{ik}, & \text{in} \quad Q,\\ N_k=0, & \text{on} \quad \partial Q. \end{cases} \tag{2.2}$$

Then we define a constant matrix $a^0=(a_{ij}^0)$ by

$$a_{ij}^0=\int_Q(a_{ij}+a_{ik}\frac{\partial N_j}{\partial y_k})dy.$$

Also we define function $u^0(x)$ to be the solution of the following equation

$$\begin{cases} -\nabla\cdot(a^0\nabla u^0)=f, & \text{in} \quad \Omega,\\ u^0=0, & \text{on} \quad \Gamma_1,\\ a^0\nabla u^0\cdot n=g, & \text{on} \quad \Gamma_2. \end{cases} \tag{2.3}$$

Finally we introduce the notation g_{ij} defined by

$$g_{ij} = a_{ij} + a_{ik}\frac{\partial N_j}{\partial y_k} - a^0_{ij}.$$

From the definition of L^ϵ we know

$$\begin{aligned} L^\epsilon &= -(\frac{\partial}{\partial x_i} + \epsilon^{-1}\frac{\partial}{\partial y_i})a_{ij}(y)(\frac{\partial}{\partial x_j} + \epsilon^{-1}\frac{\partial}{\partial y_j}) \\ &= \epsilon^{-2}L_1 + \epsilon^{-1}L_2 + L_3, \end{aligned} \tag{2.4}$$

where variables x and y are independent and

$$\begin{aligned} L_1 &= -\frac{\partial}{\partial y_i}(a_{ij}\frac{\partial}{\partial y_j}), \\ L_2 &= -\frac{\partial}{\partial y_i}(a_{ij}\frac{\partial}{\partial x_j}) - \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial y_j}), \\ L_3 &= -a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \end{aligned}$$

3. Dual Approximate Error Estimate

Define dual approximate expression of the solution u^ϵ of problem (2.1) by

$$u^\epsilon_1 = u^0 + \epsilon u^1 \equiv u^0 + \epsilon N_k \frac{\partial u^0}{\partial x_k}, \tag{3.1}$$

where u^0 and N_k are defined by (2.3) and (2.2) respectively. In this section, we discuss the error $\|u^\epsilon - u^\epsilon_1\|_1$. To this end in view, we first derive the equation satisfied by $u^\epsilon - u^\epsilon_1$.

Lemma 3.1. *Assume $u^0 \in H^3$, then $u^\epsilon - u^\epsilon_1$ is the solution of the following mixed boundary value problem*

$$\begin{cases} L^\epsilon(u^\epsilon - u^\epsilon_1) = -\nabla\cdot(a^\epsilon \nabla(u^\epsilon - u^\epsilon_1)) = f^*, & \text{in} \quad \Omega, \\ u^\epsilon - u^\epsilon_1 = 0, & \text{on} \quad \Gamma_1, \\ a^\epsilon \nabla(u^\epsilon - u^\epsilon_1)\cdot n = g^*, & \text{on} \quad \Gamma_2, \end{cases}$$

where

$$\begin{aligned} f^* &= g_{ij}\frac{\partial^2 u^0}{\partial x_i \partial x_j} + \frac{\partial}{\partial y_i}(a_{ij}N_k)\frac{\partial^2 u^0}{\partial x_j \partial x_k} + \epsilon a_{ij}N_k\frac{\partial^3 u^0}{\partial x_i \partial x_j \partial x_k}, \\ g^* &= -g_{ij}\frac{\partial u^0}{\partial x_j}n_i. \end{aligned}$$

Proof. From (2.4) and (3.1) it follows that

$$\begin{aligned} L^\epsilon u^\epsilon_1 &= (\epsilon^{-2}L_1 + \epsilon^{-1}L_2 + L_3)(u^0 + \epsilon u^1) \\ &= \epsilon^{-2}L_1u^0 + \epsilon^{-1}(L_1u^1 + L_2u^0) + (L_2u^1 + L_3u^0) + \epsilon L_3 u^1. \end{aligned} \tag{3.2}$$

Since variables x and y are independent and N_k satisfies (2.2), we have

$$L_1u^0 = -\frac{\partial}{\partial y_i}(a_{ij}\frac{\partial u^0}{\partial y_j}) = 0, \tag{3.3}$$

$$\begin{aligned} L_1u^1 + L_2u^0 &= -\frac{\partial}{\partial y_i}(a_{ij}\frac{\partial N_k}{\partial y_j}\frac{\partial u^0}{\partial x_k}) - \frac{\partial}{\partial y_i}(a_{ij}\frac{\partial u^0}{\partial x_j}) - \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u^0}{\partial y_j}) \\ &= -\{\frac{\partial}{\partial y_i}(a_{ij}\frac{\partial N_k}{\partial y_j}) + \frac{\partial}{\partial y_i}(a_{ik})\}\frac{\partial u^0}{\partial x_k} \\ &= 0. \end{aligned} \tag{3.4}$$

Using (3.2)-(3.4) yields

$$\begin{aligned} L^\epsilon u_1^\epsilon &= L_2 u^1 + L_3 u^0 + \epsilon L_3 u^1 \\ &= -\frac{\partial}{\partial y_i}(a_{ij} N_k \frac{\partial^2 u^0}{\partial x_j \partial x_k}) - \frac{\partial}{\partial x_i}(a_{ij} \frac{\partial N_k}{\partial y_j}\frac{\partial u^0}{\partial x_k}) \\ &\quad - a_{ij}\frac{\partial^2 u^0}{\partial x_i \partial x_j} - \epsilon a_{ij} N_k \frac{\partial^3 u^0}{\partial x_i \partial x_j \partial x_k}. \end{aligned}$$

By equality

$$L^\epsilon u^\epsilon = f = -\frac{\partial}{\partial x_i}(a_{ij}^0 \frac{\partial u^0}{\partial x_j}),$$

we deduce that

$$\begin{aligned} L^\epsilon(u^\epsilon - u_1^\epsilon) &= \{a_{ij} + a_{ik}\frac{\partial N_j}{\partial y_k} - a_{ij}^0\}\frac{\partial^2 u^0}{\partial x_i \partial x_j} \\ &\quad + \frac{\partial}{\partial y_i}(a_{ij} N_k)\frac{\partial^2 u^0}{\partial x_j \partial x_k} + \epsilon a_{ij} N_k \frac{\partial^3 u^0}{\partial x_i \partial x_j \partial x_k} \\ &= f^*, \quad \text{in} \quad \Omega. \end{aligned}$$

From (2.1)-(2.3) it follows that

$$u^\epsilon - u_1^\epsilon = u^\epsilon - u^0 - \epsilon N_k \frac{\partial u^0}{\partial x_k} = 0, \quad \text{on} \quad \Gamma_1.$$

Finally noting

$$\begin{aligned} a^\epsilon \bigtriangledown u_1^\epsilon \cdot n &= a_{ij}(\frac{\partial}{\partial x_j} + \epsilon^{-1}\frac{\partial}{\partial y_j})(u^0 + \epsilon N_k \frac{\partial u^0}{\partial x_k}) \cdot n_i \\ &= a_{ij}(\frac{\partial u^0}{\partial x_j} + \epsilon N_k \frac{\partial^2 u^0}{\partial x_j \partial x_k} + \epsilon^{-1}\frac{\partial u^0}{\partial y_j} + \frac{\partial N_k}{\partial y_j}\frac{\partial u^0}{\partial x_k}) \cdot n_i \\ &= (a_{ij} + a_{ik}\frac{\partial N_j}{\partial y_k})\frac{\partial u^0}{\partial x_j} n_i + \epsilon a_{ij} N_k \frac{\partial^2 u^0}{\partial x_j \partial x_k} n_i \\ &= a_{ij}^0 \frac{\partial u^0}{\partial x_j} n_i + g_{ij}\frac{\partial u^0}{\partial x_j} n_i + \epsilon a_{ij} N_k \frac{\partial^2 u^0}{\partial x_j \partial x_k} n_i, \end{aligned}$$

combining this equality with (2.1)-(2.3) we obtain

$$a^\epsilon \bigtriangledown (u^\epsilon - u_1^\epsilon) \cdot n = -g_{ij}\frac{\partial u^0}{\partial x_j} n_i = g^*, \quad \text{on} \quad \Gamma_2.$$

Thus the proof is completed.

Let $N_k^\epsilon(x) = N_k(\frac{x}{\epsilon})$, we have the following estimate about $N_k^\epsilon(x)$.

Lemma 3.2. $\|N_k^\epsilon\|_0 \le C_1 \epsilon |N_k^\epsilon|_1 \le C_2$ *and* $|N_k^\epsilon|_2 \le C\epsilon^{-2}$.

Proof. On the one hand, since $N_k \in H_0^1(Q)$, by Poincare inequality we have

$$\|N_k\|_{L^2(Q)} \le C|N_k|_{H^1(Q)}.$$

From $x = \epsilon y$ it follows that

$$\|N_k^\epsilon\|_{L^2(\epsilon Q)} \le C\epsilon |N_k^\epsilon|_{H^1(\epsilon Q)}.$$

Summing over all inequalities for $\epsilon(z + Q)$ with respect to $z \in I^\epsilon$ yields

$$\|N_k^\epsilon\|_0 \le C\epsilon |N_k^\epsilon|_1.$$

On the other hand, N_k^ϵ satisfies the following equation

$$\begin{cases} -\frac{\partial}{\partial x_i}(a_{ij}^\epsilon \frac{\partial N_k^\epsilon}{\partial x_j}) = \epsilon^{-1}\frac{\partial}{\partial x_i}(a_{ik}^\epsilon), & \text{in} \quad \epsilon Q, \\ N_k^\epsilon = 0, & \text{on} \quad \partial(\epsilon Q). \end{cases}$$

Then N_k^ϵ satisfies the following variational equation

$$(a_{ij}^\epsilon \frac{\partial N_k^\epsilon}{\partial x_j}, \frac{\partial v}{\partial x_i})_{\epsilon Q} = -\epsilon^{-1}(a_{ik}^\epsilon, \frac{\partial v}{\partial x_i})_{\epsilon Q}, \quad \forall v(x) \in H_0^1(\epsilon Q).$$

Taking $v = N_k^\epsilon$ in the above variational equation we obtain

$$\begin{aligned} C_1|N_k^\epsilon|^2_{H^1(\epsilon Q)} &\leq (a_{ij}^\epsilon \frac{\partial N_k^\epsilon}{\partial x_j}, \frac{\partial N_k^\epsilon}{\partial x_i})_{\epsilon Q} \\ &= -\epsilon^{-1}(a_{ik}^\epsilon, \frac{\partial N_k^\epsilon}{\partial x_i})_{\epsilon Q} \\ &\leq \epsilon^{-1}(\sum_{i=1}^{2}\|a_{ik}^\epsilon\|^2_{L^2(\epsilon Q)})^{1/2}|N_k^\epsilon|_{H^1(\epsilon Q)}. \end{aligned}$$

Summing over all inequalities for $\epsilon(z+Q)$ with respect to $z \in I^\epsilon$ yields

$$|N_k^\epsilon|_1 \leq C\epsilon^{-1}.$$

Finally, since $a_{ij}(y)$ is one order smooth, problem (2.2) has H^2 regularity (see [8]), i.e.,

$$\|N_k\|_{H^2(Q)} \leq C_1\|\frac{\partial}{\partial y_i}a_{ik}\|_{L^2(Q)} \leq C,$$

which follows that

$$|N_k^\epsilon|^2_{H^2(\epsilon Q)} \leq C\epsilon^{-2}.$$

Summing over all inequalities for $\epsilon(z+Q)$ with respect to $z \in I^\epsilon$, we obtain

$$|N_k^\epsilon|_2 \leq C\epsilon^{-2}.$$

Let $g_{ij}^\epsilon(x) = g_{ij}(\frac{x}{\epsilon})$, we have the following result.

Lemma 3.3. $\|g_{ij}^\epsilon\|_0 \leq C_1$ *and* $|g_{ij}^\epsilon|_1 \leq C_2\epsilon^{-1}$.

Proof. From the definition of g_{ij} we know

$$g_{ij}^\epsilon = a_{ij}^\epsilon + \epsilon a_{ik}^\epsilon \frac{\partial N_j^\epsilon}{\partial x_k} - a_{ij}^0.$$

By Lemma 3.2,

$$\begin{aligned} \|g_{ij}^\epsilon\|_0 &\leq \|a_{ij}^\epsilon\|_0 + \epsilon\|a_{ik}^\epsilon\|_{L^\infty}\|\frac{\partial N_j^\epsilon}{\partial x_k}\|_0 + C\|a_{ij}^0\|_{L^\infty} \\ &\leq C_1. \end{aligned}$$

Using the assumption on a_{ij} and Lemma 3.2 yields

$$\begin{aligned} |g_{ij}^\epsilon|_1 &\leq |a_{ij}^\epsilon|_1 + \epsilon(|a_{ik}^\epsilon|_{W^1_\infty}\|\frac{\partial N_j^\epsilon}{\partial x_k}\|_0 + |a_{ik}^\epsilon|_{L^\infty}|\frac{\partial N_j^\epsilon}{\partial x_k}|_1) \\ &\leq C_1\epsilon^{-1} + C_2\epsilon(\epsilon^{-1}|N_j^\epsilon|_1 + |N_j^\epsilon|_2) \\ &\leq C\epsilon^{-1}. \end{aligned}$$

Let $l_t^j = \{y \mid y_j = t, 0 < y_i < 1, i \neq j\}$, we also have the following result about g_{ij}.

Lemma 3.4. *Function g_{ij} satisfies the following relations*

$$\int_Q g_{ij} dy = 0, \quad i,j = 1,2; \tag{3.5}$$

$$\frac{\partial}{\partial y_i} g_{ik} = 0, \quad k = 1,2; \tag{3.6}$$

$$\int_{l_t^j} g_{jk} ds = 0, \quad \forall t \in \mathcal{R}, \quad j,k = 1,2. \tag{3.7}$$

Here in (3.7) there is no summation over j.

Proof. Relations (3.5)-(3.6) follow directly from the definitions of a_{ij}^0 and N_j. Next we prove (3.7). To this end, denote by $Q_{t_1 t}^j$ the set

$$Q_{t_1 t}^j = \{y \mid t_1 < y_j < t, 0 < y_i < 1, i \neq j\}.$$

Multiplying (3.6) by $y_j - t_1$ and integrating over $Q_{t_1 t}^j$ yields

$$\begin{aligned} 0 &= \int_{Q_{t_1 t}^j} \frac{\partial}{\partial y_i} g_{ik}(y_j - t_1) dy \\ &= \int_{\partial Q_{t_1 t}^j} g_{ik}(y_j - t_1) n_i ds - \int_{Q_{t_1 t}^j} g_{jk} dy. \end{aligned}$$

Since g_{ik} is a periodic function with period 1, we derive that

$$\int_{l_t^j} g_{jk}(t - t_1) ds = \int_{Q_{t_1 t}^j} g_{jk} dy.$$

Choosing $t_1 = t - 1$ and takig into account (3.5) we obtain (3.7).

Lemma 3.5. *For any $v(x) \in H^1$, the following inequalities hold*

$$|\int_{\Gamma_2} g_{ij} n_i v ds| \leq C\epsilon^{1/2} |v|_1, \quad j = 1,2,$$

$$|\int_\Omega g_{ij} v dx| \leq C\epsilon |v|_1.$$

Proof. From the definition of Γ_2 it follows that Γ_2 consists of 1-dimensional surfaces of the squares $\epsilon(z + Q)$ for $z \in I_2^\epsilon$ which lie on $\partial\Omega$. Denote by $\sigma_j^1, \cdots, \sigma_j^{l_j}$ the 1-dimensional faces of the squares $\epsilon(z + Q)$ for $z \in I_2^\epsilon$ such that σ_j^k is parallel to $x_j = 0$ and lies on Γ_2, $j = 1,2$. Thus

$$\Gamma_2 = \bigcup_{j=1}^{2} \bigcup_{k=1}^{l_j} \sigma_j^k.$$

Since the component n_i of unit outer normal vector on σ_j^k equals zero for $i \neq j$ while $|n_j| = 1$, from (3.7) we get

$$\int_{\sigma_j^k} g_{il} n_i ds = \int_{\sigma_j^k} g_{jl} n_j ds = 0, \tag{3.8}$$

here there is no summation over i and j.

Denote by q_j^k the square $\epsilon(z + Q)$ whose surface contains σ_j^k. Define operator $P|_{q_j^k} = P_j^k$, where P_j^k is defined by

$$P_j^k v = \frac{1}{|q_j^k|} \int_{q_j^k} v dx, \quad k = 1,2,\cdots,l_j; \quad j = 1,2.$$

Using scaling argument, trace inequality, and Bramble-Hilbert Lemma, we can prove

$$\|v - P_j^k v\|_{L^2(\sigma_j^k)} \leq C\epsilon^{1/2}|v|_{H^1(q_j^k)}, \quad k = 1, 2, \cdots, l_j; \quad j = 1, 2.$$

By the definition of P, above inequality, equality (3.8), and Lemma 3.3, noting that g_{ij} is a periodic function, we deduce that

$$\begin{aligned} |\int_{\Gamma_2} g_{il} n_i v ds|^2 &= |\int_{\Gamma_2} (g_{il} - Pg_{il}) n_i (v - Pv) ds|^2 \\ &\leq \sum_{i=1}^{2} \|g_{il} - Pg_{il}\|^2_{L^2(\Gamma_2)} \|v - Pv\|^2_{L^2(\Gamma_2)} \\ &\leq C\epsilon^2 \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{l_j} |g_{il}|^2_{H^1(q_j^k)} |v|^2_{H^1(\Omega)} \\ &\leq C\epsilon^3 \sum_{i=1}^{2} |g_{il}|^2_{H^1(\Omega)} |v|^2_{H^1(\Omega)} \\ &\leq C\epsilon |v|_1^2. \end{aligned}$$

Thus we obtain the first inequality. In order to prove the second inequality, define local L^2-projection operator Π by

$$\Pi|_{\epsilon(z+Q)} = \Pi_z, \quad \Pi_z v = \frac{1}{|\epsilon(z+Q)|}\int_{\epsilon(z+Q)} v dx, \quad \forall z \in I^\epsilon.$$

By scaling argument and Bramble-Hilbert Lemma

$$\|v - \Pi_z v\|^2_{L^2(\epsilon(z+Q))} \leq C\epsilon^2 |v|^2_{H^1(\epsilon(z+Q))}, \quad \forall v \in H^1, \quad \forall z \in I^\epsilon.$$

Summing over all $z \in I^\epsilon$ yields

$$\|v - \Pi v\|_0 \leq C\epsilon |v|_1, \quad \forall v \in H^1.$$

Using (3.5), the definition of Π, and Lemma 3.3, we obtain

$$\begin{aligned} |\int_\Omega g_{ij} v dx| &= |\int_\Omega (g_{ij} - \Pi g_{ij})(v - \Pi v) dx| \\ &\leq \|g_{ij} - \Pi g_{ij}\|_0 \|v - \Pi v\|_0 \\ &\leq C\epsilon^2 |g_{ij}|_1 |v|_1 \\ &\leq C\epsilon |v|_1. \end{aligned}$$

Hence the proof is completed.

Now we prove the main result of this section.

Theorem 3.6. *Assume $u^0 \in W^3_\infty$, then the following inequality holds*

$$|u^\epsilon - u_1^\epsilon|_1 \leq C(\epsilon\|u^0\|_{W^3_\infty} + \epsilon^{1/2}\|u^0\|_{W^2_\infty}). \tag{3.9}$$

Proof. From Lemma 3.1 we know $u^\epsilon - u_1^\epsilon$ satisfies the following variational equation

$$(a^\epsilon \nabla (u^\epsilon - u_1^\epsilon), \nabla v) = (f^*, v) + \int_{\Gamma_2} g^* v ds, \quad \forall v \in H^1_{0,\Gamma_1}(\Omega), \tag{3.10}$$

where and hereafter $H^1_{0,\Gamma_1}(\Omega) = \{v \in H^1 \ | \ v|_{\Gamma_1} = 0\}$.

In (3.10), taking $v = u^\epsilon - u_1^\epsilon$, using integration by parts yields

$$\begin{aligned} C|u^\epsilon - u_1^\epsilon|_1^2 &\leq (a^\epsilon \nabla (u^\epsilon - u_1^\epsilon), \nabla(u^\epsilon - u_1^\epsilon)) \\ &\leq |\int_\Omega f^*(u^\epsilon - u_1^\epsilon) dx| + |\int_{\Gamma_2} g^*(u^\epsilon - u_1^\epsilon) ds| \end{aligned}$$

$$
\begin{aligned}
&\leq \quad |\int_\Omega g_{ij}\frac{\partial^2 u^0}{\partial x_i \partial x_j}(u^\epsilon - u_1^\epsilon)dx| \\
&\quad +\epsilon|\int_\Omega a_{ij}N_k\frac{\partial}{\partial x_i}(\frac{\partial^2 u^0}{\partial x_j \partial x_k}(u^\epsilon - u_1^\epsilon))dx| \\
&\quad +\epsilon|\int_\Omega a_{ij}N_k\frac{\partial^3 u^0}{\partial x_i \partial x_j \partial x_k}(u^\epsilon - u_1^\epsilon)dx| \\
&\quad +|\int_{\Gamma_2} g_{ij}\frac{\partial u^0}{\partial x_j}n_i(u^\epsilon - u_1^\epsilon)ds|.
\end{aligned}
$$

By Poincare inequality, Lemma 3.2 and Lemma 3.5 we obtain

$$
\begin{aligned}
|u^\epsilon - u_1^\epsilon|_1^2 &\leq C_1\epsilon|u^0|_{W^3_\infty}\|u^\epsilon - u_1^\epsilon\|_0 + C_2\epsilon|u^0|_{W^2_\infty}|u^\epsilon - u_1^\epsilon|_1 \\
&\quad +C_3\epsilon^{1/2}|u^0|_{W^2_\infty}\|u^\epsilon - u_1^\epsilon\|_0 + C_4\epsilon^{1/2}|u^0|_{W^1_\infty}|u^\epsilon - u_1^\epsilon|_1 \\
&\leq C(\epsilon\|u^0\|_{W^3_\infty} + \epsilon^{1/2}\|u^0\|_{W^2_\infty})|u^\epsilon - u_1^\epsilon|_1.
\end{aligned}
$$

From this it follows that

$$
|u^\epsilon - u_1^\epsilon|_1 \leq C(\epsilon\|u^0\|_{W^3_\infty} + \epsilon^{1/2}\|u^0\|_{W^2_\infty}).
$$

4. High-low Order Coupled FEM

According to the expression of u_1^ϵ, in order to obtain numerical solution of equation (2.1), we only need to compute u^0 and N_k by finite element method. u^0 is computed in whole domain Ω, while N_k is computed only in one periodic subdomain. The grid size in computing u^0 is generally larger than that in computing N_k. To match numerical approximate errors of u^0 and N_k, one can use high order finite element method to compute u^0, but low order finite element method to compute N_k. Here we use quadratic Lagrange finite element method to compute u^0, while linear finite element method to compute N_k.

First we consider the finite element approximation of periodic solution N_k. The weak form of problem (2.2) is to find $N_k \in H_0^1(Q)$ such that

$$
(a_{ij}\frac{\partial N_k}{\partial y_j}, \frac{\partial v}{\partial y_i})_Q = -(a_{ik}, \frac{\partial v}{\partial y_i})_Q, \quad \forall v \in H_0^1(Q). \tag{4.1}
$$

Let $\mathcal{T}_h$ be quasi-uniform partition of Q with grid size h, $V_h \subset H_0^1(Q)$ be linear conforming finite element space defined on $\mathcal{T}_h$. The finite element approximation of problem (4.1) is to find $N_k^h \in V_h$ such that

$$
(a_{ij}\frac{\partial N_k^h}{\partial y_j}, \frac{\partial v}{\partial y_i})_Q = -(a_{ik}, \frac{\partial v}{\partial y_i})_Q, \quad \forall v \in V_h. \tag{4.2}
$$

Using standard argument, we can obtain the following result.

Lemma 4.1. *Assume N_k and N_k^h to be the solutions of problems (4.1) and (4.2) respectively, $N_k \in W^2_\infty(Q)$, then we have*

$$
\begin{aligned}
&\|N_k - N_k^h\|_{L^2(Q)} + h|N_k - N_k^h|_{H^1(Q)} \leq Ch^2|N_k|_{H^2(Q)}, \\
&\|N_k - N_k^h\|_{L^\infty(Q)} + h|N_k - N_k^h|_{W^1_\infty(Q)} \leq Ch^2|\ln h||N_k|_{W^2_\infty(Q)}.
\end{aligned}
$$

Next we consider finite element approximation of the homogenization solution u^0. The weak form of problem (2.3) is to find $u^0 \in H^1_{0,\Gamma_1}(\Omega)$ such that

$$
(a^0 \bigtriangledown u^0, \bigtriangledown v) = (f, v) + \int_{\Gamma_2} gvds, \quad \forall v \in H^1_{0,\Gamma_1}(\Omega). \tag{4.3}
$$

According to the definition of a^0, a^0 can be computed by a and N_j. Unfortunately, N_j is unknown. But we can use N_j^h to approximate N_j. Based on this idea, approximate a^0 is defined as follows.

$$a_h^0 = (a_{ij}^{0,h}), \quad a_{ij}^{0,h} = \int_Q (a_{ij} + a_{ik}\frac{\partial N_j^h}{\partial y_k})dy.$$

Lemma 4.2. *If $N_j \in W_\infty^1(Q)$, then the matrix a^0 is bounded symmetric positive definite, moreover if $N_j \in H^2(Q)$, then for sufficiently small h, a_h^0 is also bounded symmetric positive definite.*

Proof. First we show a^0 is symmetric. In (4.1) taking $v = N_l$, we obtain

$$\int_Q (\frac{\partial N_l}{\partial y_i} a_{ij} \frac{\partial N_k}{\partial y_j} + \frac{\partial N_l}{\partial y_i} a_{ik})dy = 0.$$

From this equality, it follows that

$$\begin{aligned} I_{lk} &\equiv \int_Q \frac{\partial (N_l + y_l)}{\partial y_i} a_{ij} \frac{\partial (N_k + y_k)}{\partial y_j} dy \\ &= \int_Q (\frac{\partial N_l}{\partial y_i} a_{ij} \frac{\partial N_k}{\partial y_j} + \frac{\partial N_l}{\partial y_i} a_{ij}\delta_{kj} + \delta_{li} a_{ij} \frac{\partial N_k}{\partial y_j} + \delta_{li} a_{ij} \delta_{kj})dy \\ &= \int_Q (\frac{\partial N_l}{\partial y_i} a_{ij} \frac{\partial N_k}{\partial y_j} + \frac{\partial N_l}{\partial y_i} a_{ik} + a_{lj} \frac{\partial N_k}{\partial y_j} + a_{lk})dy \\ &= \int_Q (a_{lk} + a_{lj}\frac{\partial N_k}{\partial y_j})dy \\ &= a_{lk}^0, \end{aligned}$$

where δ_{kj} and δ_{li} are Kronecker symbols. This implies

$$a_{kl}^0 = I_{kl} = I_{lk} = a_{lk}^0,$$

i.e., a^0 is symmetric. Similarly we can show a_h^0 is symmetric.

Next we prove that there exist positive constants η_i $(1 \le i \le 4)$, such that

$$\eta_1(\xi_1^2 + \xi_2^2) \le \xi_i a_{ij}^0 \xi_j \le \eta_2(\xi_1^2 + \xi_2^2), \tag{4.4}$$

$$\eta_3(\xi_1^2 + \xi_2^2) \le \xi_i a_{ij}^{0,h} \xi_j \le \eta_4(\xi_1^2 + \xi_2^2). \tag{4.5}$$

Noting

$$\begin{aligned} \xi_i a_{ij}^0 \xi_j &= \int_Q \frac{\partial}{\partial y_k}[\xi_i(N_i + y_i)] a_{kl} \frac{\partial}{\partial y_l}[\xi_j(N_j + y_j)]dy \\ &\ge C \sum_{k=1}^{2} \| \frac{\partial}{\partial y_k}[\xi_i(N_i + y_i)]\|_{L^2(Q)}^2, \end{aligned}$$

if $\xi_i a_{ij}^0 \xi_j = 0$, then we have

$$\xi_i(N_i + y_i) = \text{constant}.$$

Since N_i is a periodic function, we must have $\xi_i = 0$, i.e., a^0 is positive definite.

Using the hypothesis $N_j \in W_\infty^1(Q)$, we obtain

$$\begin{aligned} \xi_i a_{ij}^0 \xi_j &= \int_Q (\xi_i a_{ij} \xi_j + \xi_i a_{ik} \frac{\partial N_j}{\partial y_k} \xi_j)dy \\ &\le C(\xi_1^2 + \xi_2^2) \max_{1 \le j \le 2} |N_j|_{W_\infty^1(Q)} \\ &\le C(\xi_1^2 + \xi_2^2). \end{aligned}$$

Thus (4.4) holds. To prove (4.5), taking $v = N_l^h - N_l$ in (4.1) yields

$$\begin{aligned}(a_{ij}\frac{\partial N_k}{\partial y_j}, \frac{\partial (N_l^h - N_l)}{\partial y_i})_Q &= -(a_{ik}, \frac{\partial (N_l^h - N_l)}{\partial y_i})_Q \\ &= a_{kl}^0 - a_{kl}^{0,h}. \end{aligned}\tag{4.6}$$

Subtracting (4.1) from (4.2) and taking $v = N_l^h$, we obtain

$$(a_{ij}\frac{\partial (N_k^h - N_k)}{\partial y_j}, \frac{\partial N_l^h}{\partial y_i})_Q = 0. \tag{4.7}$$

From (4.6) and (4.7) it follows that

$$\begin{aligned}|a_{ij}^{0,h} - a_{ij}^0| &= |\int_Q \frac{\partial (N_i^h - N_i)}{\partial y_k} a_{kl} \frac{\partial (N_j^h - N_j)}{\partial y_l} dy| \\ &\leq C|N_i^h - N_i|_{H^1(Q)} |N_j^h - N_j|_{H^1(Q)} \\ &\leq Ch^2 |N_i|_{H^2(Q)} |N_j|_{H^2(Q)}. \end{aligned}\tag{4.8}$$

For sufficiently small h, from (4.4) and (4.8), we know (4.5) holds.

Substituting a^0 by a_h^0 in problm (4.3), we obtain the following problem: find $\bar{u}^0 \in H^1_{0,\Gamma_1}(\Omega)$ such that

$$(a_h^0 \bigtriangledown \bar{u}^0, \bigtriangledown v) = (f, v) + \int_{\Gamma_2} gv ds, \quad \forall v \in H^1_{0,\Gamma_1}(\Omega). \tag{4.9}$$

From Lemma 4.2 we know that problem (4.3) or (4.9) has a unique solution.

Let $\mathcal{T}_{h_0}$ be quasi-uniform partition of Ω with grid size h_0, $V_{h_0} \subset H^1_{0,\Gamma_1}(\Omega)$ be usual quadratic Lagrange finite element space defined on $\mathcal{T}_{h_0}$. The finite element approximation of problem (4.9) is to find $\bar{u}^0_{h_0} \in V_{h_0}$ such that

$$(a_h^0 \bigtriangledown \bar{u}^0_{h_0}, \bigtriangledown v) = (f, v) + \int_{\Gamma_2} gv ds, \quad \forall v \in V_{h_0}. \tag{4.10}$$

Obviously, (4.10) has a unique solution. Using standard argument, we can show the following result.

Lemma 4.3. *Assume $\bar{u}^0$ and $\bar{u}^0_{h_0}$ to be the solutions of problems (4.9) and (4.10) respectively, $\bar{u}^0 \in H^3$, then we have*

$$\|\bar{u}^0 - \bar{u}^0_{h_0}\|_0 + h_0 |\bar{u}^0 - \bar{u}^0_{h_0}|_1 \leq Ch_0^3 |\bar{u}^0|_3.$$

Now we estimate $\|u^0 - \bar{u}^0_{h_0}\|_1$.

Lemma 4.4. *Assume u^0 and $\bar{u}^0_{h_0}$ to be the solutions of problems (4.3) and (4.10) respectively, $\bar{u}^0 \in H^3$ and $N_k \in H^2(Q)$ to be the solutions of problems (4.9) and (4.1) respectively, then we have*

$$\|u^0 - \bar{u}^0_{h_0}\|_1 \leq C(h_0^2 |\bar{u}^0|_3 + h^2 |\bar{u}^0|_1).$$

Proof. From (4.3) and (4.9) it follows that

$$(a^0 \bigtriangledown (u^0 - \bar{u}^0), \bigtriangledown v) = ((a_h^0 - a^0) \bigtriangledown \bar{u}^0, \bigtriangledown v), \quad \forall v \in H^1_{0,\Gamma_1}(\Omega).$$

Taking $v = u^0 - \bar{u}^0$ in the above inequality, using Lemma 4.2, inequality (4.8), and H^2 regularity of problem (4.1), we obtain

$$\begin{aligned} C_1 |u^0 - \bar{u}^0|_1^2 &\leq (a^0 \bigtriangledown (u^0 - \bar{u}^0), \bigtriangledown (u^0 - \bar{u}^0)) \\ &= ((a_h^0 - a^0) \bigtriangledown \bar{u}^0, \bigtriangledown (u^0 - \bar{u}^0)) \\ &\leq C_2 h^2 \max_{1 \leq j \leq 2} |N_j|^2_{H^2(Q)} |\bar{u}^0|_1 |u^0 - \bar{u}^0|_1 \\ &\leq C_2 h^2 \max_{1 \leq j \leq 2} |\frac{\partial}{\partial y_i} a_{ij}|^2_{L^2(Q)} |\bar{u}^0|_1 |u^0 - \bar{u}^0|_1 \\ &\leq C_2 h^2 |\bar{u}^0|_1 |u^0 - \bar{u}^0|_1, \end{aligned}$$

which implies

$$|u^0-\bar{u}^0|_1\le Ch^2|\bar{u}^0|_1.$$

By Lemma 4.3 we get

$$\begin{aligned}|u^0-\bar{u}^0_{h_0}|_1 &\le |u^0-\bar{u}^0|_1+|\bar{u}^0-\bar{u}^0_{h_0}|_1\\ &\le C(h_0^2|\bar{u}^0|_3+h^2|\bar{u}^0|_1).\end{aligned}$$

An application of Poincare inequality completes the proof.

Finally we give a numerical computational formula of u^ϵ as follows.

$$u^{\epsilon,h}_{h_0}=\bar{u}^0_{h_0}+\epsilon N_k^h\frac{\partial \bar{u}^0_{h_0}}{\partial x_k}.$$

For any $v|_\tau\in H^1(\tau)$, $\tau\in\mathcal{T}_{h_0}$, define

$$|v|_{1,h_0}=(\sum_{\tau\in\mathcal{T}_{h_0}}|v|^2_{H^1(\tau)})^{1/2}.$$

The main result of this section is the following Theorem.

Theorem 4.5. *Assume $u^0\in W^3_\infty$ to be the solution of problem (4.3), $\bar{u}^0\in H^3$ to be the solution of problem (4.9), and $N_k(y)\in W^2_\infty(Q)$ to be the solution of problem (4.1), for simplicity we also assume $\epsilon\le h_0$. Then*

$$\begin{aligned}|u^\epsilon-u^{\epsilon,h}_{h_0}|_{1,h_0} \le\ & C\{\epsilon\|u^0\|_{W^3_\infty}+\epsilon^{1/2}\|u^0\|_{W^2_\infty}+(h_0^2|\bar{u}^0|_3+h^2|\bar{u}^0|_1\\ &+h|u^0|_{W^2_\infty}+\epsilon h_0|u^0|_3)\max_{1\le k\le 2}|N_k|_{W^2_\infty(Q)}\}.\end{aligned}$$

Proof. Let $\tilde{V}_{h_0}$ be linear conforming finite element space defined on $\mathcal{T}_{h_0}$. Assume Π_{h_0} : $H^2\to\tilde{V}_{h_0}$ to be usual interpolation operator. According to the definitions of u_1^ϵ and $u^{\epsilon,h}_{h_0}$, using inverse inequality, Lemma 4.1, Lemma 4.4, and the estimates of interpolation errors (see [4]), we deduce that

$$\begin{aligned}|u_1^\epsilon-u^{\epsilon,h}_{h_0}|_{1,h_0}\le\ & |u^0-\bar{u}^0_{h_0}|_1+\epsilon|(N_k-N_k^h)\frac{\partial u^0}{\partial x_k}|_1+\epsilon|N_k^h(\frac{\partial u^0}{\partial x_k}-\frac{\partial \bar{u}^0_{h_0}}{\partial x_k})|_{1,h_0}\\
\le\ & |u^0-\bar{u}^0_{h_0}|_1+\epsilon\max_{1\le k\le 2}\{|N_k-N_k^h|_1|u^0|_{W^1_\infty}\\
&+|N_k-N_k^h|_0|u^0|_{W^2_\infty}\}+\epsilon\max_{1\le k\le 2}|N_k^h|_{W^1_\infty}|u^0-\bar{u}^0_{h_0}|_1\\
&+\epsilon|N_k^h|_{L^\infty}\{|\frac{\partial u^0}{\partial x_k}-\Pi_{h_0}(\frac{\partial u^0}{\partial x_k})|_1+Ch_0^{-1}|\Pi_{h_0}(\frac{\partial u^0}{\partial x_k})-\frac{\partial \bar{u}^0_{h_0}}{\partial x_k}|_0\}\\
\le\ & |u^0-\bar{u}^0_{h_0}|_1+\epsilon\max_{1\le k\le 2}\{\epsilon^{-1}|N_k-N_k^h|_{H^1(Q)}|u^0|_{W^1_\infty}\\
&+|N_k-N_k^h|_{L^2(Q)}|u^0|_{W^2_\infty}\}+\max_{1\le k\le 2}(|N_k^h-N_k|_{W^1_\infty(Q)}\\
&+|N_k|_{W^1_\infty(Q)})|u^0-\bar{u}^0_{h_0}|_1+\epsilon(|N_k^h-N_k|_{L^\infty(Q)}+|N_k|_{L^\infty(Q)})\\
&\cdot\{|\frac{\partial u^0}{\partial x_k}-\Pi_{h_0}(\frac{\partial u^0}{\partial x_k})|_1+Ch_0^{-1}(|\Pi_{h_0}(\frac{\partial u^0}{\partial x_k})-\frac{\partial u^0}{\partial x_k}|_0+|\frac{\partial u^0}{\partial x_k}-\frac{\partial \bar{u}^0_{h_0}}{\partial x_k}|_0)\}\\
\le\ & C\max_{1\le k\le 2}\{h_0^2|\bar{u}^0|_3+h^2|\bar{u}^0|_1+h|N_k|_{H^2(Q)}|u^0|_{W^1_\infty}\\
&+\epsilon h^2|N_k|_{H^2(Q)}|u^0|_{W^2_\infty}+(h|\ln h||N_k|_{W^2_\infty(Q)}+|N_k|_{W^1_\infty(Q)})\\
&\cdot(h_0^2|\bar{u}^0|_3+h^2|\bar{u}^0|_1)+\epsilon(h^2|\ln h||N_k|_{W^2_\infty(Q)}+|N_k|_{L^\infty(Q)})\\
&\cdot\{h_0|u^0|_3+h_0^{-1}(h_0^2|u^0|_3+h_0^2|\bar{u}^0|_3+h^2|\bar{u}^0|_1)\}\\
\le\ & C(h_0^2|\bar{u}^0|_3+h^2|\bar{u}^0|_1+h|u^0|_{W^2_\infty}+\epsilon h_0|u^0|_3)\max_{1\le k\le 2}|N_k|_{W^2_\infty(Q)}.\end{aligned}$$

For simplicity, many terms have been enlarged in the last estimate. Combining the above estimate with (3.9), we complete the proof.

Remark. From Theorem 4.5 we know that the final error order is $O(\epsilon^{1/2} + h + h_0^2)$. Since h_0 is greater than h, one can see the error of u^0 matches that of N_k in part. That is the reason why we use high order finite elements to compute u^0.

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