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ALGORITHMS FOR IMPLEMENTATION OF GENERAL LIMIT REPRESENTATIONS OF GENERALIZED INVERSES*

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Abstract

In this paper we investigate three various algorithms for computation of generalized inverses which are contained in the limit expressions $\lim_{z\to 0} V\,(DT+z\mathbf{I})^{-l}\,U$ and $\lim_{z\to 0} V\,(DT+z\mathbf{I})^{-l}\,z^e$. These algorithms are extensions of the algorithms developed by various authors in [2], [3-4], [7-9], [16-18].

Key words: Generalized inverses, Limit representation, Finite algorithm, Imbedding method.

1. Introduction and Preliminaries

The set of all $m\times n$ complex matrices of rank r is denoted by $\mathbb{C}_r^{m\times n}$. By $\mathbf{I}$ we denote an appropriate identity matrix. Also, $\mathrm{Tr}(A)$ denotes the trace of a square matrix A. By $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are denoted the range and the null space of A, respectively. Finally, $\mathrm{adj}(A)$ and $\det(A)$ denote the adjoint of the matrix A and the determinant of A, respectively.

For any matrix $A\in\mathbb{C}^{m\times n}$ consider the following equations in X:

$$(1)\quad AXA=A,\quad (2)\quad XAX=X,\quad (3)\quad (AX)^*=AX,\quad (4)\quad (XA)^*=XA$$

and if $m=n$, also

$$(5)\quad AX=XA,\quad (1^k)\quad A^{k+1}X=A^k.$$

For a sequence $\mathcal{S}$ of $\{1,2,3,4,5\}$ the set of matrices obeying the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$. If X satisfies (1) and (2), it is said to be a reflexive g-inverse of A, whereas $X=A^\dagger$ is said to be the Moore-Penrose inverse of A if it satisfies (1)–(4). The group inverse $A^\#$ is the unique $\{1,2,5\}$ inverse of A, and exists if and only if $\mathrm{ind}(A)=\min\{k:\ \mathrm{rank}(A^{k+1})=\mathrm{rank}(A^k)\}=1$. A matrix $G=A^D$ is said to be the Drazin inverse of A if (1^k) (for some positive integer k), (2) and (5) are satisfied.

Let there be given positive definite matrices M and N of the order m and n, respectively. For any $m\times n$ matrix A, the weighted Moore-Penrose inverse of A is the unique solution $X=A^\dagger_{M,N}$ of the matrix equations (1), (2) and the following equations in X:

$$(3M)\qquad (MAX)^*=MAX\qquad\qquad (4N)\qquad (NXA)^*=NXA.$$

In this paper we investigate three methods for implementation of the following limit expressions, related to a given matrix A of the order $m\times n$:

$$L=\lim_{z\to 0} V(DT+z\mathbf{I})^{-l}U,\qquad L_1=\lim_{z\to 0} V(DT+z\mathbf{I})^{-l}z^e,\tag{1.1}$$

where D, T, U and V are appropriate variable complex matrices of the order $q\times p$, $p\times q$, $q\times m$ and $n\times q$, respectively, $l\ge 1$ and e is an arbitrary integer. These limit expressions contain all so far known limit representations of generalized inverses investigated in [1], [5], [6], [8], [10-15], [18-20]. Moreover, in the case $D=U$, $V=\mathbf{I}$ we obtain the limit expression investigated in [16].

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The paper is organized as follows. In the second section we establish and investigate a general imbedding method for computing the generalized inverses included in the limit expressions (1.1). This method deals with a system of first-order ordinary differential equations associated to the matrices $F^l(z)=(\mathrm{adj}(DT+z\mathbf{I}))^l$, $H^l(z)=F^l(z)z^l$ and the scalar $g^l(z)=(\det(DT+z\mathbf{I}))^l$. In certain particular cases we obtain the results originated in [9], [17] and [18].

In the third section is investigated implementation of the limit representations (1.1) by means of several sets of orthogonal vectors. This implementation is an extension of the method introduced in [9] for implementation of the known limit representation of the Moore-Penrose inverse.

In the last section, using a generalization of the method from [8] and [16], we introduce a more condensed form of the Leverrier-Faddeev finite algorithm for computation of various generalized inverses. Introduced algorithm contains known generalizations of the Leverrier-Faddeev algorithm, available in [2], [4], [7-9] and [16-17]. A part of this method which concerns the limit L in the single case $V=\mathbf{I}$, $D=T$ reduces to the known generalization of the Leverrier-Faddeev algorithm, introduced in [16].

2. A Generalized Imbedding Method

In this section we develop a generalization of the imbedding methods, introduced in [9], [17] and [18]. This generalization of the imbedding method can be used in implementation of the limit expressions (1.1). This method is based on the integration of the first-order ordinary differential equations associated to the matrix powers $F^l=F^l(z)=(\mathrm{adj}(DT+z\mathbf{I}))^l$, $H^l=H^l(z)=F^l(z)z^l$ and the scalar $g^l=g^l(z)=(\det(DT+z\mathbf{I}))^l$.

Theorem 2.1. *Consider arbitrary matrices $D\in\mathbb{C}^{q\times p}$, $T\in\mathbb{C}^{p\times q}$, $U\in\mathbb{C}^{q\times m}$ and $V\in\mathbb{C}^{n\times q}$, an integer $l\geq 1$ and an arbitrary integer e. For the matrix $B(z)=DT+z\mathbf{I}$, let the matrices $F(z)$, $H(z)$ and the scalar $g(z)$ are defined by*

$$\begin{aligned} &F=F(z)=\mathrm{adj}(B(z))=(B_{ij}), \quad H=H(z)=F(z)z,\\ &g=g(z)=\det(B(z)). \end{aligned} \tag{2.1}$$

Then $F^l(z)$, $H^l(z)$ and $g^l(z)$ satisfy the following ordinary differential equations:

$$\begin{aligned} \frac{d(F^l)}{dz}&=lF^l\frac{g^{l-1}\,\mathrm{Tr}(F)-B^{l-1}F^l}{g^l},\\ \frac{d(g^l)}{dz}&=lg^{l-1}\,\mathrm{Tr}(F),\\ \frac{d(H^l)}{dz}&=z^{l-1}\frac{g^l-lzF^lB^{l-1}-lzg^{l-1}\,\mathrm{Tr}(F)}{g^l}F^l. \end{aligned} \tag{2.2}$$

Assume that the matrices $F^l(z)$, $H^l(z)$ and the scalar $g^l(z)$ satisfy the following initial conditions:

$$F^l(z_0)=(\mathrm{adj}(DT+z_0\mathbf{I}))^l,\quad H^l(z_0)=F^l(z_0)z_0^l,\quad g^l(z_0)=(\det(DT+z_0\mathbf{I}))^l$$

where

$$z_0>0,\ |z_0|\leq\min_{z_i\in S}|z_i|,\quad S=\{z_i\,|\,z_i>0 \ \textit{is the eigenvalue of} \ -DT\}. \tag{2.3}$$

In this case is

$$\begin{aligned} L&=\lim_{z\to 0}V(DT+z\mathbf{I})^{-l}U\\ &=V\lim_{z\to 0}\frac{F^l(z_0)+l\int\limits_{z_0}^{z}F^l\dfrac{g^{l-1}\,\mathrm{Tr}(F)-B^{l-1}F^l}{g^l}dz}{g^l(z_0)+l\int\limits_{z_0}^{z}g^{l-1}\,\mathrm{Tr}(F)dz}U. \end{aligned}$$

Also, $L_1 = \lim_{z\to 0} V(DT + z\mathbf{I})^{-l} z^e$ *exist if and only if* $e \geq 0$. *In this case is*

$$L_1=\begin{cases} V\lim\limits_{z\to 0}\dfrac{H^l(z_0)+\int_{z_0}^{z} z^{l-1}\dfrac{g^l-lzF^lB^{l-1}-lzg^{l-1}\mathrm{Tr}(F)}{g^l}F^l dz}{g^l(z_0)+l\int_{z_0}^{z} g^{l-1}\mathrm{Tr}(F)dz}, & e=l\\ 0, & e>l \text{ or } l>e>0\\ V\lim\limits_{z\to 0}\dfrac{F^l(z_0)+l\int_{z_0}^{z} F^l\dfrac{g^{l-1}\mathrm{Tr}(F)-B^{l-1}F^l}{g^l}dz}{g^l(z_0)+l\int_{z_0}^{z} g^{l-1}\,\mathrm{Tr}(F)dz}, & l>e=0. \end{cases}$$

Proof. Using the results from [14], we conclude that the expressions $V(DT + z\mathbf{I})^{-l}U$ and $V(DT + z\mathbf{I})^{-l}z^e$ always exist for any positive integer l in a deleted neighborhood of $z = 0$. According to the used notations, we get

$$B^{-l} = \left[\frac{F(z)}{g(z)}\right]^l = \left[\frac{F}{g}\right]^l. \tag{2.4}$$

Premultiplyng both sides in the last equation by the matrix B^l and postmultiplyng both sides by g^l, we get

$$\mathbf{I}g^l = B^l F^l, \tag{2.5}$$

On the other hand, postmultiplyng both sides of (2.3) by $B^l g^l$ we obtain

$$\mathbf{I}g^l = F^l B^l. \tag{2.6}$$

Differentiation of both sides in (2.5) with respect to the parameter z and multiplication of the obtained equation by F^l from the left produces the following:

$$F^l(B^l)'_z F^l + F^l B^l (F^l)'_z = F^l (g^l)'_z. \tag{2.7}$$

Using (2.6) in the second term of (2.7) we obtain

$$F^l(B^l)'_z F^l + g^l(F^l)'_z = F^l(g^l)'_z.$$

Hence

$$(F^l)'_z = \frac{d(F^l)}{dz} = \frac{F^l(g^l)'_z - F^l(B^l)'_z F^l}{g^l}. \tag{2.8}$$

Differentiating g^l with respect to z we obtain

$$(g^l)'_z = lg^{l-1}g'_z. \tag{2.9}$$

It is clear that

$$B'_z = \frac{d(DT + z\mathbf{I})}{dz} = \mathbf{I} \tag{2.10}$$

which means

$$\frac{db_{ij}(z)}{dz} = \delta_{ij} = \begin{cases} 1, & i = j,\\ 0, & i \neq j. \end{cases}$$

Now, we obtain

$$g'_z = \sum_{i,j=1}^{n} \frac{\partial g(z)}{\partial b_{ij}}\frac{db_{ij}(z)}{dz} = \sum_{i,j=1}^{n} B_{ij}\delta_{ij} = \sum_{i=1}^{n} B_{ii} = \mathrm{Tr}(F). \tag{2.11}$$

From (2.9) and (2.11) we have

$$(g^l)'_z = lg^{l-1}\,\mathrm{Tr}(F). \tag{2.12}$$

By substituting (2.12) into the right hand side of (2.8) and using (2.10), we have

$$(F^l)'_z = \frac{F^l l g^{l-1}\,\mathrm{Tr}(F) - F^l l B^{l-1} F^l}{g^l} = l F^l \frac{g^{l-1}\,\mathrm{Tr}(F) - B^{l-1}F^l}{g^l}.$$

We now derive a differential equation which characterizes the matrix $H^l(z)$. Using (2.5), (2.6) and $F^l = \frac{1}{z^l}H^l$, we obtain

$$\mathbf{I}g^l = \frac{1}{z^l}B^l H^l, \tag{2.5'}$$

$$\mathbf{I}g^l = \frac{1}{z^l}H^l B^l. \tag{2.6'}$$

Differentiating the equation (2.5') with respect to the variable z and multiplying obtained equation by H^l from the left, we obtain

$$H^l(g^l)'_z = -\frac{l}{z^{l+1}}H^l B^l H^l + \frac{1}{z^l}H^l(B^l)'_z H^l + \frac{1}{z^l}H^l B^l (H^l)'_z.$$

Using (2.10), (2.12) and (2.6'), one can verify

$$g^l(H^l)'_z = \frac{1}{z}g^l H^l - \frac{1}{z^l}H^l l B^{l-1}H^l - l g^{l-1}\,\mathrm{Tr}(F)H^l,$$

which implies

$$\begin{aligned}(H^l)'_z &= \frac{z^{l-1}g^l H^l - lH^l B^{l-1}H^l - l z^l g^{l-1}\,\mathrm{Tr}(F)H^l}{z^l g^l}\\ &= \frac{z^{l-1}g^l - l z^l F^l B^{l-1} - l z^l g^{l-1}\,\mathrm{Tr}(F)}{g^l}F^l.\end{aligned} \tag{2.13}$$

For a value of $z = z_0 > 0$ suitably greater than the zero (according to the condition (2.3)), we can determine the determinant and the adjoint of the matrix $B(z_0)$ accurately. This provides initial conditions

$$F^l(z_0) = (\mathrm{adj}(DT + z_0\mathbf{I}))^l, \quad H^l(z_0) = F^l(z_0)z_0^l, \quad g^l(z_0) = (\det(DT + z_0\mathbf{I}))^l$$

at $z = z_0$ for the differential equations in (2.2). Using these initial conditions, (2.2) and (2.4), we obtain

$$\begin{aligned} L &= \lim_{z\to 0} V(DT + z\mathbf{I})^{-l}U = V\lim_{z\to 0}\left(\frac{F(z)}{g(z)}\right)^l U\\ &= V\lim_{z\to 0}\frac{F^l(z_0) + \int_{z_0}^{z}\frac{d(F^l)}{dz}dz}{g^l(z_0) + \int_{z_0}^{z}\frac{d(g^l)}{dz}dz}U\\ &= V\lim_{z\to 0}\frac{F^l(z_0) + l\int_{z_0}^{z}F^l\frac{g^{l-1}\,\mathrm{Tr}(F) - B^{l-1}F^l}{g^l}dz}{g^l(z_0) + l\int_{z_0}^{z}g^{l-1}\,\mathrm{Tr}(F)dz}U.\end{aligned}$$

Moreover, it is clear that $L_1 = \lim\limits_{z\to 0} V(DT + z\mathbf{I})^{-l}z^e$ exists if and only if $e \geq 0$. In the case $e \geq l$, we get

$$L_1 = \lim_{z\to 0} V\left((DT + z\mathbf{I})^{-1}z\right)^l z^{e-l} = V\lim_{z\to 0}\left(\frac{H(z)}{g(z)}\right)^l z^{e-l}$$

For $e > l$ we get $L_1 = 0$. Taking $e = l$ and using (2.2), one can verify the following:

$$L_1 = V \lim_{z\to 0} \left(\frac{H(z)}{g(z)}\right)^l = V \lim_{z\to 0} \frac{H^l(z_0) + \int_{z_0}^{z} \frac{d(H^l)}{dz} dz}{g^l(z_0) + \int_{z_0}^{z} \frac{d(g^l)}{dz} dz}$$

$$= V \lim_{z\to 0} \frac{H^l(z_0) + \int_{z_0}^{z} z^{l-1} \frac{g^l - lzF^l B^{l-1} - lzg^{l-1}\operatorname{Tr}(F)}{g^l} F^l dz}{g^l(z_0) + l\int_{z_0}^{z} g^{l-1}\operatorname{Tr}(F) dz}.$$

In the case $l > e$, in view of (2.2) and (2.4), we obtain the following:

$$L_1 = V \lim_{z\to 0} \left(\frac{F(z)}{g(z)}\right)^l z^e$$

$$= \begin{cases} 0, & e > 0 \\ V \lim_{z\to 0} \dfrac{F^l(z_0) + \int_{z_0}^{z} \frac{d(F^l)}{dz} dz}{g^l(z_0) + \int_{z_0}^{z} \frac{d(g^l)}{dz} dz} U, & e = 0 \end{cases}$$

$$= \begin{cases} 0, & e > 0 \\ V \lim_{z\to 0} \dfrac{F^l(z_0) + l\int_{z_0}^{z} F^l \frac{g^{l-1}\operatorname{Tr}(F) - B^{l-1}F^l}{g^l} dz}{g^l(z_0) + l\int_{z_0}^{z} g^{l-1}\operatorname{Tr}(F) dz} U, & e = 0. \end{cases}$$

Remark 2.1. The imbedding method defined in Theorem 2.1 for implementation of the limit value L represents a generalization of various modifications of the imbedding method, used for computing various generalized inverses, which are considered in [9], [17] and [18]. First generalization is application of an arbitrary integer $l \geq 1$ in Theorem 2.1. The second generalization is possibility to use arbitrary, and possible different, matrices D, T, U and V, instead of a single matrix D. Even in the case $l = 1$, $D = U$, $V = \mathbf{I}$, $T = A$, defined generalized imbedding method for the limit representation L contains the results from [9], [17] and [18]. In [9] is investigated only the case $D = A^*$ and implementation of the Moore-Penrose inverse. In [17] are considered the cases $D = A^*$, $D = N^{-1}A^*M$, $D = A^k$, $k = \operatorname{ind}(A)$ and an imbedding method for the Moore-Penrose, weighted Moore-Penrose inverse and the Drazin inverse, respectively. In the case $l = 1$, $T = A$, $D = U = G$, where $\mathcal{R}(G) = R$, $\mathcal{N}(G) = S$, we obtain the imbedding method for computing the generalized inverse $A^{(2)}_{R,S}$, introduced in [18]. Moreover, a generalization of the imbedding method introduced in Theorem 2.1 contains very wide class of generalized inverses, contained in the limit expressions L and L_1.

3. Limit Representation and Orthogonal Systems

For the sake of completeness we restate known results from [9].

Proposition 3.1. *[9] Let A be an arbitrary $m \times n$ complex matrix of rank r. Then A can be written in the form*

$$A = \sum_{i=1}^{r} a_i \alpha_i \beta_i^*, \quad a_1 \geq a_2 \geq \cdots \geq a_r > 0 \tag{3.1}$$

where $\alpha_1, \ldots, \alpha_r$ form an orthogonal set of vectors in $\mathbb{C}^m$, and $\beta_1, \ldots, \beta_r$ form an orthogonal system in $\mathbb{C}^n$.

The Moore-Penrose inverse of A is equal to [9]

$$A^{\dagger}=\sum_{i=1}^{r} a_i^{-1}\alpha_i\beta_i^*.$$

Theorem 3.1. *Let D, T, U and V are arbitrary matrices of the order $q\times p$, $p\times q$, $q\times m$ and $n\times q$, respectively, whose ranks are r_D, r_T, r_U and r_V, respectively. Assume that $\alpha_1,\dots,\alpha_{\max\{r_D,r_T\}}$ form an orthogonal set of vectors in $\mathbb{C}^p$, $\beta_1,\dots,\beta_{\max\{r_D,r_T\}}$ form an orthogonal system in $\mathbb{C}^q$. Also, assume that $\gamma_1,\dots,\gamma_{r_V}$ is an orthogonal system in $\mathbb{C}^m$ and $\delta_1,\dots,\delta_{r_U}$ is an orthogonal system in $\mathbb{C}^n$. Consider a real number z which satisfies*

$$z>0,\ |z|\le\min_{z_i\in S}|z_i|,\quad S=\{z_i\,|\,z_i>0 \text{ is the eigenvalue of } -DT\}. \tag{3.2}$$

Let the matrices D, T, U and V are expressed in this way:

$$\begin{aligned}
D&=\sum_{i=1}^{r_D} d_i\beta_i\alpha_i^*,\quad d_1\ge\dots\ge d_{r_D}>0,\\
T&=\sum_{i=1}^{r_T} t_i\alpha_i\beta_i^*,\quad t_1\ge\dots\ge t_{r_T}>0,\\
U&=\sum_{i=1}^{r_U} u_i\beta_i\gamma_i^*,\quad u_1\ge\dots\ge u_{r_U}>0,\\
V&=\sum_{i=1}^{r_V} v_i\delta_i\beta_i^*,\quad v_1\ge\dots\ge v_{r_V}>0.
\end{aligned}\tag{3.3}$$

If $r=\min\{r_D,r_T\}$, then for an arbitrary integer $l\ge 1$, the limit expression $L=\lim\limits_{z\to 0}V(DT+z\mathbf{I})^{-l}U$ exists in the case

$$\min\{r_U,r_V\}<r+1. \tag{3.4}$$

In this case is

$$L=\sum_{i=1}^{\min\{r,r_U,r_V\}}\frac{u_iv_i\delta_i\gamma_i^*}{d_i^lt_i^l}. \tag{3.5}$$

Also, $L_1=\lim\limits_{z\to 0}V(DT+z\mathbf{I})^{-l}z^e$ exists in the case $r_V<r+1$ or in the case $e\ge 0$. In these cases is

$$L_1=\begin{cases}\sum\limits_{i=r+1}^{\min\{r_V,q\}} v_i\delta_i\beta_i^*, & e=l,\ r_V\ge r+1\\ 0, & e>l \text{ or } l\ge e>0,\ r_V<r+1.\end{cases}\tag{3.6}$$

Proof. According to representations (3.1) and (3.3), the matrix DT can be represented in the form

$$DT=\sum_{i=1}^{r} d_it_i\beta_i\beta_i^*. \tag{3.7}$$

The vectors $\beta_1,\dots\beta_r$ form an orthogonal system in $\mathbb{C}^q$. If $r<q$, there is possible to add $q-r$ vectors $\beta_{r+1},\dots,\beta_q$, such that $\beta_1,\dots\beta_q$ are orthogonal and span $\mathbb{C}^q$. Then, it is not difficult to verify

$$\mathbf{I}=\sum_{i=1}^{q}\beta_i\beta_i^*. \tag{3.8}$$

From (3.7) and (3.8) we have

$$\begin{aligned} DT + z\mathbf{I} &= \sum_{i=1}^{r} d_i t_i \beta_i \beta_i^* + z \sum_{i=1}^{q} \beta_i \beta_i^* \\ &= \sum_{i=1}^{r} (d_i t_i + z) \beta_i \beta_i^* + z \sum_{i=r+1}^{q} \beta_i \beta_i^*. \end{aligned}$$

According to the condition (3.2), the matrix $DT + z\mathbf{I}$ is nonsingular and we get

$$(DT + z\mathbf{I})^{-1} = \sum_{i=1}^{r} \frac{\beta_i \beta_i^*}{d_i t_i + z} + \sum_{i=r+1}^{q} \frac{\beta_i \beta_i^*}{z}. \tag{3.9}$$

Also, for an arbitrary integer $l \geq 1$ we obtain

$$(DT + z\mathbf{I})^{-l} = \sum_{i=1}^{r} \frac{\beta_i \beta_i^*}{(d_i t_i + z)^l} + \sum_{i=r+1}^{q} \frac{\beta_i \beta_i^*}{z^l} \tag{3.10}$$

as it can be proved by induction.

Now, we obtain

$$\begin{aligned} V(DT+z\mathbf{I})^{-l}U &= \sum_{i=1}^{r_V} v_i \delta_i \beta_i^* \left(\sum_{j=1}^{r} \frac{\beta_j \beta_j^*}{(d_j t_j + z)^l} + \sum_{j=r+1}^{q} \frac{\beta_j \beta_j^*}{z^l} \right) \sum_{k=1}^{r_U} u_k \beta_k \gamma_k^* \\ &= \sum_{i=1}^{r_V} \sum_{j=1}^{r} \sum_{k=1}^{r_U} \frac{v_i u_k \delta_i \beta_i^* \beta_j \beta_j^* \beta_k \gamma_k^*}{(d_j t_j + z)^l} + \sum_{i=1}^{r_V} \sum_{j=r+1}^{q} \sum_{k=1}^{r_U} \frac{v_i u_k \delta_i \beta_i^* \beta_j \beta_j^* \beta_k \gamma_k^*}{z^l}. \end{aligned}$$

Considering the second term in the last equation, we conclude that $L = \lim\limits_{z \to 0} V(DT + z\mathbf{I})^{-l}U$ exists in the case $\min\{r_U, r_V\} < r + 1$. In this case is

$$L = \lim_{z \to 0} \sum_{i=1}^{\min\{r, r_U, r_V\}} \frac{u_i v_i \delta_i \gamma_i^*}{(d_i t_i + z)^l} = \sum_{i=1}^{\min\{r, r_U, r_V\}} \frac{u_i v_i \delta_i \gamma_i^*}{(d_i t_i)^l}.$$

We now investigate the limit L_1. In the case $e \geq l$, in view of (3.10), one can verify the following:

$$(DT + z\mathbf{I})^{-l} z^l = \sum_{i=1}^{r} \frac{\beta_i \beta_i^*}{(d_i t_i + z)^l} z^l + \sum_{i=r+1}^{q} \beta_i \beta_i^* \to \sum_{i=r+1}^{q} \beta_i \beta_i^*, \ \text{ as } z \to 0.$$

This implies

$$\begin{aligned} L_1 &= \lim_{z \to 0} \sum_{i=1}^{r_V} v_i \delta_i \beta_i^* \sum_{j=r+1}^{q} \beta_j \beta_j^* \cdot z^{e-l} \\ &= \begin{cases} 0, & e > l \text{ or } e = l, r_V < r + 1 \\ \sum\limits_{i=r+1}^{\min\{r_V, q\}} v_i \delta_i \beta_i^*, & r_V \geq r + 1, e = l. \end{cases} \end{aligned}$$

In the case $l > e \geq 0$ we get

$$\begin{aligned} L_1 &= \lim_{z \to 0} \sum_{i=1}^{r_V} v_i \delta_i \beta_i^* \left(\sum_{j=1}^{r} \frac{\beta_j \beta_j^*}{(d_j t_j + z)^l} + \sum_{j=r+1}^{q} \frac{\beta_j \beta_j^*}{z^l} \right) z^e \\ &= \lim_{z \to 0} \sum_{i=1}^{r_V} \sum_{j=r+1}^{q} \frac{v_i \delta_i \beta_i^* \beta_j \beta_j^*}{z^{l-e}}. \end{aligned}$$

Consequently, L_1 exists in the case $r_V < r + 1$, in which case is $L_1 = 0$.

Remark 3.1. Consider a given matrix $\mathbb{C}^{m\times n}$. In the case $l=1$, if $V=\mathbf{I}$, $D=U=A^T$, $T=A$, we get known representation of the Moore-Penrose inverse from [9].

Moreover, in Theorem 3.1 are defined representations for various classes of generalized inverses which possess the limit representations L and L_1.

4. A Generalization of the Leverrier-Faddeev Algorithm

In this section we introduce a finite algorithm which is based on the limit expressions (1.1). This algorithm is a generalization of all known modifications of the well-known algorithm, attributed as Leverrier-Faddeev algorithm (also called Souriau-Frame algorithm).

Theorem 4.1. *Let there be given arbitrary matrices D, T, U and V of the order $q\times p$, $p\times q$, $q\times m$ and $n\times q$, respectively. Let*

$$\begin{aligned} F(z) &= \operatorname{adj}(DT+z\mathbf{I}) = F_1 z^{q-1} + \cdots F_{q-1}z + F_q \\ g(z) &= \det(DT+z\mathbf{I}) = g_0 z^q + g_1 z^{q-1} + \cdots + g_q, \end{aligned} \tag{4.1}$$

where $F_1=\mathbf{I}, F_2, \cdots, F_q$ are fixed $q\times q$ matrices and $g_0=1, g_1, \ldots, g_q$ are scalars. Then the following statement is valid:

$$L = \lim_{z\to 0} V(DT+z\mathbf{I})^{-l}U = V\frac{F_r^l}{g_r^l}U, \tag{4.2}$$

where r is the largest index satisfying $g_r\neq 0$.

Also, the limit $L_1=\lim_{z\to 0} V(DT+z\mathbf{I})^{-l}z^e$ exists if and only if $e\geq 0$, and

$$L_1 = \begin{cases} V\left(\mathbf{I}-\dfrac{F_r}{g_r}DT\right)^l, & e=l \\ 0, & e>l \text{ or } l>e>0 \\ V\left(\dfrac{F_r}{g_r}\right)^l, & l>e=0. \end{cases} \tag{4.3}$$

The quantities $F_1, g_1, \ldots, F_r, g_r$ are determined by the following recursive relations:

$$\begin{aligned} F_{j+1} &= g_j\mathbf{I} - DTF_j, \quad j=1,\ldots,r-1, \\ g_{j+1} &= \frac{1}{j+1}\operatorname{Tr}(DTF_{j+1}), \quad j=1,\ldots,r-1 \end{aligned} \tag{4.4}$$

and by the following initial conditions:

$$F_1=\mathbf{I}, \quad g_1 = \operatorname{Tr}(DT) \tag{4.5}$$

Proof. Note that $g_0=1$, so at least one member of the sequence $g_0, g_1, \ldots, g_q$ is different from zero. If r is the largest integer which satisfies $g_r\neq 0$, from (4.1) we get

$$\begin{aligned} L &= \lim_{z\to 0} V(DT+z\mathbf{I})^{-l}U = \lim_{z\to 0}\left(\frac{F(z)}{g(z)}\right)^l U \\ &= \lim_{z\to 0} V\left(\frac{F_1 z^{q-1}+\cdots F_{r-1}z^{q-r+1}+F_r z^{q-r}}{g_0 z^q + g_1 z^{q-1}+\cdots+g_{r-1}z^{q-r+1}+g_r z^{q-r}}\right)^l U \\ &= \lim_{z\to 0} V\left(\frac{F_1 z^{r-1}+\cdots F_{r-1}z+F_r}{z^r+g_1 z^{r-1}+\cdots+g_{r-1}z+g_r}\right)^l U \\ &= V\frac{F_r^l}{g_r^l}U. \end{aligned} \tag{4.6}$$

The equations (4.4)–(4.6) can be proved generalizing the principles from [9] and [17], as follows. Using

$$(DT+z\mathbf{I})(F_1 z^{q-1}+\cdots F_{q-1}z+F_q) = \mathbf{I}(g_0 z^q + g_1 z^{q-1} + \cdots + g_q),$$

and comparing like powers of z on both sides of this equation, we see that the first equation in (4.4) holds. In order to obtain the second equation in (4.4), we set

$$B = DT + z\mathbf{I}, \quad B = (B_{ij}).$$

From (2.12) in the case $l = 1$, we have

$$\frac{dg}{dz} = \operatorname{Tr}(F).$$

We can write this equality as

$$qz^{q-1} + (q-1)g_1 z^{q-2} + \cdots + g_{q-1} = z^{q-1}\operatorname{Tr}(F_1) + z^{q-2}\operatorname{Tr}(F_2) + \cdots + \operatorname{Tr}(F_q).$$

Equating coefficient of like powers of z we see that

$$(q-j)g_j = \operatorname{Tr}(F_{j+1}) \quad j = 0, \ldots, r-1.$$

Taking the trace of both sides in the first identity in (4.4), we obtain

$$\operatorname{Tr}(F_{j+1}) = qg_j - \operatorname{Tr}(DTF_j)$$

From the last two equations we get

$$qg_j - \operatorname{Tr}(DTF_j) = qg_j - jg_j, \quad j = 0, \ldots, r,$$

which implies the second equation in (4.4).

In the case $e \geq l$, the limit expression L_1 can be transformed as follows:

$$L_1 = \lim_{z\to 0} V(DT + z\mathbf{I})^{-l} z^e = \lim_{z\to 0} V\left((DT + z\mathbf{I})^{-1}(DT + z\mathbf{I} - DT)\right)^l z^{e-l}.$$

Therefore, in this case is

$$L_1 = \begin{cases} \lim\limits_{z\to 0} V\left(\mathbf{I} - (DT + z\mathbf{I})^{-1}DT\right)^l, & e = l \\ 0, & e > l \end{cases}$$

$$= \begin{cases} V\left(\mathbf{I} - \dfrac{F_r}{g_r}DT\right)^l, & e = l \\ 0, & e > l. \end{cases}$$

In the case $l > e$, in a similar way as in (4.6), the limit L_1 can be expressed as follows:

$$L_1 = \lim_{z\to 0} V\left(\frac{F_r}{g_r}\right)^l z^e.$$

Consequently, the limit L_1 exists in the case $e \geq 0$, and it is equal to

$$L_1 = \begin{cases} V\left(\dfrac{F_r}{g_r}\right)^l, & e = 0 \\ 0, & e > 0. \end{cases}$$

Remark 4.1. Finite algorithm defined in Theorem 4.1, which is related to the limit representation L, contains various modifications the Leverrier-Faddeev algorithm for computation of various generalized inverses, which are considered in [2], [4], [7-9], [16], [17].

In the case $V = \mathbf{I}$, $D = U = A^*$, $T = A$, $l = 1$, from (4.4) and (4.5) we obtain well-known finite algorithm for computation of the Moore-Penrose inverse, introduced in [4]

For $V = \mathbf{I}$, $D = U = A^k$, $k = \operatorname{ind}(A)$, $T = A$ and $l = 1$, we obtain a modification of the Leverrier-Faddeev algorithm for computation of the Drazin inverse A^D, originated in [7]. Similarly, using substitutions $V = \mathbf{I}$, $DT = A$, $U = A^k$, $l = k+1$, where $k \geq \operatorname{ind}(A)$, we obtain a modification of the Leverrier-Faddeev algorithm for computation of the Drazin inverse, introduced in [8].

Also, if D satisfies $D = U$, $\mathcal{R}(D) = R$ and $\mathcal{N}(D) = S$, in the case $l = 1$, $V = \mathbf{I}$, $T = A$, from (4.4) and (4.5) we obtain an elegant proof of well-known finite algorithm, introduced in [2], for computation of generalized inverses $A^{(2)}_{R,S}$.

In the case $V = \mathbf{I}$, $U = D$ we obtain a generalization of the Leverrier-Faddeev algorithm, introduced in [16].

Moreover, from Theorem 4.1 it is not difficult to define additional set of finite algorithms for computing various classes of generalized inverses, contained in the limit representations L and L_1.

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