Journal of Computational Mathematics, Vol.19, No.6, 2001, 639–650.

PARALLEL COMPOUND METHODS FOR SOLVING PARTITIONED STIFF SYSTEMS*

Li-rong Chen De-gui Liu

(*Beijing Institute of Computer Application and Simulation Technology, Beijing 100854, China*)

Abstract

This paper deals with the solution of partitioned systems of nonlinear stiff differential equations. Given a differential system, the user may specify some equations to be stiff and others to be nonstiff. For the numerical solution of such a system Parallel Compound Methods(PCMs) are studied. Nonstiff equations are integrated by a parallel explicit RK method while a parallel Rosenbrock method is used for the stiff part of the system.

Their order conditions, their convergence and their numerical stability are discussed, and the numerical tests are conducted on a personal computer and a parallel computer.

Key words: Parallel compound methods, Stiff systems, Order conditions, Convergence, Stability.

1. Introduction

Many stiff systems occurring in practice have a special structure, and they can be split into two coupled subsystems

$$\begin{aligned} y'_S(x) &= f_S(x, y_S(x), y_N(x)), \quad y_S(x_0) = y_{S0}, \quad y_S \in R^{n_S} \\ y'_N(x) &= f_N(x, y_S(x), y_N(x)), \quad y_N(x_0) = y_{N0}, \quad y_S \in R^{n_N} \end{aligned} \tag{1}$$

where y_S denotes the vector of stiff components and y_N denotes the vector of nonstiff components. For such partitioned systems a partitioned discretization method is used, i.e. the stiff subsystem is solved by a "stiff" method and the nonstiff by a classical method (see [5–8]). This paper deals with a class of parallel compound methods. The compound method consists of a parallel explicit Runge-Kutta method [2] for the solution of the nonstiff subsystem and a parallel Rosenbrock method [3] for the solution of the stiff subsystem. The internal stages of RK and Rosenbrock methods can be computed in parallel.

The paper discusses order conditions, convergence and numerical stability as well as the implementation and usage of such compound methods. Test results for three partitioned stiff initial value problems are given with respect to speedup and efficiency.

* Received August 16, 1998.

2. Parallel Compound Methods

For simplicity, consider the autonomous partitioned stiff systems

$$\begin{aligned} y'_S(x) &= f_S(y_S(x), y_N(x)), \quad y_S(x_0) = y_{S0}, \quad y_S \in R^{n_S} \\ y'_N(x) &= f_N(y_S(x), y_N(x)), \quad y_N(x_0) = y_{N0}, \quad y_S \in R^{n_N} \end{aligned} \tag{2}$$

An s-stage Parallel Compound Method(PCM) is defined by:

$$\begin{aligned} y_{Sn+1} &= y_{Sn} + \sum_{i=1}^{s} c_i l_{in} \\ y_{Nn+1} &= y_{Nn} + \sum_{i=1}^{s} c_i k_{in} \\ k_{in} &= h f_N(y_{Sn} + \sum_{j=1}^{i-1} \alpha_{ij} l_{jn-1},\ y_{Nn} + \sum_{j=1}^{i-1} \alpha_{ij} k_{jn-1}) \\ (I - h\gamma J) l_{in} &= h f_S(y_{Sn} + \sum_{j=1}^{i-1} \alpha_{ij} l_{jn-1},\ y_{Nn} + \sum_{j=1}^{i-1} \alpha_{ij} k_{jn-1}) + hJ \sum_{j=1}^{i-1} \gamma_{ij} l_{jn-1} \\ & \qquad i = 1, 2, \ldots, s \end{aligned} \tag{3}$$

where γ, α_{ij}, γ_{ij}, c_i are the real coefficients, I denotes $n_S \times n_S$ identity matrix, $J = \dfrac{\partial f_S}{\partial y_S}(y_{Sn}, y_{Nn})$. The method (3) can be briefly characterized:

- The nonstiff components y_N are computed explicitly, the stiff components y_S semi-implicitly. At each integration step a system of linear equations of $n_S \leq n_S + n_N$ must be solved,
- Through a frontal approach the internal stages of RK and Rosenbrock methods k_{in} and $l_{in}(i = 1, 2, \ldots, s)$ can be computed in parallel on $2s$ processors.

The following abbreviations are used:

$$\begin{aligned} &\alpha_{ij} = 0 \qquad j \geq i, \qquad \gamma_{ij} = 0 \qquad j > i \\ &\beta'_{ij} = \alpha_{ij} + \gamma_{ij}, \qquad \gamma_{ii} = \gamma \\ &\beta_{ij} = \begin{cases} \beta'_{ij}, & i > j \\ 0, & i \leq j \end{cases} \\ &\alpha_i = \sum_{j=1}^{s} \alpha_{ij}, \quad \beta'_i = \sum_{j=1}^{s} \beta'_{ij}, \quad \beta_i = \sum_{j=1}^{s} \beta_{ij} \end{aligned}$$

As the quantities y_n, k_{in-1}, l_{in-1} $(i = 1, 2, \ldots, s-1)$ are known, k_{in}, l_{in} $(i = 1, 2, \ldots, s)$ can be evaluated on $2s$ processors in parallel, and more y_{n+1} obtained. The information flow that describes the parallel execution of two-stage formula on four processors is showed in Fig. 1.

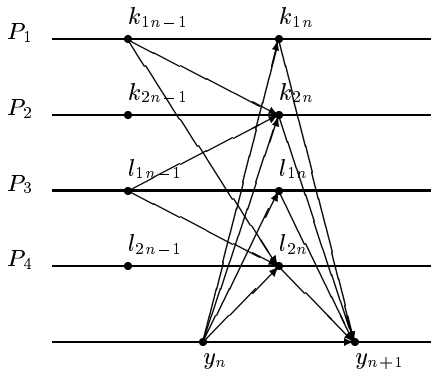


Fig. 1. the information flow of PCM(s=2)

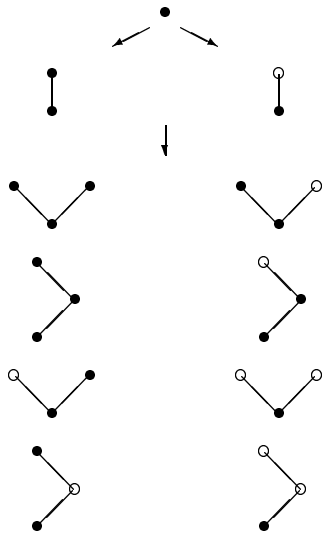
3. Order Conditions of PCMs

Assume that $f = (f_S, f_N)^T\colon\ U \to R^{n_S+n_N}$ is arbitrarily often differentiable, where U is an open set in $R^{n_S+n_N}$. For the order conditions the power series in h of $(L_{in},\ K_{in})^T$ and $(y_{Sn+1},\ y_{Nn+1})^T$ must be studied. These turn out to be a generalization of the Butcher series, the so-called P-series, as defined in Hairer[6]. The notation from [6] stands for:

- TP is the set of rooted P-trees.
- For $z \in \{S, N\}$ we denote by
  - ϕ_z is the P-tree of order 0 with root index z,
  - τ_z is the P-tree of order 1 with root index z,
  - $t =_z [t_1, t_2, \ldots, t_m]$ that the roots of $t_1, t_2, \ldots, t_m$ are connected with a new node by new arcs; this new node becomes the root of the new P-tree t and is labelled by z.
- $\rho(t)$ is the order of t, i.e. the number of nodes of t.
- $w(t)$ is the root index of t.
- $\alpha(t)$ is the cardinality of t, i.e. the number of possibilities of monotonically labelling the nodes of t with the numbers $1, \ldots, \rho(t)$, starting at the root.

In the PCM (3) the higher derivatives of y_S and y_N are strongly connected with monotonically labelled trees using two different knots $\bullet$ and $\circ$. The knot $\bullet$ stands for f_S, a $\bullet$ with one leg for a partial derivative of f_S, etc.; the knot $\circ$ stands for f_N, etc. In the following sketch examples for the derivation of some trees is given.

$$
\begin{aligned}
y'_S &= f_S \\
y''_S &= \frac{\partial f_S}{\partial y_S} f_S + \frac{\partial f_S}{\partial y_N} f_N \\
y'''_S &= \frac{\partial^2 f_S}{\partial y_S^2}(f_S, f_S) + \frac{\partial^2 f_S}{\partial y_S \partial y_N}(f_S, f_N) \\
&\quad + \frac{\partial f_S}{\partial y_S}\frac{\partial f_S}{\partial y_S} f_S + \frac{\partial f_S}{\partial y_S}\frac{\partial f_S}{\partial y_N} f_N \\
&\quad + \frac{\partial^2 f_S}{\partial y_N \partial y_S}(f_N, f_S) + \frac{\partial^2 f_S}{\partial y_N^2}(f_N, f_N) \\
&\quad + \frac{\partial f_S}{\partial y_N}\frac{\partial f_N}{\partial y_S} f_S + \frac{\partial f_S}{\partial y_N}\frac{\partial f_N}{\partial y_N} f_N
\end{aligned}
$$



Assume that $y(x) = (y_S(x), y_N(x))^T$ be the exact solution of (2), and furthermore assume

$$
\begin{aligned}
y^*_{Sn+1} &= y_S(x_n) + \sum_{i=1}^{s} c_i L_{in} \\
y^*_{Nn+1} &= y_N(x_n) + \sum_{i=1}^{s} c_i K_{in} \\
K_{in} &= h f_N(y_S(x_n) + \sum_{j=1}^{i-1} \alpha_{ij} L_{jn-1},\ y_N(x_n) + \sum_{j=1}^{i-1} \alpha_{ij} K_{jn-1}) \\
(I - h\gamma D f_S) L_{in} &= h f_S(y_S(x_n) + \sum_{j=1}^{i-1} \alpha_{ij} L_{jn-1},\ y_N(x_n) + \sum_{j=1}^{i-1} \alpha_{ij} K_{jn-1}) \\
&\quad + h D f_S \sum_{j=1}^{i-1} \gamma_{ij} L_{jn-1}, \qquad i = 1, \ldots, s
\end{aligned}
\tag{4}
$$

where $Df_S = \dfrac{\partial f_S}{\partial y_S}(y_S(x_n), y_N(x_n))$. The PCM is of order p if and only if the local truncation errors satisfy

$$
\begin{aligned}
y_S(x_n + h) - y^*_{Sn+1} &= O(h^{p+1}) \\
y_N(x_n + h) - y^*_{Nn+1} &= O(h^{p+1})
\end{aligned}
\tag{5}
$$

The application of [6,8] to PCMs leads to a recursive description of the coefficients of the map $\Phi_i(t)$ associated with $(L_{in}, K_{in})^T$.

Lemma 1. *$(L_{in}, K_{in})^T$ can be represented as a P-series:*

$$\begin{pmatrix} L_{in} \\ K_{in} \end{pmatrix} \sim P(\Phi_i, y(x_n)) = \begin{pmatrix} \sum_{t\in TP, w(t)=S} \\ \sum_{t\in TP, w(t)=N} \end{pmatrix} \Phi_i(t)\alpha(t)F(t)(y(x_n))\frac{h^{\rho(t)}}{\rho(t)!} \tag{6}$$

where the coefficients are recursively given by:

$$\begin{aligned} &\forall z \in \{S, N\} \qquad \Phi_i(\phi_z) = 0, \qquad \Phi_i(\tau_z) = 1 \\ &\Phi_i(t) = \begin{cases} \rho(t) \sum_{j_1\cdots j_m} \prod_{s=1}^{m} \alpha_{ij_s}\Gamma_{j_s}(t_s), & \text{if } t =_z [t_1, \ldots, t_m], \quad m \geq 2 \\ & \quad t =_N [t_1], \text{ or } (t =_S [t_1], \ w(t_1) = N) \\ \rho(t)[\sum_j \beta_{ij}\Gamma_j(t_1) + \gamma\Phi_i(t_1)], & \text{if } t =_S [t_1], \text{ and } w(t_1) = S \end{cases} \end{aligned} \tag{7}$$

The application of the linearity of P-series leads to

$$\begin{pmatrix} y^*_{Sn+1} \\ y^*_{Nn+1} \end{pmatrix} = P(\Phi, y(x_n)) \tag{8}$$

with

$$\Phi(t) = \begin{cases} 1, & \rho(t) = 0 \\ \sum_{i=1}^{s} c_i\Phi_i(t), & \rho(t) \geq 1 \end{cases} \tag{9}$$

Lemma 2. *The exact solution $y(x)$ can be represented as a P-series:*

$$y(x + h) \sim P(II, y(x)) \tag{10}$$

with $II(t) = 1$, $\forall t \in TP$

From (5),(8),(10) we can obtain directly the following theorem:

Theorem 1. *(Consistency order of PCMs) The parallel compound method (3) has order of consistency p, if $\Phi(t) = 1$, $\forall t \in TP$, $\rho(t) \leq p$, $\Phi(t)$ is defined by (7),(9).*

4. Construction of PCMs

From Theorem 1 we can get the order condition equations that are listed in Table 1.

Table 1. Order condition equations for PCMs up to 3

Order	Trees	Equation
1		$\sum c_i = 1$
2		$\sum c_i \alpha_i = \frac{1}{2}$
		$\sum c_i \beta'_i = \frac{1}{2}$
3		$\sum c_i \alpha_i^2 = \frac{1}{3}$
		$\sum c_i \alpha_{ij} \alpha_j = \frac{2}{3}$
		$\sum c_i \beta'_{ij} \alpha_j = \frac{2}{3} - \gamma$
		$\sum c_i \alpha_{ij} \beta'_j = \frac{2}{3}$
		$\sum c_i \beta'_{ij} \beta'_j = \frac{2}{3} - \gamma$

Remark. All summation indices i, j are in range $1, 2, \ldots, s$, $\star$ stands for a $\bullet$ or $\circ$.

we can easily obtain the following corollaries by virtue of the order condition equations:

Corollary 1. *There exist PCMs with order $p = 1$, stage number $s = 1$.*

Corollary 2. *There exist PCMs with order $p = 2$, stage number $s = 2$.*

For the implementation of an efficient stepsize control, embedded methods are of special interests. in order to get the necessary information about the local truncation error after one integration step, two solutions of different orders are compared. A PCM with order p and stage number s is embedded by a PCM with order $p-1$ and stage number $s-1$:

$$\begin{aligned}
\hat{y}_{Sn+1} &= y_{Sn} + \sum_{i=1}^{s-1} \hat{c}_i l_{in} \\
\hat{y}_{Nn+1} &= y_{Nn} + \sum_{i=1}^{s-1} \hat{c}_i k_{in} \\
k_{in} &= h f_N(y_{Sn} + \sum_{j=1}^{i-1} \alpha_{ij} l_{jn-1},\ y_{Nn} + \sum_{j=1}^{i-1} \alpha_{ij} k_{jn-1}) \\
(I - h\gamma J) l_{in} &= h f_S(y_{Sn} + \sum_{j=1}^{i-1} \alpha_{ij} l_{jn-1},\ y_{Nn} + \sum_{j=1}^{i-1} \alpha_{ij} k_{jn-1}) \\
&\quad + hJ \sum_{j=1}^{i-1} \gamma_{ij} l_{jn-1}, \qquad i = 1, \ldots, s-1
\end{aligned}$$

For the PCMs with embedded technique we have

Corollary 3. *There exist PCMs(1)2 with stage $s = 2$. Rosenbrock part is A-stable, if* $\gamma = 1 + \frac{1}{\sqrt{3}}$.

Two sets of the coefficients of PCMs(1)2 are shown in Table 2.

Table 2. Coefficients of PCMs(1)2

α_2	γ	γ_{21}	c_1	c_2	$\hat{c}_1$
$\frac{1}{2}$	$1+\frac{1}{\sqrt{3}}$	$-(1+\frac{1}{\sqrt{3}})$	0	1	1
1	$1+\frac{1}{\sqrt{3}}$	$-2(1+\frac{1}{\sqrt{3}})$	$\frac{1}{2}$	$\frac{1}{2}$	1

Corollary 4. *There exist PCMs(2)3 with stage $s = 4$ and 3 function evalutions per step. The parameters $\gamma,\ \alpha_2,\ \alpha_3,\ \beta_{32},\ \beta_{43},\ c_4$ are free.*

A set of coefficients of PCMs(2)3 with stage $s = 4$ are listed in Table 3.

Table 3. Coefficients of PCMs(2)3

$\gamma = 3.20$		
$\gamma_{21} = -1.2$	$\gamma_{31} = 191.3297297$	$\gamma_{32} = 0$
$\gamma_{41} = 117.2450434$	$\gamma_{42} = -28.76035714$	$\gamma_{43} = -0.01982142857$
$\alpha_{21} = 0.5$	$\alpha_{31} = -7.0$	$\alpha_{32} = 8.0$
$\hat{c}_1 = -0.01032366071$	$\hat{c}_2 = 1.020647321$	$\hat{c}_3 = -0.01032366071$
$c_1 = 0.1666666667$	$c_2 = 0.6666666667$	
$c_3 = -0.1666666667$	$c_4 = 0.3333333333$	

In the following discussion we will focus on the convergence of PCMs. Denoting

$$
\begin{aligned}
&e_{Sn} = y_S(x_n) - y_{Sn}, \qquad e_{Nn} = y_N(x_n) - y_{Nn}, \qquad n = 0, 1, \ldots \\
&\delta_{Sn} = \max\{\|e_{Sn}\|, \ldots, \|e_{Sn-s+1}\|\} \\
&\delta_{Nn} = \max\{\|e_{Nn}\|, \ldots, \|e_{Nn-s+1}\|\}, \qquad n = s-1, s, \ldots \\
&\Delta l_{in} = L_{in} - l_{in}, \qquad \Delta k_{in} = K_{in} - k_{in}
\end{aligned}
$$

Subtracting (3) from (4) we obtain

$$
\begin{aligned}
e_{Sn+1} &= e_{Sn} + \sum_{i=1}^{s} c_i \Delta l_{in} + T_{Sn+1} \\
e_{Nn+1} &= e_{Nn} + \sum_{i=1}^{s} c_i \Delta k_{in} + T_{Nn+1}
\end{aligned}
\tag{11}
$$

where T_{Sn+1}, T_{Nn+1} denotes the local truncation error.

Using the mathematical induction we get

Lemma 3. *There exist positive numbers* $d_{Sk},\ b_{Sk},\ d_{Nk},\ b_{Nk}\ (k=1,2,\ldots,s)$, *such that*

$$\begin{aligned}\|\sum_{i=1}^{s} c_i \Delta l_{in}\| &\le h\sum_{k=1}^{s} d_{Sk}\|e_{Sn-k+1}\| + h\sum_{k=1}^{s} b_{Sk}\|e_{Nn-k+1}\| \\ \|\sum_{i=1}^{s} c_i \Delta k_{in}\| &\le h\sum_{k=1}^{s} d_{Nk}\|e_{Sn-k+1}\| + h\sum_{k=1}^{s} b_{Nk}\|e_{Nn-k+1}\|\end{aligned} \tag{12}$$

From (11) and the above lemma it can be easily verified that the following theorem holds:

Theorem 2. *For PCMs if*

(1) *the local truncation errors* T_{Sn}, T_{Nn} *satisfy* $T_{Sn}=O(h^{p+1})$, $T_{Nn}=O(h^{p+1})$

(2) *the errors of starting values satisfy*

$$\begin{aligned}\delta_{Ss-1} &= \max\{\|e_{Ss-1}\|,\ldots,\|e_{S0}\|\} = O(h^p) \\ \delta_{Ns-1} &= \max\{\|e_{Ns-1}\|,\ldots,\|e_{N0}\|\} = O(h^p)\end{aligned}$$

then for the global errors it holds that

$$\|e_{Sn}\| = O(h^p), \qquad \|e_{Nn}\| = O(h^p), \qquad n = s, s+1,\ldots$$

5. Numerical Stability of PCMs

Consider the linear test differential system

$$\begin{aligned} y_S' &= \mu y_S + a y_N, \qquad y_S(x_0) = y_{S0} \\ y_N' &= b y_S + \kappa y_N, \qquad y_N(x_0) = y_{N0}, \qquad x \in [x_0, x_M]\end{aligned} \tag{13}$$

where

$$\begin{aligned} &(1) \qquad \mu \ll \kappa < 0 \\ &(2) \qquad a \cdot b := c < \mu\kappa, \qquad a, b \in R\end{aligned} \tag{14}$$

Definition 1. *A PCM is called absolutely stable for the stepsize* h *if, when applied to (13),* $y_{Sn} \to 0,\ y_{Nn} \to 0\ (n \to \infty)$ *hold for any initial values* $y_{S0},\ y_{N0}$.

Definition 2. h_{max} *is called an absolutely stable bound of a PCM, if the PCM is absolutely stable for all* $h \in (0, h_{max})$.

Since the parallel compound method PCM1 has the same form as CRKR1, please refer to [4,5] for the stability of PCM1. In the following discussion we shall investigate the stability behavior of PCMs(1)2.

Application of PCMs(1)2 to (13) yields

$$\begin{aligned}
&y_{Sn+1} = y_{Sn} + c_1 l_{1n} + c_2 l_{2n}\\
&y_{Nn+1} = y_{Nn} + c_1 k_{1n} + c_2 k_{2n}\\
&k_{1n} = hby_{Sn} + h\kappa y_{Nn}\\
&k_{2n} = hby_{Sn} + h\kappa y_{Nn} + hb\alpha_{21} l_{1n-1} + h\kappa\alpha_{21} k_{1n-1}\\
&l_{1n} = \frac{h\mu}{1-h\gamma\mu} y_{Sn} + \frac{ha}{1-h\gamma\mu} y_{Nn}\\
&l_{2n} = \frac{h\mu}{1-h\gamma\mu} y_{Sn} + \frac{ha}{1-h\gamma\mu} y_{Nn} + \frac{h\mu}{1-h\gamma\mu}\beta_{21} l_{1n-1} + \frac{ha}{1-h\gamma\mu}\alpha_{21} k_{1n-1}
\end{aligned} \tag{15}$$

Using the consistency condition of order 2, $c_2\alpha_{21} = \frac{1}{2}$, $c_2\beta_{21} = \frac{1}{2} - \gamma$, and denoting

$$u = \frac{h\mu}{1-h\gamma\mu}, \qquad v = h\kappa, \qquad w = \frac{h^2 c}{1-h\gamma\mu}$$

we obtain

$$\begin{aligned}
&y_{Sn+1} = (1+u)y_{Sn} + ((\tfrac{1}{2}-\gamma)u^2 + \tfrac{w}{2})y_{Sn-1} + \frac{ha}{1-h\gamma\mu}[y_{Nn} + ((\tfrac{1}{2}-\gamma)u + \tfrac{v}{2})y_{Nn-1}]\\
&y_{Nn+1} = hb(y_{Sn} + \tfrac{1}{2}(u+v)y_{Sn-1}) + (1+v)y_{Nn} + \tfrac{1}{2}(w+v^2)y_{Nn-1}
\end{aligned} \tag{16}$$

By virtue of z-transformation we have

$$\begin{aligned}
A_{11}(z)U^*(z) + A_{12}(z)V^*(z) &= S_1(z)\\
A_{21}(z)U^*(z) + A_{22}(z)V^*(z) &= S_2(z)
\end{aligned}$$

where $U^*(z)$, $V^*(z)$ are the z-transformatiom of sequences $\{y_{Sn}\}$, $\{y_{Nn}\}$ respectively. $S_1(z)$, $S_2(z)$ are polynomials with the degree not greater than two. $A_{ij}(z)$ have the forms:

$$\begin{aligned}
A_{11}(z) &= z^2 - (1+u)z - (\tfrac{1}{2}-\gamma)u^2 - \tfrac{1}{2}w\\
A_{12}(z) &= -\frac{ha}{1-h\gamma\mu}(z + (\tfrac{1}{2}-\gamma)u + \tfrac{1}{2}v)\\
A_{21}(z) &= -hb(z + \tfrac{1}{2}(u+v))\\
A_{22}(z) &= z^2 - (1+v)z - \tfrac{1}{2}(w+v^2)
\end{aligned}$$

We can easily obtain the following theorem:

Theorem 3. *A parallel compound method PCM(1)2 is absolutely stable if and only if the zero points z_j of determinant $|A(z)|$ of matrix $A(z) = (A_{ij}(z))$ satisfy $\|z_j\| < 1$, namely, the roots z_j of equation of polynomial*

$$\begin{aligned}
&(z^2 - (1+u)z - (\tfrac{1}{2}-\gamma)u^2 - \tfrac{1}{2}w)(z^2 - (1+v)z - \tfrac{1}{2}(w+v^2))\\
&\quad -w(z + (\tfrac{1}{2}-\gamma)u + \tfrac{1}{2}v)(z + \tfrac{1}{2}(u+v)) = 0
\end{aligned} \tag{17}$$

satisfy $\|z_j\| < 1$.

Fig.2 shows the absolutely stable bound h_{max} for $\mu = -100,\ \kappa = -2$ as function of the coupling-parameter c for the PCM(1)2 with $\gamma = 1 + \frac{1}{\sqrt{3}}$.

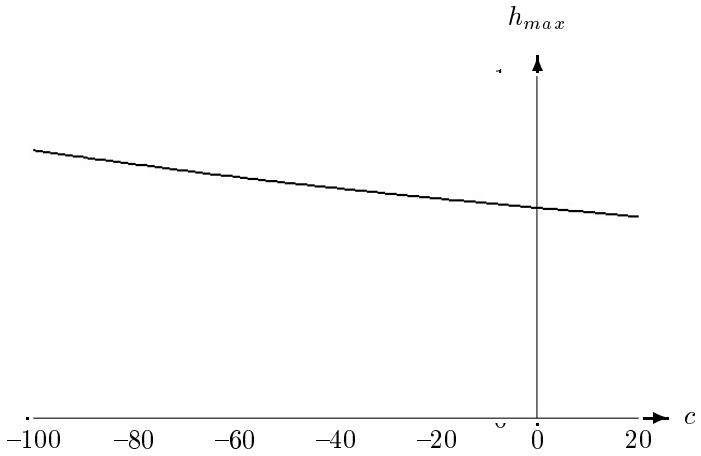


Fig. 2. The absolutely stable bound h_{max} of PCM(1)2

6. Numerical Results

Example 1. (see [7]).

$$y_i' = i - 0.1\sum_{j=1}^{20} y_j - 0.01 y_{i+1} y_{i-1} + r_i y_i$$
$$y_i(0) = 10, \qquad i = 1, 2, \ldots, 20$$

with

$$y_0 = y_{20}, \quad y_{21} = y_1, \quad r_{20} = -1000, \quad r_i = 0.1,\ i = 1, 2, \ldots, 19$$
$$x_M = 10.0, \qquad y_S = y_1, \qquad y_N = (y_2, y_3, \ldots, y_{20})^T$$

Example 2. (see [8]).

$$y_1' = 250((R-1)y_1 + y_2), \qquad y_1(0) = 1$$
$$y_2' = 0.1(y_1 - y_2), \qquad y_2(0) = 1$$
$$y_3' = 93y_1 - 0.26(y_3 - y_4), \qquad y_3(0) = 660.2$$
$$y_4' = 0.87(y_3 - y_4) - 11(y_4 - y_5), \qquad y_4(0) = 302.2$$
$$y_5' = 1.8(y_4 - y_5) - 13(y_5 - 270), \qquad y_5(0) = 273.9$$

with $R = -0.0048(y_3 - 660.2) - 0.032(y_5 - 273.9)$.

$x_M = 1.0,\ y_S = y_1,\ y_N = (y_2, y_3, y_4, y_5)^T$.

Example 3. (see [8]).

$$\begin{aligned}
y_1' &= -10^4 y_1 y_3 + 10^4 y_2 y_6, & y_1(0) &= 1\\
y_2' &= -10^4 y_1 y_6 - 10^4 y_2 y_3, & y_2(0) &= 1\\
y_3' &= -y_3 - y_4 + 1, & y_3(0) &= 1\\
y_4' &= -2y_4, & y_4(0) &= 1\\
y_5' &= 2 - y_5, & y_5(0) &= -1\\
y_6' &= -y_6 - 0.5y_5 + 0.5, & y_6(0) &= 0
\end{aligned}$$

with $x_M = 10.0, \qquad y_S = (y_1, y_2)^T, y_N = (y_3, y_4, y_5, y_6)^T$.

Because the parallel methods are constructed by aiming at large systems, we increase intentionally the number of equations in testing speedup and efficiency. For instance, we shall use the following system instead of Example 2

$$\begin{aligned}
&for(i = 0;\ i < N;\ ++i)\{\\
&y_1' = 250((R-1)y_1 + y_2), && y_1(0) = 1\\
&.....\\
&y_5' = 1.8(y_4 - y_5) - 13(y_5 - 270), && y_5(0) = 273.9\\
&\}
\end{aligned}$$

where $N > 1$ is cycle index representing the computation complex of right-hand functions.

The computations are performed in double precision on a parallel computer S10. T_{PPCM} and T_{PCM} stand for the computing time to solve the problems by a PCM(1)2 using four processors and one processor respectively; T_{GEAR} stands for the computing time by the GEAR version "DRIVE" using one processor. The test results are listed in Table 4. Two kinds of speedup and efficiency shown in the table are defined by

$$S_{PP} = \frac{T_{PCM}}{T_{PPCM}}, \quad S_{PG} = \frac{T_{GEAR}}{T_{PPCM}}, \quad E_{PP} = \frac{S_{PP}}{4}, \quad E_{PG} = \frac{S_{PG}}{4}$$

Table 4. Speedup and efficiency of PCM(1)2 , N=1000

Example	S_{PP}	S_{PG}	E_{PP}	E_{PG}
1	2.71	3.87	68%	97%
2	2.76	3.30	69%	82%
3	2.54	2.82	64%	70%

On a personal computer a PCM(2)3 is performed in parallel in a six-processor simulated circumstance. In Table 5 the speedup and efficiency, in comparison with PCM(2)3 performed in serial, are listed

Table 5. Speedup and efficiency of PCM(2)3

N		500	1000	2000
Example 1	Speedup	4.39	4.49	4.57
	Efficiency	73%	75%	76%
Example 2	Speedup	3.62	4.04	4.32
	Efficiency	60%	67%	72%
Example 3	Speedup	4.31	5.10	5.09
	Efficiency	72%	85%	85%

At each integration step the PCM has to solve systems of linear equations of dimension n_S, GEAR or other implicit and linearly implicit methods have to solve systems of dimension $n_S + n_N$. So, PCM is especially efficient for systems of higher dimension with a small number of stiff components. This is shown by the results of Example 1 from Tables 4,5.

References

[1] Burrage, K., Parallel and Sequential Methods for Ordinary Differential Equations, Clarendon Press, Oxford, 1995.

[2] Fei Jinggao, A class of parallel explicit Runge-Kutta formulas, *Chinese Journal Numerical Mathematics Application,* **16**:1 (1994), 23–36.

[3] Chen Lirong, Liu Degui, A Class of Parallel Rosenbrock Formulas, Proceeding of 3rd BICSC'95.

[4] Chen Lirong, Parallel Methods for the Numerical Simulation of Stiffly Large Systems, Ph.D Dissertation of the Second Academy of China Aerospace Industry Corporation.

[5] Chen Lirong, Liu Degui, Combined methods for a class of partitioned stiffly large systems, *Chinese Journal of Computational Physics,* **14**:4-5 (1997), 509-512.

[6] Hairer, E., Order conditions for numerical methods for partitioned ordinary differential equations, *Numer. Math.,* **36** (1981), 431–445.

[7] Büttner, M., Schmitt, B.A., Weiner, R., automatic partitioning in Linearly-Implicit Runge-Kutta methods, *Appl. Numer. Math.,* **13** (1993), 41–55.

[8] Strehmel, K., Weiner, R., partitioned adaptive Runge-Kutta methods and their stability, *Numer. Math.,* **45** (1984), 283–300.