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CHEBYSHEV SPECTRAL-FINITE ELEMENT METHOD FOR TWO-DIMENSIONAL UNSTEADY NAVIER-STOKES EQUATION[*1)]

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Abstract

A mixed Chebyshev spectral-finite element method is proposed for solving two-dimensional unsteady Navier-Stokes equation. The generalized stability and convergence are proved. The numerical results show the advantages of this method.

Key words: Navier-Stokes equation, Chebyshev spectral-finite element method.

1. Introduction

Spectral method has been used successfully in computational fluid dynamics. For semi-periodic problems, we can use mixed Fourier-Chebyshev spectral method, Fourier spectral-finite difference method and Fourier spectral-finite element method (see[1–5]). As we know, many problems are fully non-periodic. But the sections of domains might be rectangular in certain directions. For example, the fluid flow in a cylindrical container. So we proposed Chebyshev spectral-finite element method(see[6]). In this paper, we develop mixed Chebyshev spectral-finite element method for two-dimensional unsteady Navier-Stokes equation.

2. The Scheme

Let $I_x = \{x \,/\, -1 < x < 1\}$, $I_y = \{y \,/\, 0 < y < 1\}$ and $\Omega = I_x \times I_y$ with the boundary $\partial\Omega$. The speed vector and the pressure are denoted by $U = (U_1, U_2)$ and P respectively. $\nu > 0$ is the kinetic viscosity. $U_0(x, y)$ and $f(x, y, t)$ are given functions. Let $T > 0, \partial_t = \frac{\partial}{\partial t}, \partial_x = \frac{\partial}{\partial x}$, and $\partial_y = \frac{\partial}{\partial y}$. The Navier-Stokes equation is as follows

$$\begin{cases} \partial_t U + \partial_x(U_1 U) + \partial_y(U_2 U) + \nabla P - \nu\nabla^2 U = f, & \text{in } \Omega \times (0, T], \\ \nabla^2 P + \Phi(U) = \nabla \cdot f, & \text{in } \Omega \times (0, T], \\ U\,|_{t=0} = U_0, & \text{in } \Omega \bigcup \partial\Omega \end{cases} \tag{2.1}$$

where

$$\Phi(U) = 2(\partial_y U_1 \partial_x U_2 - \partial_x U_1 \partial_y U_2).$$

Suppose that the boundary is a non-slip wall and so $U = 0$ on $\partial\Omega$. There is no boundary condition for the pressure. But if we use the second equation of (2.1) to evaluate the pressure,

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then we need a non-standard boundary condition. We assume approximately that $\frac{\partial P}{\partial n} = 0$ on $\partial\Omega$. For fixing the value of pressure, we require that

$$\mu(P,t) \equiv \int\int_{\Omega} P(x,y,t)\,\mathrm{d}x\mathrm{d}y = 0, \qquad \forall t \in [0,T].$$

Clearly for each time t and U, the second equation of (2.1) is a Neumann problem for P. It can be verified that $\mu(\nabla \cdot f - \Phi(U), t) \equiv 0$ and so this problem is consistent(see[7]). The main advantage of this model is that the derivation of the second formula of (2.1) implies the incompressible condition automatically.

Let $\mathcal{D}$ be an interval (or a domain) in R^1(or R^2). $L^2(\mathcal{D}), H^r(\mathcal{D})$ and $H_0^r(\mathcal{D})(r > 0)$ denote the usual Hilbert spaces with the usual inner products and norms. We also define

$$L_0^2(\mathcal{D}) = \{\eta \in L^2(\mathcal{D}) \,/\, \int_{\mathcal{D}} \eta\,\mathrm{d}\mathcal{D} = 0\}.$$

Let $\omega(x) = (1-x^2)^{-\frac{1}{2}}$ and

$$(u,v)_{\omega,I_x} = \int_{I_x} uv\omega\,\mathrm{d}x, \qquad \|v\|_{\omega,I_x} = (v,v)_{\omega,I_x}^{\frac{1}{2}},$$

$$L_\omega^2(I_x) = \{v \,/\, v \text{ is measurable and } \|v\|_{\omega,I_x} < \infty\}.$$

Furthermore

$$(u,v)_\omega = \int\int_{\Omega} uv\omega\,\mathrm{d}x\mathrm{d}y, \quad \|v\|_\omega = (v,v)_\omega^{\frac{1}{2}},$$

$$L_\omega^2(\Omega) = \{v \,/\, v \text{ is measurable and } \|v\|_\omega < \infty\}.$$

Now we construct the scheme. For any positive integer N, we denote by $\mathcal{P}_N$ the set of all polynomials of degree $\le N$, defined on R^1. Let

$$V_N(I_x) = \{v(x) \in \mathcal{P}_N \ / \quad v(-1) = v(1) = 0\},$$

$$W_N(I_x) = \{v(x) \in \mathcal{P}_N \ / \quad \frac{dv}{dx}(-1) = \frac{dv}{dx}(1) = 0\}.$$

Next, we divide I_y into M_h subintervals with the nodes $0 = y_0 < y_1 < \cdots < y_{M_h} = 1$. Let $I_l = (y_{l-1}, y_l), h_l = y_l - y_{l-1}$, $h = \max_{1\le l\le M_h} h_l$ and $h' = \min_{1\le l\le M_h} h_l$. Assume that there exists a positive constant d independent of the divisions of I_y, such that $h/h' \le d$. Let

$$\tilde{S}_h^k(I_y) = \{v(y) \,/\, v(y)\,|_{I_l} \in \mathcal{P}_k, 1 \le l \le M_h\}, \quad S_h^k(I_y) = \tilde{S}_h^k(I_y) \bigcap H_0^1(I_y).$$

The trial function space $X_{N,h}^k(\Omega)$ for the speed and the trial function space $Y_{N,h}^k(\Omega)$ for the pressure are defined by

$$X_{N,h}^k(\Omega) = V_N(I_x) \otimes S_h^k(I_y), \quad Y_{N,h}^k(\Omega) = \{W_N(I_x) \otimes (\tilde{S}_h^k(I_y) \bigcap H^1(I_y))\} \bigcap L_0^2(\Omega).$$

In addition, let

$$Z_{N,h}^k(\Omega) = \{\mathcal{P}_{N-2}(I_x) \otimes (\tilde{S}_h^k(I_y) \bigcap H^1(I_y))\} \bigcap L_0^2(\Omega).$$

We denote by P_N^0 the $L_\omega^2(I_x)-$ orthogonal projection from $L_\omega^2(I_x)$ onto $V_N(I_x)$, Π_h^k is the piecewise Lagrange interpolation of order $k \ge 1$, from $C(\bar{I}_y)$ onto $\tilde{S}_h^k(I_y) \bigcap H^1(I_y)$. Furthermore let $P_{N,h} : L_\omega^2(\Omega) \to X_{N,h}^k(\Omega)$ be the orthogonal projection, i.e., for any $v \in L_\omega^2(\Omega)$, the projection $P_{N,h}v \in X_{N,h}^k(\Omega)$ and

$$(v - P_{N,h}v, u)_\omega = 0, \quad \forall u \in X_{N,h}^k(\Omega).$$

Let τ be the mesh size in time t and $S_\tau = \{t = l\tau \ / \ 0 \le l \le [\frac{T}{\tau}]\}$. Let

$$u_t(t) = \frac{1}{\tau}(u(t+\tau) - u(t)).$$

A fully discrete Chebyshev spectral-finite element scheme for (2.1) is to find the pair $(u(t), p(t)) \in (X^k_{N,h}(\Omega))^2 \times Y^k_{N,h}(\Omega)$ for all $t \in S_\tau$ such that

$$\begin{cases} (u_t, v)_\omega + (\partial_x(u_1 u), v)_\omega + (\partial_y(u_2 u), v)_\omega + \nu a_\omega(u + \sigma\tau u_t, v) + (\nabla p, v)_\omega \\ \qquad = (f, v)_\omega, \qquad\qquad \forall\, v \in (X^k_{N,h}(\Omega))^2, \\ a_\omega(p, v) = (\Phi(u) - \nabla\cdot f, v)_\omega, \qquad \forall\, v \in Z^k_{N,h}(\Omega), \\ u(0) = P_{N,h}U_0, \end{cases} \tag{2.2}$$

where σ is a parameter, $0 \le \sigma \le 1$, and

$$a_\omega(u, v) = \int_\Omega \nabla u(x,y)\cdot\nabla(\omega(x)v(x,y))\,\mathrm{d}x\mathrm{d}y.$$

We now give some numerical results. Take the test functions

$$U = (Ae^{Bt}(x^2-1)^2 y(y-1)(2y-1), -2Ae^{Bt}(x^3-x)y^2(y-1)^2),$$

$$P = 4Ae^{2Bt}(x^3-3x)(2y^3-3y^2+0.5).$$

We use the scheme CSFM, i.e., Scheme (2.2) with $k = 1$, in which the interval I_y is uniformly partitioned with the mesh size $h = 1/M$. For comparison, we also consider the bilinear finite element scheme FEM. In this case, the domain is divided uniformly into rectangular subdomains with the length $h_x = 2/N$ in x-direction and $h_y = 1/M$ in y-direction. For describing the errors of numerical solutions, let

$$\begin{aligned} \hat I_x &= \{x_j /\, x_j = \cos(j\pi/N),\ 0 \le j \le N\}, \quad \text{for Scheme CSFM}, \\ \hat I_x &= \{x_j /\, x_j = -1 + jh_x,\ 0 \le j \le N\}, \quad \text{for Scheme FEM}, \\ \hat I_y &= \{y_j /\, y_j = jh_y,\ 0 \le j \le M\}, \quad \text{for Scheme CSFM and FEM} \end{aligned}$$

and

$$E(U(t)) = \left(\frac{\sum_{i=1}^{2}\sum_{x\in\hat I_x}\sum_{y\in\hat I_y} |\, u_i(x,y,t) - U_i(x,y,t)\,|^2}{\sum_{i=1}^{2}\sum_{x\in\hat I_x}\sum_{y\in\hat I_y} |\, U_i(x,y,t)\,|^2} \right)^{\frac12},$$

$$E(P(t)) = \left(\frac{\sum_{x\in\hat I_x}\sum_{y\in\hat I_y} |\, p(x,y,t) - P(x,y,t)\,|^2}{\sum_{x\in\hat I_x}\sum_{y\in\hat I_y} |\, P(x,y,t)\,|^2} \right)^{\frac12}.$$

The numerical results are shown in Tables I. Clearly Scheme CSFM gives better results than Scheme FEM. In particular, the spectral approach used in x-direction has higher accuracy and so we only need a relatively small N to resolve the solutions.

Table I. $A = 0.2,\ B = 0.1,\ \tau = 0.005,\ \sigma = 0,\ \nu = 0.0001.$

	Scheme CSFM, $M = 10,\ N = 4$		Scheme FEM, $M = N = 10$	
t	$E(U(t))$	$E(P(t))$	$E(U(t))$	$E(P(t))$
0.5	0.1919E-2	0.1902E-2	0.3132E-2	0.9178E-2
1.0	0.3735E-2	0.2101E-2	0.5945E-2	0.9614E-2
1.5	0.5446E-2	0.2322E-2	0.8512E-2	0.1020E-2
2.0	0.7065E-2	0.2567E-2	0.1088E-1	0.1108E-1
2.5	0.8570E-2	0.2836E-2	0.1306E-1	0.1169E-1

3. Some Lemmas

We first introduce some Sobolev spaces. For integer $r \geq 0$, set

$$| v |_{r,\omega,I_x} = \left\| \frac{d^r v}{dx^r} \right\|_{\omega,I_x}, \qquad \|v\|_{r,\omega,I_x} = \left(\sum_{m=0}^{r} | v |^2_{m,\omega,I_x} \right)^{\frac{1}{2}},$$

$$H^r_\omega(I_x) = \{ v \ / \ \|v\|_{r,\omega,I_x} < \infty \}.$$

For real $r \geq 0$, $H^r_\omega(I_x)$ is defined by the complex interpolation between the spaces $H^{[r]}_\omega(I_x)$ and $H^{[r+1]}_\omega(I_x)$. Next, let B be a Banach space with the norm $\|\cdot\|_B$. Define

$$\begin{aligned} L^2(\mathcal{D},B) &= \left\{ v(z) : \mathcal{D} \to B \ / \ v \ \text{ is strongly measurable, } \ \|v\|_{L^2(\mathcal{D},B)} < \infty \right\}, \\ C(\mathcal{D},B) &= \left\{ v(z) : \mathcal{D} \to B \ / \ v \ \text{ is strongly continuous, } \ |||v|||_B < \infty \right\} \end{aligned}$$

where

$$\|v\|_{L^2(\mathcal{D},B)} = \left(\int_{\mathcal{D}} \|v(z)\|^2_B \, \mathrm{d}z \right)^{\frac{1}{2}}, \qquad |||v|||_B = \max_{z \in \mathcal{D}} \|v(z)\|_B.$$

Moreover for all integer $\mu \geq 0$, define

$$H^\mu(\mathcal{D},B) = \left\{ v(z) \in L^2(\mathcal{D},B) \ / \ \|v\|_{H^\mu(\mathcal{D},B)} < \infty \right\}$$

with the norm

$$\|v\|_{H^\mu(\mathcal{D},B)} = \left(\sum_{k=0}^{\mu} \| \frac{\partial^k v}{\partial z^k} \|^2_{L^2(D,B)} \right)^{\frac{1}{2}}.$$

For real $\mu \geq 0$, we define the space $H^\mu(\mathcal{D},B)$ by the complex interpolation as before.

We also introduce some non-isotropic spaces. Let

$$H^{r,s}_\omega(\Omega) = L^2(I_y, H^r_\omega(I_x)) \bigcap H^s(I_y, L^2_\omega(I_x)), \qquad r,s \geq 0$$

equipped with

$$\|v\|_{H^{r,s}_\omega(\Omega)} = \left(\|v\|^2_{L^2(I_y,H^r_\omega(I_x))} + \|v\|^2_{H^s(I_y,L^2_\omega(I_x))} \right)^{\frac{1}{2}}.$$

Also let

$$\begin{aligned} M^{r,s}_\omega(\Omega) &= H^{r,s}_\omega(\Omega) \bigcap H^{s-1}(I_y, H^1_\omega(I_x)), \quad r \geq 0, \ s \geq 1, \\ A^{r,s}_\omega(\Omega) &= H^{\frac{7}{6}}(I_y, H^1_\omega(I_x)) \bigcap H^s(I_y, H^{r+1}_\omega(I_x)) \bigcap H^{s+1}(I_y, H^r_\omega(I_x)), \quad r,s \geq 0. \end{aligned}$$

Their norms are defined in the way similar to $\|\cdot\|_{H^{r,s}_\omega(\Omega)}$. Furthermore, let $H^{r,s}_{0,\omega}(\Omega)$ and $M^{r,s}_{0,\omega}(\Omega)$ be the closures of $C^\infty_0(\Omega)$ in the spaces $H^{r,s}_\omega(\Omega)$ and $M^{r,s}_\omega(\Omega)$ respectively. For simplicity,let $H^r_{0,\omega}(\Omega) = H^{r,r}_{0,\omega}(\Omega)$ and $\|\cdot\|_{H^{r,r}_\omega(\Omega)} = \|\cdot\|_{r,\omega}$. Besides, we denote by $L^\infty(I_x), L^\infty(\Omega)$ and $W^{1,\infty}(\Omega)$ the usual Sobolev spaces with the norms $\|\cdot\|_{\infty,I_x}, \|\cdot\|_\infty$ and $\|\cdot\|_{1,\infty}$ respectively. The corresponding semi-norms are denoted by $|\cdot|_{\infty,I_x}, |\cdot|_\infty$ and $|\cdot|_{1,\infty}$, etc..

Denote by c a generic positive constant independent of N, h, τ and any function. Let $\bar{s} = \min(s, k+1)$. For some lemmas, we require that there exist some suitably big and positive constants c_1 and c_2 independent of N, h, τ and any function such that

$$c_1 h^{-\frac{4}{3}} \leq N \leq c_2 h^{-\frac{4}{3}}. \tag{3.1}$$

Lemma 1. *If $v(x,y,t) \in L^2_\omega(\Omega)$ for $t \in S_\tau$, then*

$$2(v(t), v_t(t))_\omega = (\|v(t)\|^2_\omega)_t - \tau \|v_t(t)\|^2_\omega.$$

Lemma 2. *(Lemma 1 of [6]) For any $u, v \in H^1_{0,\omega}(\Omega)$, we have*

$$a_\omega(v,v) \geq \frac{1}{4}\|v\|^2_{1,\omega}, \qquad | a_\omega(u,v) | \leq 2 \mid u \mid_{1,\omega} \mid v \mid_{1,\omega} .$$

Lemma 3. *(Lemma 2 of [6]) Let $v \in H^{r,1}_{0,\omega}(\Omega) \bigcap H^{r,s}_\omega(\Omega)$ with $0 \leq r \leq 1$ and $s \geq 0$, or $v \in H^1_{0,\omega}(\Omega) \bigcap H^{r,s}_\omega(\Omega)$ with $r > 1$ and $s \geq 0$. Then*

$$\|v - P_{N,h}v\|_\omega \leq c(N^{-r} + h^{\bar{s}})\|v\|_{H^{r,\bar{s}}_\omega(\Omega)}.$$

In order to obtain better error estimation, we introduce the projection $P^*_{N,h} : (H^1_{0,\omega}(\Omega))^2 \to (X^k_{N,h}(\Omega))^2$, i.e., for any $v \in H^1_{0,\omega}(\Omega)$,

$$a_\omega(v - P^*_{N,h}v, u) = 0, \quad \forall u \in (X^k_{N,h}(\Omega))^2.$$

Lemma 4. *Let (3.1) hold and $v \in H^1_{0,\omega}(\Omega) \bigcap M^{r,s}_\omega(\Omega)$ with $r, s \geq 1$. Then*

$$\|v - P^*_{N,h}v\|_{1,\omega} \leq c(N^{1-r} + h^{\bar{s}-1})\|v\|_{M^{r,\bar{s}}_\omega(\Omega)}.$$

If $v \in H^1_{0,\omega}(\Omega) \bigcap M^{r+\frac{1}{4},s}_\omega(\Omega)$, then

$$\|v - P^*_{N,h}v\|_\omega \leq c(N^{-r} + h^{\bar{s}})\|v\|_{M^{r+\frac{1}{4},\bar{s}}_\omega(\Omega)}.$$

Proof. The first conclusion comes from Lemma 3 of [6]. By (3.1) and the means of the duality, we find out that

$$\begin{aligned}\|v - P^*_{N,h}v\|_\omega &\leq c(N^{-1} + h)\|v - P^*_{N,h}v\|_{1,\omega} \leq c(N^{-1} + h)(N^{\frac{3}{4}-r} + h^{\bar{s}-1})\|v\|_{M^{r+\frac{1}{4},\bar{s}}_\omega(\Omega)} \\ &\leq c(N^{-r} + h^{\bar{s}})\|v\|_{M^{r+\frac{1}{4},\bar{s}}_\omega(\Omega)}.\end{aligned}$$

We now denote by P_N the truncated Chebyshev projection. Let $W^{m,q}_\omega(I_x)$ be the Sobolev space with the weight $\omega(x)$. We know from (9.5.7) of [4] that for any $u \in W^{m,q}_\omega(I_x)$ with $m \geq 0$ and $1 \leq q \leq \infty$,

$$\|u - P_N u\|_{L^q_\omega(I_x)} \leq c\sigma_{N,q}N^{-m}\|u\|_{W^{m,q}_\omega(I_x)}, \tag{3.2}$$

$$\sigma_{N,q} = \begin{cases} 1 + \ln N, & \text{if } q = 1 \text{ or } q = \infty, \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 5. *(Lemma 4 of [6]). If $v \in H^r_\omega(I_x)$ with $r > \frac{1}{2}$, then*

$$\|P_N v\|_{\infty,I_x} \leq c\|v\|_{r,\omega,I_x}.$$

Lemma 6. *(Lemma 5 of [6]) For any $v \in \mathcal{P}_N(I_x) \otimes \tilde{S}^k_h(I_y)$,*

$$\|v\|_\infty \leq c\sqrt{\frac{N}{h}}\|v\|_\omega.$$

Moreover if in addition $v \in H^1_\omega(\Omega)$, then

$$\|P_N v\|_\infty \leq c(\ln N)^{\frac{1}{2}}\|v\|_{1,\omega}.$$

Remark 1. For any $v \in \tilde{X}^k_{N,h}(\Omega) \bigcap H^1_\omega(\Omega)$,

$$\|v\|_\infty \leq c(\ln N)^{\frac{1}{2}}\|v\|_{1,\omega}.$$

To estimate $\|P^*_{N,h}v\|_{1,\infty}$, we introduce the operator $P^1_N : H^1_{0,\omega}(I_x) \to V_N(I_x)$ such that for any $u \in H^1_{0,\omega}(I_x)$,

$$\int_{-1}^1 \partial_x(u - P^1_N u)\partial_x(z\omega)\,\mathrm{d}x = 0, \qquad \forall z \in V_N(I_x).$$

Also let $P_h^*: H_0^1(I_y) \to S_h^k(I_y)$ such that for any $u \in H_0^1(I_y)$,

$$\int_0^1 \partial_y(u - P_h^* u)\partial_y z \, \mathrm{d}y = 0, \quad \forall z \in S_h^k(I_y).$$

By (9.5.17) of [4] and the interpolation of spaces, for any $u \in H^1_{0,\omega}(I_x) \bigcap H^r_\omega(I_x)$ with $r \geq 1$,

$$\|u - P_N^1 u\|_{\mu,\omega,I_x} \leq cN^{\mu - r}\|u\|_{r,\omega,I_x}, \quad 0 \leq \mu \leq 1. \tag{3.3}$$

Also it can be verified as in [8] that for any $u \in H_0^1(I_y) \bigcap H^s(I_y)$ with $s \geq 1$,

$$\|u - P_h^* u\|_{H^\mu(I_y)} \leq ch^{\bar{s} - \mu}\|u\|_{H^{\bar{s}}(I_y)}, \quad 0 \leq \mu \leq 1. \tag{3.4}$$

Lemma 7. *Let (3.1) hold. Then for any* $v \in H^1_{0,\omega}(\Omega) \bigcap M_\omega^{\frac{9}{8},\frac{7}{6}}(\Omega) \bigcap H^s(I_y, H^r_\omega(I_x))$ *with* $r, s > \frac{1}{2}$,

$$\|P^*_{N,h} v\|_\infty \leq c\|v\|_{M_\omega^{\frac{9}{8},\frac{7}{6}}(\Omega) \bigcap H^s(I_y, H^r_\omega(I_x))}.$$

If in addition $v \in A^{r,s}_\omega(\Omega)$ *and* $r, s > \frac{1}{2}$, *then*

$$\|P^*_{N,h} v\|_{1,\infty} \leq c\|v\|_{A^{r,s}_\omega(\Omega)}.$$

Proof. We have

$$\|P^*_{N,h} v\|_\infty \leq \|P^*_{N,h} v - \Pi_h^k P_N v\|_\infty + \|\Pi_h^k P_N v\|_\infty.$$

By (3.1), Lemma 4, Lemma 6 and Theorem 3.2.1 of [8],

$$\begin{aligned}
&\|P^*_{N,h} v - \Pi_h^k P_N v\|_\infty \leq c\sqrt{\tfrac{N}{h}}\|P^*_{N,h} v - \Pi_h^k P_N v\|_\omega \\
&\leq c\sqrt{\tfrac{N}{h}}(\|P^*_{N,h} v - v\|_\omega + \|v - \Pi_h^k P_N v\|_\omega) \leq c\|v\|_{M_\omega^{\frac{9}{8},\frac{7}{6}}(\Omega)}.
\end{aligned}$$

By embedding theory ,Lemma 5 and Theorem 3.1.5 of [8],

$$\|\Pi_h^k P_N v\|_\infty \leq \|P_N v\|_{H^s(I_y, C(I_x))} \leq c\|v\|_{H^s(I_y, H^r_\omega(I_x))}.$$

Then the first conclusion follows. We now prove the second one. Clearly $u_* \equiv P^*_{N,h} v - P_h^* P_N^1 v \in X^k_{N,h}(\Omega)$. By Lemma 2 and the definitions of P_h^* and P_N^1, we have

$$\begin{aligned}
\tfrac{1}{4}\|u_*\|^2_{1,\omega} &\leq a_\omega(u_*, u_*) = a_\omega(v - P_h^* P_N^1 v, u_*) \\
&= a_\omega(v - P_N^1 v, u_*) + a_\omega(P_N^1(v - P_h^* v), u_*) \\
&\leq (\|\partial_y v - P_N^1(\partial_y v)\|_\omega + \|\partial_x v - P_h^*(\partial_x v)\|_\omega)\|u_*\|_{1,\omega}.
\end{aligned}$$

Hence

$$\|u_*\|_{1,\omega} \leq c(\|\partial_y v - P_N^1(\partial_y v)\|_\omega + \|\partial_x v - P_h^*(\partial_x v)\|_\omega). \tag{3.5}$$

Furthermore, we have by (3.1),(3.3)–(3.5) and Lemma 6 that

$$\|P^*_{N,h} v - P_h^* P_N^1 v\|_{1,\infty} \leq \sqrt{\frac{N}{h}}\|P^*_{N,h} v - P_h^* P_N^1 v\|_{1,\omega} \leq c\|v\|_{H^{\frac{7}{6}}(I_y, H^1_\omega(I_x))}.$$

On the other hand, we have from (3.1),(3.3),(3.4), Lemma 5, Lemma 6, Theorem 3.2.6 of [8] and (9.5.3) of [4] that

$$\begin{aligned}
&\|\partial_x P_h^* P_N^1 v\|_\infty \leq \|P_h^*(\partial_x P_N^1 v) - \Pi_h^k(\partial_x P_N^1 v)\|_\infty + \|\Pi_h^k(\partial_x P_N^1 v - P_N \partial_x v)\|_\infty + \|\Pi_h^k P_N \partial_x v\|_\infty \\
&\leq c\sqrt{\tfrac{N}{h}}\|P_h^*(\partial_x P_N^1 v) - \Pi_h^k(\partial_x P_N^1 v)\|_\omega + c\|\partial_x P_N^1 v - P_N \partial_x v\|_{H^s(I_y, C(I_x))} + c\|v\|_{H^s(I_y, H^{r+1}_\omega(I_x))} \\
&\leq c\|v\|_{H^{\frac{7}{6}}(I_y, H^1_\omega(I_x))} + c\sqrt{N}(\|\partial_x P_N^1 v - \partial_x v\|_{H^s(I_y, L^2_\omega(I_x))} + \|\partial_x v - P_N \partial_x v\|_{H^s(I_y, L^2_\omega(I_x))}) \\
&\qquad + c\|v\|_{H^s(I_y, H^{r+1}_\omega(I_x))} \\
&\leq \|v\|_{H^{\frac{7}{6}}(I_y, H^1_\omega(I_x)) \bigcap H^s(I_y, H^{r+1}_\omega(I_x))}
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_y P_h^* P_N^1 v\|_\infty \le \|\partial_y P_h^* P_N^1 v - \partial_y P_h^* P_N v\|_\infty + \|\partial_y P_h^* P_N v - \Pi_h^k P_N \partial_y v\|_\infty + \|\Pi_h^k P_N \partial_y v\|_\infty \\
&\le c\sqrt{\tfrac{N}{h}}\|\partial_y P_h^* (P_N^1 v - P_N v)\|_\omega + \tfrac{c}{\sqrt{h}}(\|\partial_y P_h^* P_N v - P_N \partial_y v\|_{L^2(I_y,C(I_x))} \\
&\qquad + \|P_N \partial_y v - \Pi_N^k P_N \partial_y v\|_{L^2(I_y,C(I_x))}) + c\|v\|_{H^{s+1}(I_y,H_\omega^r(I_x))} \\
&\le c\sqrt{\tfrac{N}{h}}\|P_N^1 v - P_N v\|_{H^1(I_y,L_\omega^2(I_x))} + c\|v\|_{H^{s+1}(I_y,H_\omega^r(I_x))} \\
&\le c\|v\|_{H^1(I_y,H_\omega^1(I_x))\bigcap H^{s+1}(I_y,H_\omega^r(I_x))}.
\end{aligned}$$

Finally, the above statements lead to that $\mid P_{N,h}^* v \mid_{1,\infty} \le c\|v\|_{A_\omega^{r,s}(\Omega)}$.

Lemma 8. *There exists a positive constant c_d depending only on the value of d, such that for all $v \in \mathcal{P}_N(I_x) \otimes (H^1(I_y) \bigcap \tilde{S}_h^k(I_y))$,*

$$\mid v \mid_{1,\omega}^2 \le (2N^4 + c_d h^{-2})\|v\|_\omega^2.$$

Proof. Let u_q and $u_q^{(1)}$ be the coefficients of Chebyshev expansions of $u \in \mathcal{P}_N$ and $\frac{du}{dx}$ respectively. By (2.4.22) of [4],

$$c_q u_q^{(1)} = 2 \sum_{\substack{l=q+1 \\ l+q \text{ odd}}}^{N} l u_l, \quad c_0 = 2, c_q = 1 \text{ for } q \ge 1,$$

and so

$$c_q (u_q^{(1)})^2 \le \frac{4}{c_q} \sum_{\substack{l=q+1 \\ l+q \text{ odd}}}^{N} \frac{l^2}{c_l} \sum_{\substack{l=q+1 \\ l+q \text{ odd}}}^{N} c_l u_l^2 \le 2N^3 \sum_{l=0}^{N} c_l u_l^2.$$

Thus for any $v \in \mathcal{P}(I_x) \otimes (\tilde{S}_h^k(I_y) \bigcap H^1(I_y))$, $\|\partial_x v\|_\omega^2 \le 2N^4 \|v\|_\omega^2$. By (3.2.30) of [8],

$$\|\partial_y v\|_\omega^2 \le c_d h^{-2} \int_{-1}^{1} \|v\|_{L^2(I_y)}^2 \omega \,\mathrm{d}x \le c_d h^{-2} \|v\|_\omega^2.$$

For real $s \ge 0$, let

$$\|v\|_{H^{-s}(\Omega)} = \sup_{u \in H^s(\Omega)} \frac{\mid (v,u)_{L^2(\Omega)} \mid}{\|u\|_{H^s(\Omega)}}.$$

Lemma 9. *Let (3.1) hold. If $v \in Y_{N,h}^k(\Omega)$ and $g \in Z_{N,h}^k(\Omega)$ satisfy*

$$a_\omega(v,u) = (g,u)_\omega, \qquad \forall u \in Z_{N,h}^k(\Omega), \tag{3.6}$$

then

$$\|v\|_{1,\omega} \le \|g\|_{H^{-\frac{3}{4}}(\Omega)}.$$

Proof. Let $\{\varphi_l(y) \,/\, l = 1, 2, \cdots, M_h'\}$ be the normalized $L^2(I_y)-$ orthogonal base of $\tilde{S}_h^k(I_y) \bigcap H^1(I_y)$. We can set $\varphi_1(y) \equiv 1$. For any $z(x) \in \mathcal{P}_{N-2}(I_x) \bigcap L_0^2(I_x)$, we have from (4.6) that

$$-\int_\Omega \frac{\partial^2 v}{\partial x^2} z\omega \,\mathrm{d}x\mathrm{d}y = \int_\Omega g z \omega \,\mathrm{d}x\mathrm{d}y.$$

Since both $\int_0^1 \frac{\partial^2 v}{\partial x^2}\,\mathrm{d}y$ and $\int_0^1 g\,\mathrm{d}y$ are in the space $\mathcal{P}_{N-2}(I_x) \bigcap L_0^2(I_x)$, the above equation implies

$$-\int_0^1 \frac{\partial^2 v}{\partial x^2}\,\mathrm{d}y = \int_0^1 g\,\mathrm{d}y. \tag{3.7}$$

Similarly,by taking $u(x,y) = z(x)\varphi_l(y)$ for any $z(x) \in \mathcal{P}_{N-2}(I_x)$ $(l = 2, 3, \cdots, M_h')$ in (3.6), we get

$$-\int_0^1 \frac{\partial^2 v}{\partial x^2}\varphi_l\,\mathrm{d}y + \int_0^1 (P_{N-2}\frac{\partial v}{\partial y})\frac{\partial \varphi_l}{\partial y}\,\mathrm{d}y = \int_0^1 g\varphi_l\,\mathrm{d}y. \tag{3.8}$$

Then (3.7) and (3.8) lead to

$$-\int_\Omega \frac{\partial^2 v}{\partial x^2} u\,\mathrm{d}x\mathrm{d}y + \int_\Omega (P_{N-2}\frac{\partial v}{\partial y})\frac{\partial u}{\partial y}\,\mathrm{d}x\mathrm{d}y = \int_\Omega gu\,\mathrm{d}x\mathrm{d}y, \quad \forall u \in \mathcal{P}_N(I_x)\otimes S_h^k(I_y). \tag{3.9}$$

By taking $u = v$ in (3.9) and integrating by parts, we find that

$$|\, v\,|^2_{H^1(\Omega)} + (P_{N-2}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}, \frac{\partial v}{\partial y})_{L^2(\Omega)} = (g, v)_{L^2(\Omega)}. \tag{3.10}$$

By the fact that $H^{\frac14}(I_x) \hookrightarrow L^2_\omega(I_x)$(Theorem 4.1 of [9]),(3.2) and an inverse inequality (Theorem 3.2.6 of [8]),we have

$$\begin{aligned} &|\,(P_{N-2}\tfrac{\partial v}{\partial y} - \tfrac{\partial v}{\partial y}, \tfrac{\partial v}{\partial y})_{L^2(\Omega)}\,| \le cN^{-\frac34}\|\tfrac{\partial v}{\partial y}\|_{L^2(I_y,H_\omega^{\frac34}(I_x))}\|\tfrac{\partial v}{\partial y}\|_{L^2(\Omega)} \\ &\le ch^{-2}N^{-\frac32}\|\tfrac{\partial v}{\partial x}\|^2_{L^2(\Omega)} + (\tfrac12 + cN^{-\frac34})\|\tfrac{\partial v}{\partial y}\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.11}$$

Thus we have from (3.1),(3.10) and (3.11) that

$$|\, v\,|^2_{H^1(\Omega)} \le c\,|\,(g,v)_{L^2(\Omega)}\,|\,.$$

On the other hand, $v \in Y^k_{N,h}(\Omega) \subset L^2_0(\Omega)$, and so by Poincaré inequality, we have $\|v\|_{L^2(\Omega)} \le c\,|\,v\,|_{H^1(\Omega)}$. Therefore

$$\|v\|_{H^1(\Omega)} \le c\|g\|_{H^{-1}(\Omega)}. \tag{3.12}$$

Similarly, by taking $u = -\frac{\partial^2 v}{\partial x^2}$ in (3.9), we get

$$\|v\|_{L^2(I_y,H^2(I_x))\bigcap H^1(I_y,H^1(I_x))} \le c\|g\|_{L^2(\Omega)}. \tag{3.13}$$

Let $B^r(\Omega) = L^2(I_y, H^r(I_x)) \bigcap H^1(I_y, H^{r-1}(I_x)),\ \ 1 \le r \le 2$. Then (3.12) and (3.13) imply that $\|v\|_{B^r(\Omega)} \le c\|g\|_{H^{r-2}(\Omega)}$ for $r = 1, 2$. Therefore, by the interpolation of spaces, $\|v\|_{B^{1+\theta}(\Omega)} \le c\|g\|_{H^{-1+\theta}(\Omega)}$ for $0 \le \theta \le 1$. Taking $\theta = \frac14$, we get

$$\|v\|_{L^2(I_y,H^{\frac54}(I_x))\bigcap H^1(I_y,H^{\frac14}(I_x))} \le c\|g\|_{H^{-\frac34}(\Omega)}.$$

Finally, we obtain from Theorem 4.1 of [9] that

$$\|v\|_{1,\omega} \le \|v\|_{L^2(I_y,H^{\frac54}(I_x))\bigcap H^1(I_y,H^{\frac14}(I_x))} \le c\|g\|_{H^{-\frac34}(\Omega)}.$$

4. Error Estimations

We first analyze the generalized stability of (2.2). Assume that $u(0)$ and $f(t)$ have the errors $\tilde u(0)$ and $\tilde f(t)$, which induce the errors of $u(t)$ and $p(t)$, denoted by $\tilde u(t)$ and $\tilde p(t)$ respectively. They satisfy

$$\begin{cases} (\tilde u_t, v)_\omega + (\partial_x(\tilde u_1 u + u_1\tilde u + \tilde u_1\tilde u), v)_\omega + (\partial_y(\tilde u_2 u + u_2\tilde u + \tilde u_2\tilde u), v)_\omega \\ \ + (\nabla\tilde p, v)_\omega + \nu a_\omega(\tilde u + \sigma\tau\tilde u_t, v) = (\tilde f, v)_\omega, \quad \forall v \in (X^k_{N,h}(\Omega))^2, \\ a_\omega(\tilde p, v) = (\Phi(\tilde u) + \Phi^*(u,\tilde u) - \nabla\cdot\tilde f, v)_\omega, \quad \forall v \in Z^k_{N,h}(\Omega) \end{cases} \tag{4.1}$$

where

$$\Phi^*(u,\tilde u) = 2(\partial_y u_1\partial_x\tilde u_2 + \partial_y\tilde u_1\partial_x u_2 - \partial_x u_1\partial_y\tilde u_2 - \partial_x\tilde u_1\partial_y u_2).$$

Let $\varepsilon > 0$, and m be an undetermined positive constant. By taking $v = 2\tilde u(t) + m\tau\tilde u_t(t)$ in the first formula of (4.1), we have from Lemma 1 and Lemma 2 that

$$\begin{aligned} &(\|\tilde u\|^2_\omega)_t + \tau(m-1-\varepsilon)\|\tilde u_t\|^2_\omega + \tfrac{\nu}{2}\|\tilde u\|^2_{1,\omega} + \tfrac{\nu\sigma m\tau^2}{4}\|\tilde u_t\|^2_{1,\omega} + 2\nu\sigma\tau a_\omega(\tilde u_t,\tilde u) \\ &+\nu m\tau a_\omega(\tilde u,\tilde u_t) + \sum_{j=1}^6 F_j \le \|\tilde u\|^2_\omega + (1 + \tfrac{\tau m^2}{4\varepsilon})\|\,\tilde f\|^2_\omega \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} F_1 &= 2(\partial_x(\tilde{u}_1 u + u_1\tilde{u}) + \partial_y(\tilde{u}_2 u + u_2\tilde{u}), \tilde{u})_\omega, \\ F_2 &= m\tau(\partial_x(\tilde{u}_1 u + u_1\tilde{u}) + \partial_y(\tilde{u}_2 u + u_2\tilde{u}), \tilde{u}_t)_\omega, \\ F_3 &= 2(\partial_x(\tilde{u}_1\tilde{u}) + \partial_y(\tilde{u}_2\tilde{u}), \tilde{u})_\omega, \qquad F_4 = m\tau(\partial_x(\tilde{u}_1\tilde{u}) + \partial_y(\tilde{u}_2\tilde{u}), \tilde{u}_t)_\omega, \\ F_5 &= 2(\nabla\tilde{p}, \tilde{u})_\omega, \qquad F_6 = m\tau(\nabla\tilde{p}, \tilde{u}_t)_\omega. \end{aligned}$$

Obviously

$$2\nu\sigma\tau a_\omega(\tilde{u}_t, \tilde{u}) = A_1 + A_2, \quad \nu m\tau a_\omega(\tilde{u}, \tilde{u}_t) = B_1 + B_2$$

where

$$\begin{aligned} A_1 &= 2\nu\sigma\tau(\nabla\tilde{u}_t, \nabla\tilde{u})_\omega, \qquad A_2 = 2\nu\sigma\tau(\partial_x\tilde{u}_t, x\omega^2\tilde{u})_\omega, \\ B_1 &= \nu m\tau(\nabla\tilde{u}, \nabla\tilde{u}_t)_\omega, \qquad B_2 = \nu m\tau(\partial_x\tilde{u}, x\omega^2\tilde{u}_t)_\omega. \end{aligned}$$

It is easy to verify that

$$A_1 + B_1 = \nu\tau(\sigma + \frac{m}{2})\left[(|\ \tilde{u}\ |_{1,\omega}^2)_t - \tau\ |\ \tilde{u}_t\ |_{1,\omega}^2\right].$$

We have that(see [5])

$$\|\omega^2 v\|_{\omega,I_x} \leq |\ v\ |_{1,\omega,I_x}, \qquad \forall v \in H_{0,\omega}^1(I_x). \tag{4.3}$$

Hence

$$\begin{aligned} |\ A_2\ | &\leq 2\nu\sigma\tau\|\partial_x\tilde{u}_t\|_\omega\|\partial_x\tilde{u}\|_\omega \leq \frac{\nu\sigma}{4}\|\partial_x\tilde{u}\|_\omega^2 + 4\nu\sigma\tau^2\|\partial_x\tilde{u}_t\|_\omega^2, \\ |\ B_2\ | &\leq \nu m\tau\|\partial_x\tilde{u}\|_\omega\|\partial_x\tilde{u}_t\|_\omega \leq \frac{\nu m}{8}\|\partial_x\tilde{u}\|_\omega^2 + 2\nu m\tau^2\|\partial_x\tilde{u}_t\|_\omega^2. \end{aligned}$$

Thus (4.2) reads

$$\begin{aligned} &(\|\tilde{u}\|_\omega^2)_t + \tau(m-1-\varepsilon)\|\tilde{u}_t\|_\omega^2 + \tfrac{\nu}{8}(4-m-2\sigma)\ |\ \tilde{u}\ |_{1,\omega}^2 \\ &+\nu\tau(\sigma+\tfrac{m}{2})(|\ \tilde{u}\ |_\omega^2)_t + \tfrac{\nu\sigma m\tau^2}{4}\|\tilde{u}_t\|_{1,\omega}^2 - 5\nu\tau^2(\sigma+\tfrac{m}{2})\ |\ \tilde{u}_t\ |_{1,\omega}^2 \\ &+\sum_{j=1}^{6} F_j \leq \|\tilde{u}\|_\omega^2 + (1+\tfrac{\tau m^2}{4\varepsilon})\|\ \tilde{f}\|_\omega^2. \end{aligned} \tag{4.4}$$

We now turn to estimate $|\ F_j\ |$. By integrating by parts and (4.3),

$$\begin{aligned} |\ F_1\ | &\leq c\|u\|_\infty\|\tilde{u}\|_\omega\ |\ \tilde{u}\ |_{1,\omega} \leq \frac{\varepsilon\nu}{8}\ |\ \tilde{u}\ |_{1,\omega}^2 + \frac{c}{\varepsilon\nu}\|u\|_\infty^2\|\tilde{u}\|_\omega^2, \\ |\ F_2\ | &\leq cm\tau\|u\|_\infty\|\tilde{u}\|_\omega\ |\ \tilde{u}_t\ |_{1,\omega} \leq \frac{\varepsilon\nu m\tau^2}{6}\ |\ \tilde{u}_t\ |_{1,\omega}^2 + \frac{cm}{\varepsilon\nu}\|u\|_\infty^2\|\tilde{u}\|_\omega^2. \end{aligned}$$

By Lemma 6,

$$\begin{aligned} |\ F_3\ | &\leq c\|\tilde{u}\|_\infty\|\tilde{u}\|_\omega\ |\ \tilde{u}\ |_{1,\omega} \leq \frac{\varepsilon\nu}{8}\ |\ \tilde{u}\ |_{1,\omega}^2 + \frac{c\ln N}{\varepsilon\nu}\|\tilde{u}\|_\omega^2\ |\ \tilde{u}\ |_{1,\omega}^2, \\ |\ F_4\ | &\leq cm\tau\|\tilde{u}\|_\infty\|\tilde{u}\|_\omega\ |\ \tilde{u}_t\ |_{1,\omega} \leq \frac{\varepsilon\nu m\tau^2}{6}\ |\ \tilde{u}_t\ |_{1,\omega}^2 + \frac{cm\ln N}{\varepsilon\nu}\|\tilde{u}\|_\omega^2\ |\ \tilde{u}\ |_{1,\omega}^2\ . \end{aligned}$$

By applying Lemma 9 to the second formula of (4.1), we have that

$$|\ \tilde{p}\ |_{1,\omega} \leq c\left(\|\nabla\cdot\tilde{f}\|_{H^{-\frac{3}{4}}(\Omega)} + \|\Phi(\tilde{u})\|_{H^{-\frac{3}{4}}(\Omega)} + \|\Phi(u,\tilde{u})\|_{H^{-\frac{3}{4}}(\Omega)}\right).$$

Obviously

$$\|\Phi(u,\tilde{u})\|_{H^{-\frac{3}{4}}(\Omega)} \leq c\|\Phi^*(u,\tilde{u})\|_\omega \leq c\ |\ u\ |_{1,\infty}|\ \tilde{u}\ |_{1,\omega}\ .$$

By Lemma 6,

$$\|\Phi(\tilde{u})\|_{L^2(\Omega)} \leq \|\Phi(\tilde{u})\|_\omega \leq c\ |\ \tilde{u}\ |_{1,\infty}|\ \tilde{u}\ |_{1,\omega} \leq c\sqrt{\frac{N}{h}}\ |\ \tilde{u}\ |_{1,\omega}^2\ . \tag{4.5}$$

On the other hand, we have that

$$\begin{aligned}\|\Phi(\tilde u)\|_{H^{-1}(\Omega)} &= \sup_{v\in H^1(\Omega)}\frac{|(\Phi(\tilde u),v)_{L^2(\Omega)}|}{\|v\|_{H^1(\Omega)}} = \sup_{v\in H^1(\Omega)}\frac{|-(\tilde u_1\partial_x\tilde u_2,\partial_y v)_{L^2(\Omega)}+(\tilde u_1\partial_y\tilde u_2,\partial_x v)_{L^2(\Omega)}|}{\|v\|_{H^1(\Omega)}}\\ &\le c\|\tilde u\|_\infty \mid \tilde u\mid_{1,\omega}\le c(\ln N)^{\frac12}\mid \tilde u\mid_{1,\omega}^2 .\end{aligned}\tag{4.6}$$

Therefore we obtain from (4.5),(4.6) and Proposition 2.3 and Theorem 12.2 of [10] that

$$\|\Phi(\tilde u)\|_{H^{-\frac34}(\Omega)}\le c\|\Phi(\tilde u)\|^{\frac14}_{L^2(\Omega)}\|\Phi(\tilde u)\|^{\frac34}_{H^{-1}(\Omega)}\le c\left(\frac{N}{h}\right)^{\frac18}(\ln N)^{\frac38}\mid\tilde u\mid^2_{1,\omega} .$$

Thus

$$\mid\tilde p\mid_{1,\omega}\le c(\|\nabla\cdot\tilde f\|_{H^{-\frac34}(\Omega)}+\left(\frac{N}{h}\right)^{\frac18}(\ln N)^{\frac38}\mid\tilde u\mid^2_{1,\omega}+\mid u\mid_{1,\infty}\mid\tilde u\mid_{1,\omega})$$

and

$$\begin{aligned}\mid F_5\mid &\le \frac{\varepsilon\nu}{8}\mid\tilde u\mid^2_{1,\omega}+c(1+\frac{1}{\varepsilon\nu}\mid u\mid^2_{1,\infty})\|\tilde u\|^2_\omega\\ &\quad+\frac{c}{\varepsilon\nu}N^{\frac14}h^{-\frac14}(\ln N)^{\frac34}\|\tilde u\|^2_\omega\mid\tilde u\mid^2_{1,\omega}+c\|\nabla\cdot\tilde f\|^2_\omega.\end{aligned}$$

By Lemma 8,

$$\begin{aligned}\mid F_6\mid &\le \varepsilon\tau\|\tilde u_t\|^2_\omega+\frac{c\tau m^2}{\varepsilon}\mid u\mid^2_{1,\infty}\mid\tilde u\mid^2_{1,\omega}+\frac{c\tau m^2}{\varepsilon}\|\nabla\cdot\tilde f\|^2_{H^{-\frac34}(\Omega)}\\ &\quad+\frac{c\tau m^2}{\varepsilon}N^{\frac14}h^{-\frac14}(\ln N)^{\frac34}(N^4+c_dh^{-2})\|\tilde u\|^2_\omega\mid\tilde u\mid^2_{1,\omega} .\end{aligned}$$

Let $|||u|||_{1,\infty}=\max_{t\in S_\tau}\|u(t)\|_{1,\infty}$, etc.. By substituting the above estimations into (4.4), we have

$$\begin{aligned}&(\|\tilde u\|^2_\omega)_t+\tau(m-1-2\varepsilon)\|\tilde u_t\|^2_\omega+\tfrac{\nu}{8}(\tfrac{15}{4}-m-2\sigma-3\varepsilon)\mid\tilde u\mid^2_{1,\omega}\\ &+\nu\tau(\sigma+\tfrac{m}{2})(\mid\tilde u\mid^2_{1,\omega})_t+\nu\tau^2(\tfrac{\sigma m}{4}-5(\sigma+\tfrac{m}{2})-\tfrac{\varepsilon m}{2})\mid\tilde u_t\mid^2_{1,\omega}\\ &\le M_1\|\tilde u\|^2_\omega+B(\|\tilde u\|_\omega)\mid\tilde u\mid^2_{1,\omega}+G_1\end{aligned}$$

where

$$\begin{aligned}&M_1=c+\tfrac{c}{\varepsilon\nu}[(1+m)|||u|||^2_\infty+|||u|||^2_{1,\infty}],\\ &B(\|\tilde u\|_\omega)=-\tfrac{\nu}{32}+\tfrac{c\tau m^2}{\varepsilon}|||u|||^2_{1,\infty}+[\tfrac{c}{\varepsilon\nu}(1+m)\ln N\\ &\qquad+\tfrac{c}{\varepsilon}N^{\frac14}h^{-\frac14}(\ln N)^{\frac34}(\tfrac{1}{\nu}+\tau m^2(2N^4+c_dh^{-2}))]\|\tilde u\|^2_\omega,\\ &G_1=(1+\tfrac{\tau m^2}{4\varepsilon})\|\tilde f\|^2_\omega+c(1+\tfrac{c\tau m^2}{\varepsilon})\|\nabla\cdot\tilde f\|^2_{H^{-\frac34}(\Omega)}.\end{aligned}$$

By using Lemma 8 again, we get

$$\begin{aligned}&(\|\tilde u\|^2_\omega)_t+\tau[m-1-2\varepsilon-\nu\tau(5(\sigma+\tfrac{m}{2})+\tfrac{\varepsilon}{2}m-\tfrac{\sigma m}{4})(2N^4+c_dh^{-2})]\|\tilde u_t\|^2_\omega\\ &+\tfrac{\nu}{8}(\tfrac{15}{4}-m-2\sigma-3\varepsilon)\mid\tilde u\mid^2_{1,\omega}+\nu\tau(\sigma+\tfrac{m}{2})(\mid\tilde u\mid^2_{1,\omega})_t\\ &\le M_1\|\tilde u\|^2_\omega+B(\|\tilde u\|^2_\omega)\mid\tilde u\mid^2_{1,\omega}+G_1\end{aligned}\tag{4.7}$$

Let ε be suitably small and λ suitably large. Suppose that

$$\nu\tau(2N^4+c_dh^{-2})<\frac{2}{\lambda(5+\varepsilon-\frac{\sigma}{2})}.\tag{4.8}$$

We take

$$m=\left(\frac{33}{32}+2\varepsilon+5\sigma\nu\tau(2N^4+c_dh^{-2})\right)\left(1-\frac{1}{\lambda}\right)^{-1}.$$

Then the coefficient of the term $\|\tilde u_t\|^2_\omega$ in (4.7) is not less than $\frac{\tau}{32}$. Obviously

$$m\le\left(\frac{33}{32}+2\varepsilon+\frac{10\sigma}{\lambda(5+\varepsilon-\frac{\sigma}{2})}\right)\left(1-\frac{1}{\lambda}\right)^{-1}.$$

Thus if

$$\lambda > \left(\frac{10\sigma}{5+\varepsilon-\frac{\sigma}{2}}+\frac{7}{2}-2\sigma-3\varepsilon\right)\left(\frac{79}{32}-2\sigma-5\varepsilon\right)^{-1}, \tag{4.9}$$

then the coefficient of the term $|\,\tilde{u}\,|^2_{1,\omega}$ in (4.7) is not less than $\frac{\nu}{32}$. Thus (4.7) reads

$$\begin{aligned}&(\|\tilde{u}\|^2_\omega)_t+\tfrac{\tau}{32}\|\tilde{u}_t\|^2_\omega+\tfrac{\nu}{32}\,|\,\tilde{u}\,|^2_{1,\omega}+\nu\tau(\sigma+\tfrac{m}{2})(|\,\tilde{u}\,|^2_{1,\omega})_t\\ &\le M_1\|\tilde{u}\|^2_\omega+B(\|\tilde{u}\|_\omega)\,|\,\tilde{u}\,|^2_{1,\omega}+G_1.\end{aligned} \tag{4.10}$$

Let

$$\begin{aligned}E(t) &= \|\tilde{u}(t)\|^2_\omega+\frac{\tau}{32}\sum_{t'\in S_\tau,t'<t}(\tau\|\tilde{u}_t(t')\|^2_\omega+\nu\,|\,\tilde{u}(t')\,|^2_{1,\omega}),\\ \rho(t) &= \|\tilde{u}(0)\|^2_\omega+\nu\tau(\sigma+\frac{m}{2})\,|\,\tilde{u}(0)\,|^2_{1,\omega}+\tau\sum_{t'\in S_\tau,t'<t}G_1(t').\end{aligned}$$

By summing (4.10) for all $t'\in S_\tau$ and $t'\le t-\tau$, we get

$$E(t)\le\rho(t)+\tau\sum_{t'\in S_\tau,t'<t}(M_1E(t')+B(E(t'))\,|\,\tilde{u}(t')\,|^2_{1,\omega}).$$

By Lemma 4.16 of [11], we obtain the following conclusion.

Theorem 1. *Assume that*

(i) *(4.8) and (4.9) hold;*

(ii) *for certain suitably small positive constant* $c_3, \tau|||u|||^2_{1,\infty}<c_3\nu$;

(iii) *there exist positive constants* d_1 *and* d_2 *depending only on* $|||u|||_{1,\infty}$ *and* ν *such that for some* $t_1\in S_\tau,\ \rho(t_1)e^{d_1t_1}\le\frac{d_2h^{\frac14}}{N^{\frac14}(\ln N)^{\frac34}}$.

Then for all $t\in S_\tau, t\le t_1$,

$$E(t)\le\rho(t)e^{d_1t}.$$

We next consider the convergence of (2.2). Let $U^*=P^*_{N,h}U$. In order to get better error estimation for the pressure, we introduce the operator $P^1_h: H^1(I_y)\to\tilde{S}^k_h(I_y)\cap H^1(I_y)$ such that for any $v\in H^1(I_y)$,

$$\int_0^1\partial_yv\partial_yzdy=\int_0^1\partial_y(P^1_hv)\partial_yzdy,\qquad\forall z\in\tilde{S}^k_h(I_y)\cap H^1(I_y)$$

and

$$\int_0^1(v-P^1_hv)dy=0.$$

It can be verified as in [8] that for any $v\in H^s(I_y)$ with $s\ge1$,

$$||v-P^1_hv||_{H^\mu(I_y)}\le h^{\bar{s}-\mu}||v||_{H^{\bar{s}}(I_y)},\quad 0\le\mu\le1. \tag{4.11}$$

Then we follow the idea in Section 10.4 of [4], to define

$$P^*=P^1_hP(-1,y)+\int_{-1}^xP^1_hP^1_{N-1}\frac{\partial P}{\partial s}(s,y)ds.$$

let ϑ be the identity operator. Then

$$P-P^*=(\vartheta-P^1_h)P(-1,y)+\int_{-1}^x(\vartheta-P^1_hP^1_{N-1})\frac{\partial P}{\partial s}(s,y)ds.$$

Moreover for any $v\in Z^k_{N,h}(\Omega)$,

$$\begin{aligned}&\textstyle\int_0^1\partial_y(P-P^*)\partial_yvdy=\int_0^1\partial_y((\vartheta-P^1_h)P(-1,y))\partial_yvdy\\ &\textstyle=\int_0^1\partial_yv\partial_y((\vartheta-P^1_h)\int_{-1}^x\frac{\partial P}{\partial s}(s,y)ds)dy+\int_0^1\partial_yv\partial_y(\int_{-1}^xP^1_h(\vartheta-P^1_{N-1})\frac{\partial P}{\partial s}(s,y)ds)dy\\ &\textstyle=\int_0^1\partial_yv\partial_y(\int_{-1}^x(\vartheta-P^1_{N-1})\frac{\partial P}{\partial s}(s,y)ds)dy\\ &\textstyle=-\int_0^1v(\int_{-1}^x(\vartheta-P^1_{N-1})\frac{\partial^3P}{\partial s\partial y^2}(s,y)ds)dy.\end{aligned}$$

Furthermore

$$a_\omega(P-P^*,v)=\int_0^1\int_{-1}^1 \chi v\omega dxdy \tag{4.12}$$

where

$$\chi=-\partial_x\left((\vartheta-P_h^1P_{N-1}^1)\frac{\partial P}{\partial x}\right)-\left(\int_{-1}^x(\vartheta-P_{N-1}^1)\frac{\partial^3 P}{\partial s\partial y^2}ds\right).$$

Let $\tilde U=u-U^*$ and $\tilde P=p-P^*$. By (2.1) and (2.2), we get

$$\begin{cases}(\tilde U_t,v)_\omega+(\partial_x(U_1^*\tilde U+\tilde U_1U^*+\tilde U_1\tilde U),v)_\omega+(\partial_y(U_2^*\tilde U+\tilde U_2U^*+\tilde U_2\tilde U),v)_\omega\\ +(\nabla\tilde P,v)_\omega+\nu a_\omega(\tilde U+\sigma\tau\tilde U_t,v)=\sum\limits_{j=1}^4A_j(v),\qquad \forall v\in(X_{N,h}^k(\Omega))^2,\\ a_\omega(\tilde P,v)=(\Phi(\tilde U)+\Phi^*(U^*,\tilde U),v)_\omega+A_5(v)+A_6(v),\quad\forall v\in Z_{N,h}^k(\Omega),\\ \tilde U(0)=P_{N,h}U_0-P_{N,h}^*U_0,\end{cases} \tag{4.13}$$

where

$$\begin{aligned}&A_1(v)=(\partial_tU-U_t^*,v)_\omega,\\ &A_2(v)=(\partial_x(U_1U)+\partial_y(U_2U)-\partial_x(U_1^*U^*)-\partial_y(U_2^*U^*),v)_\omega,\\ &A_3(v)=-\nu\sigma\tau a_\omega(U_t^*,v),\qquad A_4(v)=(\nabla(P-P^*),v)_\omega,\\ &A_5(v)=a_\omega(P-P^*,v),\qquad A_6(v)=(\Phi(U)-\Phi(U^*),v)_\omega.\end{aligned}$$

Taking $v=2\tilde U$ in $A_j(v)(j=1,2,3,4)$, we have from Lemma 4 that

$$\begin{aligned}&2|A_1(\tilde U)|\le 2||\tilde U||_\omega(||\partial_tU-U_t||_\omega+||U_t-U_t^*||_\omega)\\ &\le||\tilde U||_\omega^2+c(N^{-2r}+h^{2\bar s})||U||^2_{C^1(0,T;M_\omega^{1+\frac14,\bar s}(\Omega))}+c\tau||U||^2_{H^2(t,t+\tau;L^2_\omega(\Omega))},\\ &2|A_2(\tilde U)|\le\tfrac{\varepsilon\nu}{8}|\tilde U|^2_{1,\omega}+\tfrac{c}{\varepsilon\nu}(N^{-2r}+h^{2\bar s})(||U||^2_\infty+||U^*||^2_\infty)||U||^2_{M_\omega^{r+\frac14,\bar s}(\Omega)}.\end{aligned}$$

By Lemma 2 and lemma 4,

$$\begin{aligned}2|A_3(\tilde U)|\ &\le 2\nu\sigma\tau|\tilde U|_{1,\omega}(||U_t^*-U_t||_{1,\omega}+||U_t||_{1,\omega})\\ &\le\tfrac{\varepsilon\nu}{8}|\tilde U\ |^2_{1,\omega}+\tfrac{c\nu\sigma^2\tau^2}{\varepsilon}||U||^2_{C^1(0,T;H^1_\omega(\Omega))}.\end{aligned}$$

We have $2|A_4(\tilde U)|\le D_1+D_2$ with

$$D_1=2|(\partial_x(P-P^*),\tilde U_1)_\omega|,\quad D_2=2|(P-P^*,\partial_y\tilde U_2)_\omega|.$$

Moreover (3.3), (4.11) and the trace theorem lead to

$$\begin{aligned}D_1\ &\le||\tilde U||^2_\omega+||\partial_x(P-P^*)||^2_\omega\le||\tilde U||^2_\omega+||(\vartheta-P_h^1P_{N-1}^1)\partial_xP||^2_\omega\\ &\le||\tilde U||^2_\omega+c(N^{-2r}+h^{2\bar s})||P||^2_{H^{\bar s}(I_y,H^1_\omega(I_x))\cap H^1(I_y,H_\omega^{r+1}(I_x))},\\ D_2\ &\le\tfrac{\varepsilon\nu}{8}|\tilde U|^2_{1,\omega}+\tfrac{c}{\varepsilon\nu}(||P(-1,y)-P_h^1P(-1,y)||^2_{L^2(I_y)}+||(\vartheta-P_h^1P_{N-1}^1)\partial_xP||^2_\omega)\\ &\le\tfrac{\varepsilon\nu}{8}|\tilde U|^2_{1,\omega}+\tfrac{c}{\varepsilon\nu}(N^{-2r}+h^{2\bar s})||P||^2_{H^{\bar s}(I_y,H^1_\omega(I_x))\cap H^1(I_y,H_\omega^{r+1}(I_x))}.\end{aligned}$$

For the term $A_5(v)$, we know from (4.12) and Lemma 9 that we only have to estimate $||\chi||_{H^{-\frac34}(\Omega)}$. By (3.3), (4.11) and Theorem 4.1 of [9],

$$\begin{aligned}||\chi||^2_{H^{-\frac34}(\Omega)}\ &\le||\chi||^2_\omega\le c(||(\vartheta-P_h^1)\partial_{xx}P||^2_\omega+||P_h^1(\partial_x(\vartheta-P_{N-1}^1)(\partial_xP))||^2_\omega\\ &+||(\vartheta-P_{N-1}^1)\tfrac{\partial^3P}{\partial x\partial y^2}||^2_\omega)\\ &\le c(N^{-2r}+h^{2\bar s})||P||^2_{H^{\bar s}(I_y,H^2_\omega(I_x))\cap H^2(I_y,H_\omega^{r+1}(I_x))\cap H^1(I_y,H_\omega^{r+2}(I_x))}.\end{aligned}$$

For the term $A_6(v)$, we need to estimate $||\Phi(U)-\Phi(U^*)||_{H^{-\frac34}(\Omega)}$. Firstly

$$||\Phi(U)-\Phi(U^*)||_{L^2(\Omega)}\le c(N^{-r}+h^{\bar s-1})(||U||_{1,\infty}+||U^*||_{1,\infty})||U||_{M_\omega^{r+1,\bar s}(\Omega)}.$$

Next, $\Phi(U)-\Phi(U^*)=2K_1-2K_2$ with

$$K_1=\partial_y(U_1\partial_xU_2)-\partial_y(U_1^*\partial_xU_2^*),\quad K_2=\partial_x(U_1\partial_yU_2)-\partial_x(U_1^*\partial_yU_2^*).$$

Furthermore

$$\begin{aligned}&||K_1||_{H^{-2}(\Omega)}=\sup_{v\in H^2(\Omega)}\frac{|(U_1\partial_xU_2-U_1^*\partial_xU_2^*,\partial_yv)_{L^2(\Omega)}|}{||v||_{H^2(\Omega)}}\\&\le\sup_{v\in H^2(\Omega)}\frac{|(U_1\partial_xU_2-U_1^*\partial_xU_2,\partial_yv)_{L^2(\Omega)}|+|((U_2-U_2^*)\partial_xU_1^*,\partial_yv)_{L^2(\Omega)}|+|((U_2-U_2^*)U_1^*,\partial_x\partial_yv)_{L^2(\Omega)}|}{||v||_{H^2(\Omega)}}\\&\le c(N^{-r}+h^{\bar{s}})(||U||_{1,\infty}+||U^*||_{1,\infty})||U||_{M_\omega^{r+\frac14,\bar{s}}(\Omega)}.\end{aligned}$$

We can estimate $||K_2||_{H^{-2}(\Omega)}$ similarly and so

$$||\Phi(U)-\Phi(U^*)||_{H^{-2}(\Omega)}\le c(N^{-r}+h^{\bar{s}})(||U||_{1,\infty}+||U^*||_{1,\infty})||U||_{M_\omega^{r+\frac14,\bar{s}}(\Omega)}.$$

Thus by Proposition 2.3 and Theorem 12.2 of [10],

$$\begin{aligned}||\Phi(U)-\Phi(U^*)||_{H^{-\frac34}(\Omega)}&\le c||\Phi(U)-\Phi(U^*)||_{L^2(\Omega)}^{\frac58}||\Phi(U)-\Phi(U^*)||_{H^{-2}(\Omega)}^{\frac38}\\&\le c(N^{-r}+h^{\bar{s}-\frac58})||U||_{M_\omega^{r+1,\bar{s}}(\Omega)}.\end{aligned}$$

Therefore

$$|A_6(v)|\le c||\Phi(U)-\Phi(U^*)||_{H^{-\frac34}(\Omega)}||v||_{1,\omega}\le||U||_{1,\omega}^2+c(N^{-2r}+h^{2\bar{s}-\frac54})||U||^2_{M_\omega^{r+1,\bar{s}}(\Omega)}.$$

By taking $v=m\tau\tilde{U}_t$ in $A_j(v)(j=1,2,3,4)$, we have that

$$\begin{aligned}m\tau|A_1(\tilde{U}_t)|&\le\varepsilon\tau||\tilde{U}_t||_\omega^2+\frac{cm^2\tau}{\varepsilon}(N^{-2r}+h^{2\bar{s}})||U||^2_{C^1(0,T;M_\omega^{r+\frac14,\bar{s}}(\Omega))}\\&\quad+\frac{cm^2\tau^2}{\varepsilon}||U||^2_{H^2(t,t+\tau;L^2_\omega(\Omega))},\\m\tau|A_2(\tilde{U}_t)|&\le\varepsilon\nu\tau^2|\tilde{U}_t|^2_{1,\omega}+\frac{cm^2}{\varepsilon\nu}(N^{-2r}+h^{2\bar{s}})(||U||^2_\infty+||U^*||^2_\infty)||U||^2_{M_\omega^{r+\frac14,\bar{s}}(\Omega)},\\m\tau|A_3(\tilde{U}_t)|&\le\varepsilon\nu\tau^2|\tilde{U}_t|^2_{1,\omega}+\frac{c\nu m\sigma^2\tau^2}{\varepsilon}||U||^2_{C^1(0,T;H^1_\omega(\Omega))},\\m\tau|A_4(\tilde{U}_t)|&\le\varepsilon\nu\tau^2|\tilde{U}_t|^2_{1,\omega}+\varepsilon\tau||\tilde{U}_t||^2_\omega\\&\quad+c(\frac{\tau m^2}{\varepsilon}+\frac{m}{\varepsilon\nu})(N^{-2r}+h^{2\bar{s}})||P||^2_{H^{\bar{s}}(I_y,H^1_\omega(I_x))\bigcap H^1(I_y,H_\omega^{r+1}(I_x))}.\end{aligned}$$

Moreover, by Lemma 3, Lemma 4, Lemma 8 and (4.8),

$$||\tilde{U}(0)||^2_\omega\le c(N^{-2r}+h^{2\bar{s}})||U_0||^2_{M_\omega^{r+\frac14,\bar{s}}(\Omega)}.$$

$$\tau|\tilde{U}(0)|^2_{1,\omega}\le c\tau(N^4+h^{-2})(N^{-2r}+h^{2\bar{s}})||U_0||^2_{M_\omega^{r+\frac14,\bar{s}}(\Omega)}.$$

By Lemma 7, we have that for $\alpha,\beta>\frac12$,

$$||U^*||_{1,\infty}\le||U||_{A_\omega^{\alpha,\beta}(\Omega)}.$$

Besides if (4.8) holds, then for $r>\frac{7}{32},s>\frac{11}{12}$,

$$\tau^2+N^{-2r}+h^{2\bar{s}-\frac54}=o\left(\frac{h^{\frac12}}{N^{\frac14}(\ln N)^{\frac34}}\right).$$

Finally by an argument similar to the proof of Theorem 1, we have the following result.

Theorem 2. *Assume that*

(i) *(3.1) and condition (i) of Theorem 1 hold;*

(ii) *for* $r\ge\frac34,s\ge1$ *and* $\alpha,\beta>\frac12$,

$$U\in C(0,T;W^{1,\infty}(\Omega)\bigcap M_\omega^{r+1,\bar{s}}(\Omega)\bigcap A_\omega^{\alpha,\beta}(\Omega))\bigcap C^1(0,T;M_\omega^{r+\frac14,\bar{s}}(\Omega))\bigcap H^2(0,T;L^2_\omega(\Omega)),$$

$$P \in C(0,T; H^{\bar{s}}(I_y, H^2_\omega(I_x)) \bigcap H^2(I_y, H^{r+1}_\omega(I_x)) \bigcap H^1(I_y, H^{r+2}_\omega(I_x))).$$

Then there exists a positive constant d_3 depending only on ν and the norms of U and P in the spaces mentioned in the above, such that for all $t \le T$;

$$||U(t) - u(t)||_\omega \le d_3(\tau + N^{-r} + h^{\bar{s}-\frac{5}{8}}).$$

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