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ERROR ANALYSIS AND GLOBAL SUPERCONVERGENCES FOR THE SIGNORINI PROBLEM WITH LAGRANGE MULTIPLIER METHODS*1)

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Abstract

In this paper, the finite element approximation of the Signorini problem is studied by using Lagrange multiplier methods with piecewise constant elements. Optimal error bounds are obtained and iterative algorithm and its convergence are given. Furthermore, global superconvergences are proved.

Key words: Lagrange multipliers, Elastic contact, Error estimates, Superconvergences.

1. Introduction

It is well known that contact problems have always occupied a position of special importance in the mechanics of solids. So it attracted particular attention of engineers and computational experts. In the last ten years, with the development of the theory of variational inequalities, a lot of new results and methods on contact problems have been reported. We refer to Brezzi, Hager and Raviart [3,4], Falk [8], Glowinski, Lions and Trémoliérés [10], Haslinger [11], Haslinger and Hlaváček [12,13], Kikuchi and Oden [15] for details and survey in this field.

Superconvergence estimates for the finite element methods are well studied in many papers. We refer to Křižek and Neittanmäki [16], Lin and Xu [20], Lin and Zhu [21,32], Křižek [17] and Wahlbin [30] for more details.

We shall discuss in this paper the classical example of the role of variational inequalities in the formulation of contact problems in solid mechanics – the so-called Signorini problem describing the contact of a linearly elastic body with a rigid frictionless foundation and show that the finite element approximation of the Lagrange multipliers methods with piecewise constant element is of order one in accuracy with the energy norm. Furthermore, we find that the error estimates with the energy norm are half a order higher than the usual optimal error bounds if the partition of Ω is almost a uniform piecewise strongly regular mesh and the solution is smooth enough.

We consider the deformation of a body unilaterally supported by a frictionless rigid foundation and subjected to body forces f and surface tractions t applied to a portion Γ_F of the body's surface Γ. The body is fixed along a portion Γ_D of its boundary and we denote by Γ_c a portion of body which is a candidate contact surface. Let $\mathbf{u} = (u_1, u_2)$ and $\sigma = (\sigma_{ij}), 1 \le i, j \le 2$, denote arbitrary displacement and stress fields in the body respectively. If we take on the specific form $\sigma_{ij}(\mathbf{u}) = E_{ijkl} u_{k,l}(x)$, where E_{ijkl} are the components of Hooke's tensor which may depend on x and have the symmetry properties

$$E_{ijkl} = E_{jikl} = E_{klij},\ 1 \le i, j, k, l \le 2.$$

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Then the Signorini problem consists in finding the displacement field $\mathbf{u}$ such that

$$\begin{aligned} -(E_{ijkl}u_{k,l})_{,j} &= f_i, \text{ in } \Omega, \\ u_i &= 0, \text{ on } \Gamma_D, \\ E_{ijkl}u_{k,l}n_j &= t_i, \text{ on } \Gamma_F, \\ \left.\begin{aligned} \sigma_{T_i}(\mathbf{u}) &= 0, \ \sigma_n(\mathbf{u})(u_n - g) = 0, \\ u_n - g &\le 0, \ \sigma_n(\mathbf{u}) \le 0, \end{aligned}\right\} &\text{on } \Gamma_c, \end{aligned} \tag{1}$$

where $u_{k,l} = \dfrac{\partial u_k}{\partial x_l}$, $\mathbf{n}$ and g denote the outward unit normal vector on Γ_c and the normalized initial gap, respectively, and

$$\begin{aligned} u_n &= \mathbf{u}\cdot\mathbf{n} = u_i n_i, \ \sigma_n(\mathbf{u}) = \sigma_{ij}(\mathbf{u})n_i n_j, \\ \sigma_{T_i}(\mathbf{u}) &= \sigma_{ij}(\mathbf{u})n_j - \sigma_n(\mathbf{u})n_i, \ 1 \le i,j,k,l \le 2, \end{aligned}$$

here $\sigma_{T_i}(\mathbf{u}), \sigma_n(\mathbf{u})$ denote the tangential and the normal components of the stress vector, respectively, and the usual summation convention on repeated indices is used.

Let Ω be a bounded domain with a sufficiently smooth boundary. We will use the usual Sobolev space $W^{m,p}(\Omega)$ consisting of real valued functions defined on Ω with derivatives up to order m in $L^p(\Omega)$ and the norm on $W^{m,p}(\Omega)$ is denoted by $||\cdot||_{m,p,\Omega}$. In particular, we define

$$H^m(\Omega) = W^{m,2}(\Omega) \quad \text{and} \quad ||\cdot||_{m,\Omega} = ||\cdot||_{m,2,\Omega}.$$

Let $(\cdot,\cdot)$ denote the L^2-inner product and

$$\Sigma = int(\Gamma - \Gamma_D), \ \Gamma_F \subset \Sigma, \ \Gamma_F \cap \Gamma_c = \phi, \ \overline{\Gamma}_c \subset \Sigma,$$

where int denotes interior of a set. For vector-valued function $\mathbf{v}$ with components v_i in $H^m(\Omega)$, we shall use the following notations

$$\mathbf{v} \in \mathbf{H^m}(\Omega) = \{(v_1, v_2) | v_i \in H^m(\Omega), 1 \le i \le 2\},$$

$$||\mathbf{v}||_{m,\Omega} = \{\sum_{i=1}^{2} \int_\Omega \sum_{|\alpha|\le m} |D^\alpha v_i(x)|^2 dx\}^{\frac{1}{2}},$$

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H^1}(\Omega) |\, \gamma_D(\mathbf{v}) = 0, \text{ in } H^{\frac{1}{2}}(\Gamma_D)\},$$

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{V} |\, \gamma^0_{\Sigma_n}(\mathbf{v}) - g \le 0, \quad \text{a.e. on } \Gamma_c\},$$

$$\gamma_D : \mathbf{H}^1(\Omega) \to H^{\frac{1}{2}}(\Gamma_D), \gamma_D \text{ being the trace map from } \mathbf{H}^1(\Omega),$$

$$\gamma^0_{\Sigma_n} : \mathbf{V} \to H^{\frac{1}{2}}(\Sigma), \gamma^0_{\Sigma_n} \text{ being the normal trace map from } \mathbf{V}.$$

We see that the set $\mathbf{K}$ is a nonempty closed convex subset of $\mathbf{V}$ and the normal trace operator $\gamma^0_{\Sigma_n}$ is a surjective linear continuous map. Let $H^{-\frac{1}{2}}(\Gamma_c)$ denote the dual space of $H^{\frac{1}{2}}(\Gamma_c)$ and its norm be defined by

$$||v||_{-\frac{1}{2},\Gamma_c} = \sup_{0\ne\varphi\in H^{\frac{1}{2}}(\Gamma_c)} \frac{|\int_{\Gamma_c} v\varphi|}{||\varphi||_{\frac{1}{2},\Gamma_c}},$$

where

$$||v||^2_{\frac{1}{2},\Gamma_c} = ||v||^2_{0,\Gamma_c} + |v|^2_{\frac{1}{2},\Gamma_c}, \quad |v|^2_{\frac{1}{2},\Gamma_c} = \int_{\Gamma_c}\int_{\Gamma_c} \frac{(v(x) - v(y))}{(x-y)^2} dxdy.$$

For further notation we refer to Nečas [26], Ciarlet and Lions [6], and Kikuchi and Oden [15].

2. Formulation of the Algorithm and Error Estimates

Let Ω, for simplicity, be a convex polygonal domain in R^2 and $\gamma_{c,i} = \Gamma_c \cap \gamma_i, i = 1, \cdots, n_g$, where γ_i denotes any side of the polygonal boundary. Let $a(\cdot,\cdot)$ be a symmetric and continuous bilinear form of $\mathbf{V}$, which corresponds to the virtual work in the body Ω:

$$\begin{aligned} a(\cdot,\cdot): &\qquad \mathbf{V}\times\mathbf{V}\to\mathbf{R}, \\ a(\mathbf{u},\mathbf{v}) &= \int_\Omega E_{ijkl}u_{k,l}v_{i,j}dx, \quad \forall \mathbf{u},\mathbf{v}\in\mathbf{V}. \end{aligned}$$

We suppose that $\mathbf{f}\in\mathbf{L}^2(\Omega)$ and $\mathbf{t}\in\mathbf{L}^2(\Gamma_F)$, where $\mathbf{L}^2(\Omega) = L^2(\Omega)\times L^2(\Omega)$. Then the linear functional

$$f(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}dx + \int_{\Gamma_F}\mathbf{t}\cdot\gamma_\Sigma^0(\mathbf{v})ds$$

is bounded and the trace operator γ_Σ^0 is a surjective linear continuous map. Hence the Signorini problem (1) can be given in the variational form:

$$\begin{cases} \text{Find } \mathbf{u}\in\mathbf{K} & \text{such that} \\ a(\mathbf{u},\mathbf{v}-\mathbf{u}) & \geq f(\mathbf{v}-\mathbf{u}), \forall \mathbf{v}\in\mathbf{K}. \end{cases} \tag{2}$$

Let $X(\Gamma_c) = \prod_{i=1}^{n_g} H^{-\frac{1}{2}}(\gamma_{c,i})$ with the norm

$$|||\mu|||^2_{-\frac{1}{2},\Gamma_c} = \sum_{i=1}^{n_g}||\mu||^2_{-\frac{1}{2},\gamma_{c,i}}.$$

We introduce the Lagrange multiplier p and its admissible set N defined by

$$N = \{p\in X(\Gamma_c)|\, p\leq 0 \quad \text{a.e. on} \quad \Gamma_c\}.$$

Then we may define the variational formulation with Lagrange multiplier for problem (1) as follows

$$\begin{cases} \text{Find } (\mathbf{u},p)\in\mathbf{V}\times N \quad \text{such that} \\ a(\mathbf{u},\mathbf{v}) - \sum_{i=1}^{n_g}[p, B\mathbf{v}]_{\gamma_{c,i}} = f(\mathbf{v}), \quad \forall \mathbf{v}\in\mathbf{V}, \\ \sum_{i=1}^{n_g}[q-p, B\mathbf{u}-g]_{\gamma_{c,i}} \geq 0, \quad \forall q\in N, \end{cases} \tag{3}$$

where $[\cdot,\cdot]$ denotes the duality pairing on $H^{-\frac{1}{2}}(\gamma_{c,i})\times H^{\frac{1}{2}}(\gamma_{c,i})$ and $B = \gamma^0_{\Sigma_n}$.

We suppose that the continuous bilinear form $a(\cdot,\cdot)$ on the Hilbert space $\mathbf{V}$ satisfy the following continuity and ellipticity conditions:

$$a(\mathbf{u},\mathbf{v}) \leq \delta||\mathbf{u}||_{1,\Omega}||\mathbf{v}||_{1,\Omega} \quad \text{and} \quad a(\mathbf{v},\mathbf{v}) \geq \kappa||\mathbf{v}||^2_{1,\Omega}, \tag{4}$$

where δ and κ are positive constants. Then it is easy to prove that the problem (3) is equivalent to the problem (2) (cf [15]). Hence we have

Theorem 1. *The problem (3) has a unique solution* $(\mathbf{u},p)\in\mathbf{V}\times N$. *Moreover* $\mathbf{u}\in\mathbf{K}$ *is the solution of problem (2) and* p *satisfies*

$$p = \sigma_n(\mathbf{u}), \quad on \ \gamma_{c,i}, i = 1,\cdots,n_g.$$

We suppose that there are a partition $T^h = \{e\}$ of Ω with mesh size h and a partition $\Gamma_c^H = \{\tau\}$ of Γ_c with mesh size H, where the vertices of the polygonal boundary are nodes of the partition. Let $\mathbf{V}^h$ and Q^H be the finite element spaces of piecewise linear on T^h and piecewise constants on Γ_c^H, respectively. Then the finite element approximation problem corresponding to the problem (3) is the following.

$$\begin{cases} \text{Find } (\mathbf{u}_h, p_H) \in \mathbf{V}^h \times N^H & \text{such that} \\ a(\mathbf{u}_h, \mathbf{v}_h) - \sum_{i=1}^{n_g}[p_H, B\mathbf{v}_h]_{\gamma_{c,i}} = & f(\mathbf{v}_h),\ \forall \mathbf{v}_h \in \mathbf{V}^h, \\ \sum_{i=1}^{n_g}[q_H - p_H, B\mathbf{u}_h - g]_{\gamma_{c,i}} \geq & 0,\ \forall q_H \in N^H, \end{cases} \tag{5}$$

where

$$N^H = \{q \in Q^H |\ q|_\tau \leq 0, \forall \tau \in \Gamma_c^H\}.$$

Lemma 1. *Suppose that Ω is a polygonal domain and $\frac{h}{H}$ is sufficiently small. The spaces $\mathbf{V}^h$ and N^H are defined as above. Then there exists a constant $\beta > 0$ independent of h and H such that*

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{V}^h \\ \mathbf{v}_h \neq 0}} \frac{\sum_{i=1}^{n_g}[p_H, v_{hn}]_{\gamma_{c,i}}}{||\mathbf{v}_h||_{1,\Omega}} \geq \beta |||p_H|||_{-\frac{1}{2},\Gamma_c}, \quad \forall p_H \in Q^H, \tag{6}$$

where $v_{hn} = B\mathbf{v}_h$.

Proof. Let $p_H \in Q^H$ and $\mathbf{w}$ be the solution of the following problem

$$\begin{aligned} -(w_{k,l})_{,j} + w_i &= 0, \ \text{ in } \Omega \\ w_i &= 0, \ \text{on } \Gamma_D \\ w_{k,l} n_j &= 0, \ \text{on } \Gamma_F \\ \sigma_{T_i}(\mathbf{w}) &= 0, \ \sigma_n(\mathbf{w}) = p_H, \quad \text{on } \Gamma_c. \end{aligned}$$

Then

$$(\varepsilon_{ij}(\mathbf{w}), \varepsilon_{ij}(\mathbf{v}))_\Omega + (w_i, v_i)_\Omega = (p_H, B\mathbf{v})_{\Gamma_c},\ \forall \mathbf{v} \in \mathbf{V}, \tag{7}$$

where

$$\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}), \quad i, j = 1, 2.$$

We introduce the norm $|||\cdot|||^\star$ defined by

$$|||\mathbf{v}|||^{\star 2} = (\varepsilon_{ij}(\mathbf{v}), \varepsilon_{ij}(\mathbf{v}))_\Omega + (v_i, v_i)_\Omega$$

in $\mathbf{V}$. By virtue of Korn's inequality, the norm $|||\cdot|||^\star$ is equivalent to the usual norm $||\cdot||_{1,\Omega}$ in $\mathbf{H}^1(\Omega)$. Hence from the regularity results and inverse estimation, we have

$$||\mathbf{w}||_{1+\epsilon,\Omega} \leq c||p_H||_{-\frac{1}{2}+\epsilon,\Gamma_c} \leq cH^{-\epsilon}||p_H||_{-\frac{1}{2},\Gamma_c}, \tag{8}$$

where $\epsilon \in (0,1)$. Using the well-known interpolation properties of $\mathbf{V}^h$ and (8), we see that there exists $\mathbf{z}^h \in \mathbf{V}^h$ such that

$$||\mathbf{z}^h||_{1,\Omega} \leq ||\mathbf{w}||_{1,\Omega}, \tag{9}$$

$$||\mathbf{w} - \mathbf{z}^h||_{1,\Omega} \leq ch^\epsilon ||\mathbf{w}||_{1+\epsilon,\Omega} \leq c(\frac{h}{H})^\epsilon ||p_H||_{-\frac{1}{2},\Gamma_c}. \tag{10}$$

For any $\varphi \in H^{\frac{1}{2}}(\gamma_{c,i})$, there exists a function $\mathbf{v} \in \mathbf{V}$ such that $B\mathbf{v} = \varphi$ on $\gamma_{c,i}$ and $||\mathbf{v}||_{1,\Omega} \leq c||\varphi||_{\frac{1}{2},\gamma_{c,i}}$. Hence from the definition of the dual norm, we have

$$\begin{aligned} ||p_H||_{-\frac{1}{2},\gamma_{c,i}} &= \sup_{\substack{\varphi\in H^{\frac{1}{2}}(\gamma_{c,i})\\ \varphi\neq 0}} \frac{|\int_{\gamma_{c,i}} p_H\varphi|}{||\varphi||_{1,\gamma_{c,i}}} \leq c \sup_{\substack{\varphi\in H^{\frac{1}{2}}(\gamma_{c,i})\\ \varphi\neq 0}} \frac{|\int_{\gamma_{c,i}} p_H B\mathbf{v}|}{||\mathbf{v}||_{1,\Omega}} \\ &= c \sup_{\substack{\varphi\in H^{\frac{1}{2}}(\gamma_{c,i})\\ \varphi\neq 0}} \frac{|(\varepsilon_{ij}(\mathbf{w}),\varepsilon_{ij}(\mathbf{v}))_\Omega + (w_i,v_i)_\Omega|}{||\mathbf{v}||_{1,\Omega}} \\ &\leq c||\mathbf{w}||_{1,\Omega}. \end{aligned} \tag{11}$$

From (7)-(10), we get

$$\begin{aligned} \int_{\Gamma_c} p_H B\mathbf{z}^h &\geq \int_{\Gamma_c} p_H B\mathbf{w} - \int_{\Gamma_c} p_H B(\mathbf{w}-\mathbf{z}^h) \\ &\geq c_1||\mathbf{w}||^2_{1,\Omega} - c(\frac{h}{H})^\epsilon \sum_{i=1}^{n_g} ||p_H||^2_{-\frac{1}{2},\gamma_{c,i}} \\ &\geq c||\mathbf{w}||_{1,\Omega}||\mathbf{w}||_{1,\Omega} \geq c||\mathbf{z}^h||_{1,\Omega}|||p_H|||_{-\frac{1}{2},\Gamma_c} \end{aligned}$$

which implies (6).

Let I^h denote the interpolation operator in $\mathbf{V}^h$ and R^H be a usual L^2- projection operator satisfying, for $\lambda \in L^2(\tau)$,

$$\int_\tau (\lambda - R^H\lambda) = 0, \quad \forall \tau \in \Gamma_c^H. \tag{12}$$

Then we have following results (cf [9] and [21]).

Lemma 2. *If $\lambda \in H^r(\tau)$ and $r \geq 0$, then we have*

$$||\lambda - R^H\lambda||_{-s,\tau} \leq cH^\mu||\lambda||_{r,\tau},$$

where $\mu = min(r+s, 1+s)$ and $0 \leq s \leq 1$.

Lemma 3. *Let $\mathbf{u} \in \mathbf{H}^r(\Omega)$ and $r > 1$. Then*

$$||B(\mathbf{u} - I^h\mathbf{u})||_{s,\Gamma_c} \leq ch^\mu||\mathbf{u}||_{r,\Gamma_c},$$

$$||\mathbf{u} - I^h\mathbf{u}||_{s,\Omega} \leq ch^\mu||\mathbf{u}||_{r,\Omega},$$

where $\mu = min(r-s, 2+s)$ and $0 \leq s \leq 1$.

Theorem 2. *Suppose that Ω be a convex polygonal domain in R^2. Then problem (5) has a unique solution $(\mathbf{u}_h, p_H) \in \mathbf{V}^h \times N^H$.*

Proof. This follows from Lemma 1, (4) and theorem 4.6 in [15].

Theorem 3. *Let $(\mathbf{u}, p) \in \mathbf{V} \times N$ and $(\mathbf{u}_h, p_H) \in \mathbf{V}^h \times N^H$ be solutions of (3) and (5), respectively. We suppose that $\mathbf{u} \in \mathbf{H}^2(\Omega), g \in W^{1,q}(\gamma_{c,i}), q \geq 2$. Moreover, assume that the number of points where $B\mathbf{u} - g$ changes from $B\mathbf{u} - g < 0$ to $B\mathbf{u} - g = 0$ is finite. Then we have the estimates*

$$||\mathbf{u} - \mathbf{u}_h||_{1,\Omega} \leq c(h + H^{1-\epsilon}), \tag{13}$$

$$||p - p_H||_{0,\Gamma_c} \leq c(hH^{-\frac{1}{2}} + H^{\frac{1}{2}-\epsilon}), \tag{14}$$

$$|||p - p_H|||_{-\frac{1}{2},\Gamma_c} \leq c(h + H^{1-\epsilon}), \tag{15}$$

where $\epsilon = \frac{1}{2q}$.

Proof. Let $\mathbf{v} = I^h\mathbf{u} - \mathbf{u}_h$. Noting $R^Hp \in N^H$ (since $p \in N$), $[p - p_H, B\mathbf{u} - g] \leq 0$ and $[R^Hp - p_H, g - B\mathbf{u}_h] \leq 0$ on $\gamma_{c,i}$, we obtain from (3) and (5),

$$\begin{aligned} a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) &= \textstyle\sum_{i=1}^{n_g}[p-p_H, B\mathbf{v}]_{\gamma_{c,i}} \\ &\leq \textstyle\sum_{i=1}^{n_g}\{[p-R^Hp, B\mathbf{v}]_{\gamma_{c,i}} \\ &\quad +[R^Hp-p_H, B(I^h\mathbf{u}-\mathbf{u})]_{\gamma_{c,i}} \\ &\quad +[R^Hp-p, B\mathbf{u}-g]_{\gamma_{c,i}}\}. \end{aligned} \tag{16}$$

By Lemma 2, Lemma 3, inverse estimates and a trace theorem we obtain

$$\begin{aligned} \textstyle\sum_{i=1}^{n_g}[p-R^Hp, B\mathbf{v}]_{\gamma_{c,i}} &\leq \textstyle\sum_{i=1}^{n_g}||p-R^Hp||_{-\frac{1}{2},\gamma_{c,i}}||B\mathbf{v}||_{\frac{1}{2},\gamma_{c,i}} \\ &\leq cH(\textstyle\sum_{i=1}^{n_g}||p||^2_{\frac{1}{2},\gamma_{c,i}})^{\frac{1}{2}}||\mathbf{v}||_{1,\Omega}, \end{aligned} \tag{17}$$

$$\sum_{i=1}^{n_g}[R^Hp-p_H, B(I^h\mathbf{u}-\mathbf{u})]_{\gamma_{c,i}} \leq ch|||R^Hp-p_H|||_{-\frac{1}{2},\Gamma_c}||\mathbf{u}||_{2,\Omega}. \tag{18}$$

Let $w = B\mathbf{u} - g$, $\Gamma_c^- = \{x \in \Gamma_c | B\mathbf{u} - g < 0\}$ and $\Gamma_c^0 = \{x \in \Gamma_c | B\mathbf{u} - g = 0\}$. If $\tau \subseteq \Gamma_c^-$, then $p = 0$ and $R^Hp = 0$ on τ. If $\tau \subseteq \Gamma_c^0$, then $B\mathbf{u} - g = 0$. Hence

$$(R^Hp - p, B\mathbf{u} - g)_\tau = 0, \quad \text{for all} \quad \tau \subseteq \Gamma_c^- \cup \Gamma_c^0.$$

If $\tau \not\subseteq \Gamma_c^- \cup \Gamma_c^0$, then using the assumptions of the theorem, for arbitrary $q \geq 2$,

$$\begin{aligned} \sum_{\tau\not\subseteq\Gamma_c^-\cup\Gamma_c^0}(R^Hp-p, w)_\tau &= \sum_{\tau\not\subseteq\Gamma_c^-\cup\Gamma_c^0}(R^Hp-p, w-R^Hw)_\tau \\ &\leq cH^{2-\frac{1}{q}}||p||_{\frac{1}{2},\tau}(||g||_{1,q,\tau}+||\mathbf{u}||_{2,\Omega}), \end{aligned} \tag{19}$$

where we have used the Sobolev imbedding theorem from $H^2(\Omega)$ to $W^{1,q}(\partial\Omega)$.

From (6) and Lemma 2 we see that

$$\begin{aligned} c\beta|||R^Hp-p_H|||_{-\frac{1}{2},\Gamma_c} &\leq \sup_{\substack{\mathbf{v}\in\mathbf{V}^h \\ \mathbf{v}\neq 0}} \frac{\sum_{i=1}^{n_g}[R^Hp-p_H, B\mathbf{v}]_{\gamma_{c,i}}}{||\mathbf{v}||_{1,\Omega}} \\ &= \sup_{\substack{\mathbf{v}\in\mathbf{V}^h \\ \mathbf{v}\neq 0}} \frac{\sum_{i=1}^{n_g}[R^Hp-p, B\mathbf{v}]_{\gamma_{c,i}} + a(\mathbf{u}-\mathbf{u}^h,\mathbf{v})}{||\mathbf{v}||_{1,\Omega}} \\ &\leq H(\sum_{i=1}^{n_g}||p||^2_{\frac{1}{2},\gamma_{c,i}})^{\frac{1}{2}} + h||\mathbf{u}||_{2,\Omega} + ||I^h\mathbf{u}-\mathbf{u}^h||_{1,\Omega}, \end{aligned} \tag{20}$$

which, combination $ab \leq \mu^{-1}a^2 + \mu b^2 (a, b \geq 0, \mu > 0)$, (17)–(20) with (4), implies

$$||I^h\mathbf{u} - \mathbf{u}_h||^2_{1,\Omega} \leq c(h^2 + H^{2-\frac{1}{q}}). \tag{21}$$

Hence by (21), Lemma 3, triangle inequality and taking $\epsilon = \frac{1}{2q}$, we have shown (13). From (20), (21), Lemma 2 and triangle inequality we deduce (15) immediately. The inequality (14) follows by (15), inverse estimate and triangular inequality.

Remark 1. We only require that $g \in W^{1,q}(\gamma_{c,i})$ in Theorem 2 in stead of $g \in W^{1,q}(\Gamma_c)$. If $g \in W^{1,\infty}(\gamma_{c,i})$ and $\mathbf{u} \in W^{1,\infty}(\gamma_{c,i}) \times W^{1,\infty}(\gamma_{c,i})$, then from (19) we see easily that the ϵ in inequalities (13)-(15) may be omitted.

We introduce a functional defined on $\mathbf{V}^h \times N^H$:

$$J(\mathbf{u},p) = \frac{1}{2}a(\mathbf{u},\mathbf{u}) - \sum_{i=1}^{n_g}[p, B\mathbf{u} - g]_{\gamma_{c,i}} - f(\mathbf{u}) \ ,\forall(\mathbf{u},p) \in \mathbf{V}^h \times N^H.$$

The corresponding matrix expression is

$$J(\mathbf{u},p) = \frac{1}{2}U^TAU - U^TDP - U^Tf + P^Tg, \tag{22}$$

where A is a symmetric positive definite $n \times n$ matrix and D is a $n \times m$ matrix. We consider the minimization problem:

$$\min_{\mathbf{u}\in\mathbf{V}^h} J(\mathbf{u},p).$$

It is equivalent to

$$AU - DP - f = 0.$$

Since A is a symmetric positive definite matrix, we have

$$U = A^{-1}(DP + f). \tag{23}$$

Substitute (23) into (22),

$$\widetilde{J}(p) = J(\mathbf{u},p) = -\frac{1}{2}P^TD^TA^{-1}DP - P^TD^TA^{-1}f - \frac{1}{2}f^TA^{-1}f + P^Tg.$$

Then the problem (5) is equivalent to

$$\min_{p\in N^H}(-\widetilde{J}(p)),$$

Let

$$C = D^TA^{-1}D = (C_{ij})_{i,j=1}^m, \ \ F = -D^TA^{-1}f + g = (F_i)_{i=1}^m, \ \ p = \sum_{i=1}^m P_i\varphi_i,$$

where $\{\varphi_i\}$ are the piecewise constant basis functions of the finite element space Q^H.

We construct an algorithm as follows:

(i) Given $0 < \omega < 2$ and $p^0 \in N^H$, i.e: $p^0 = \sum_{i=1}^m P_i^0\varphi_i$, where $P_i^0 \le 0$.

(ii) With p^k known, determine p^{k+1} by

$$\widetilde{\lambda}_i^{(k+1)} = -C_{ii}^{-1}(\sum_{j=1}^{i-1} C_{ij}P_j^{(k+1)} + \sum_{j=i+1}^{m} C_{ij}P_j^{(k)} - F_i),$$

$$\widetilde{P}_i^{(k+1)} = (1-\omega)P_i^{(k)} + \omega\widetilde{\lambda}_i^{(k+1)},$$

$$P_i^{(k+1)} = \min(0, \widetilde{P}_i^{(k+1)}), \ (i = 1,\dots,m), P^{k+1} = (P_1^{(k+1)},\dots,P_m^{(k+1)}),$$

(iii) When $|P^{k+1} - P^k|$ is small enough, we stop the calculation.

(iv) Let $P = P^{k+1}$, calculate $\widetilde{U} = A^{-1}(DP + f)$.

Theorem 4. *Let $0 < \omega < 2$. Then p^k, defined by the above algorithm, converges to the solution p of problem (3). Moreover $\tilde{\mathbf{u}}$ converges to the solution $\mathbf{u}$ of problem (3).*

Proof. First, we show that matrix C is a symmetric positive definite $m \times m$ matrix. Since A is a symmetric matrix, matrix C is symmetric. Moreover, if $x \neq 0$, then from Lemma 1 we have $Dx \neq 0$. In addition A is a positive definite matrix, which implies that C is a positive definite matrix. Thus, p^k converges to p by Theorem 1.3 in [10] when $0 < \omega < 2$. We see by (6) that operator D (defined by matrix D) is bounded and continuous. Hence when p^k converges to p, $\widetilde{\mathbf{u}}$ converges to $\mathbf{u}$ by (23) and (iv).

3. Global Superconvergence Estimates

In this section, we assume that $T^h = \{e\}$ is an almost uniform piecewise strongly regular mesh on the polygonal domain Ω (see [20]). $E(x)$ is a function defined on $W^{2,\infty}(\Omega)$ and $\mathbf{V}^h$ is the piecewise bilinear finite element space. Then we have the following Lemma (cf [18], [19], [21] and [30]):

Lemma 4. *Let $u \in H^3(\Omega)$ and $E(x) \in W^{2,\infty}(\Omega)$. Then there exists a positive constant c such that, for any $\mathbf{v} \in \mathbf{V}^h$,*

$$|\int_\Omega E(x)(u - I^h u)_{i,j} v_{k,l}| \leq ch^{\frac{3}{2}} ||u||_{3,\Omega} ||v||_{1,\Omega}, \quad 1 \leq i,j,k,l \leq 2.$$

We can derive the following result by the above Lemma.

Lemma 5. *Let $\mathbf{u} \in \mathbf{H}^3(\Omega)$ and $E_{ijkl} \in W^{2,\infty}(\Omega)$. Then there exists a positive constant c independent of h and H such that*

$$a(\mathbf{u} - I^h\mathbf{u}, \mathbf{v}) \leq ch^{\frac{3}{2}} ||\mathbf{u}||_{3,\Omega} ||\mathbf{v}||_{1,\Omega}, \quad \forall \mathbf{v} \in \mathbf{V}^h.$$

Theorem 5. *Let us assume that the number of points where $B\mathbf{u} - g$ changes from $B\mathbf{u} - g < 0$ to $B\mathbf{u} - g = 0$ is finite. Let $g \in W^{1,\infty}(\gamma_{c,i})(i = 1, \cdots, n_g)$ and the conditions of Lemma 5 hold. Then we have*

$$||I^h\mathbf{u} - \mathbf{u}_h||_{1,\Omega} \leq c(h^{\frac{3}{2}} + H^{\frac{5}{4}} + h^{\frac{3}{4}} H^{\frac{3}{4}}), \tag{24}$$

$$|||R^H p - p_H|||_{-\frac{1}{2},\Gamma_c} \leq c(h^{\frac{3}{2}} + H^{\frac{5}{4}} + h^{\frac{3}{4}} H^{\frac{3}{4}}). \tag{25}$$

Proof. Let $\mathbf{v} = I^h\mathbf{u} - \mathbf{u}_h$. Then similar to (18)–(21), we have

$$\sum_{i=1}^{n_g} [p - R^H p, B\mathbf{v}]_{\gamma_{c,i}} \leq cH^{\frac{3}{2}} (\sum_{i=1}^{n_g} ||p||^2_{1,\gamma_{c,i}})^{\frac{1}{2}} ||\mathbf{v}||_{1,\Omega}, \tag{26}$$

$$\sum_{i=1}^{n_g} [R^H p - p_H, B(I^h\mathbf{u} - \mathbf{u})]_{\gamma_{c,i}} \leq ch^{\frac{3}{2}} |||R^H p - p_H|||_{-\frac{1}{2},\Gamma_c} ||\mathbf{u}||_{3,\Omega}. \tag{27}$$

$$\sum_{\tau \not\subseteq \Gamma_c^- \cup \Gamma_c^0} (R^H p - p, w)_\tau \leq cH^{\frac{5}{2}} ||p||_{1,\tau} (||g||_{1,\infty,\tau} + ||\mathbf{u}||_{3,\Omega}), \tag{28}$$

$$|||R^H p - p_H|||_{-\frac{1}{2},\Gamma_c} \leq cH^{\frac{3}{2}} (\sum_{i=1}^{n_g} ||p||^2_{1,\gamma_{c,i}})^{\frac{1}{2}} + ch^{\frac{3}{2}} ||\mathbf{u}||_{3,\Omega} + c||I^h\mathbf{u} - \mathbf{u}^h||_{1,\Omega}, \tag{29}$$

where Lemma 5 is used. This completes the proof of the Theorem from (26) –(29).

We assume that T^h is obtained from T^{2h} with mesh size $2h$ by uniformly subdividing each element in T^{2h} into 4 congruent elements with mesh size h. We define the high order interpolation operator π^{2h} on the space of piecewise bilinear function space $\mathbf{V}^h$ and satisfying(see [1], [5],[21])

$$\begin{aligned} &\pi^{2h} I^h = \pi^{2h}, \qquad && \|\pi^{2h}\mathbf{v}\|_{1,q} \leq \|\mathbf{v}\|_{1,q}, \quad \forall \mathbf{v} \in \mathbf{V}^h, \\ &\|\pi^{2h}\mathbf{u} - \mathbf{u}\|_{1,q} \leq ch^{\frac{3}{2}} \|\mathbf{u}\|_{\frac{5}{2},q}, \qquad && 2 \leq q \leq \infty. \end{aligned} \tag{30}$$

Theorem 6. *Under the assumptions of Theorem 5, we obtain the following global superconvergence estimates and optimal estimates*

$$||\mathbf{u} - \pi^{2h}\mathbf{u}_h||_{1,\Omega} \le c(h^{\frac{3}{2}} + H^{\frac{5}{4}} + h^{\frac{3}{4}}H^{\frac{3}{4}}), \tag{31}$$

$$|||p - p_H|||_{-\frac{1}{2},\Gamma_c} \le c(h^{\frac{3}{2}} + H^{\frac{5}{4}} + h^{\frac{3}{4}}H^{\frac{3}{4}}), \tag{32}$$

$$||p - p_H||_{0,\Gamma_c} \le c(h^{\frac{3}{2}}H^{-\frac{1}{2}} + H^{\frac{3}{4}} + h^{\frac{3}{4}}H^{\frac{1}{4}}). \tag{33}$$

Proof. We have by (24),(30) and triangle inequality

$$\begin{aligned}
||\mathbf{u} - \pi^{2h}\mathbf{u}_h||_{1,\Omega} &\le ||\mathbf{u} - \pi^{2h}\mathbf{u}||_{1,\Omega} + ||\pi^{2h}(I^h\mathbf{u} - \mathbf{u}_h)||_{1,\Omega} \\
&\le ch^{\frac{3}{2}}||\mathbf{u}||_{\frac{5}{2},\Omega} + c||I^h\mathbf{u} - \mathbf{u}_h||_{1,\Omega} \\
&\le c(h^{\frac{3}{2}} + H^{\frac{5}{4}} + h^{\frac{3}{4}}H^{\frac{3}{4}}).
\end{aligned}$$

The inequality (32) now follows by (25) and Lemma 2. The inequality (33) follows by (25), inverse estimate and triangle inequality.

Remark 2. From (28) we see easily that if $p \in W^{1,\infty}(\gamma_{c,i})$, then the error bounds in inequality (24) is $\mathcal{O}(h^{\frac{3}{2}} + H^{\frac{3}{2}} + h^{\frac{3}{4}}H)$. Hence the error bounds in inequalities (31) and (32) are $\mathcal{O}(h^{\frac{3}{2}} + H^{\frac{3}{2}} + h^{\frac{3}{4}}H)$. Moreover we may extend the results of Theorem 1–Theorem 6 to bounded convex smooth domain as in [3], [4] and [15].

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