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EXPANSION OF STEP-TRANSITION OPERATOR OF MULTI-STEP METHOD AND ITS APPLICATIONS (I)*[1)]

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Abstract

We expand the step-transition operator of any linear multi-step method with order $s \geq 2$ up to $O(\tau^{s+5})$. And through examples we show how much the perturbation of the step-transition operator caused by the error of initial value is.

Key words: Multi-step method, Step-transition operator, Expansion.

1. Expansion of Step-Transition Operator

For an ordinarily differential equation

$$\frac{d}{dt}Z = f(Z), \quad Z \in \mathbb{R}^p, \tag{1}$$

any compatible linear m-step difference scheme

$$\sum_{k=0}^{m} \alpha_k Z_k = \tau \sum_{k=0}^{m} \beta_k f(Z_k) \quad \left(\sum_{k=0}^{m} \beta_k \neq 0\right), \tag{2}$$

can be characterized by a step-transition operator G (also denoted by G^τ): $\mathbb{R}^p \to \mathbb{R}^p$ satisfying

$$\sum_{k=0}^{m} \alpha_k G^k = \tau \sum_{k=0}^{m} \beta_k f \circ G^k, \tag{3}$$

where G^k stands for k-time composition of G: $G \circ G \cdots \circ G$ (refer to [1,2,3,5,6,7]). This operator G^τ can be represented as a power series in τ with first term equal to *identity* I.

Thus, this operator completely characterizes the multi-step scheme as: $Z_1 = G(Z_0), \cdots, Z_m = G(Z_{m-1}) = G^m(Z_0), \cdots$. For the expansion of this operator G, we give the following theorem:

Theorem 1. *If scheme (2) is of order $s \geq 2$, then the step-transition operator decided by equation (3) has the following expansion:*

$$G(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{s+1}A(Z) + \tau^{s+2}B(Z) + \tau^{s+3}C(Z) + \tau^{s+4}D(Z) + O(\tau^{s+5}), \tag{4}$$

where $Z^{[0]} = Z$, $Z^{[1]} = f(Z)$, $Z^{[k+1]} = \frac{\partial Z^{[k]}}{\partial Z} Z^{[1]}$ for $k = 1, 2, \cdots$. And

$$A = \lambda Z^{[s+1]}, \quad \lambda = \frac{\sum_{k=0}^{m} \left\{ \frac{k^s}{s!}\beta_k - \frac{k^{s+1}}{(s+1)!}\alpha_k \right\}}{\sum_{k=0}^{m} k\alpha_k}; \tag{4.1}$$

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$$B = \mu Z^{[s+2]} + \frac{\lambda}{2} Z_z^{[1]} Z^{[s+1]}, \quad \mu = \frac{\sum_{k=0}^m \left[\frac{k^{s+1}}{(s+1)!}\beta_k - \frac{k^{s+2}}{(s+2)!}\alpha_k - \frac{k^2-k}{2}\lambda\alpha_k\right]}{\sum_{k=0}^m k\alpha_k}; \tag{4.2}$$

$$\begin{aligned} C =& \nu Z^{[s+3]} + \left(\rho + \frac{\lambda}{6}\right) Z_z^{[1]} Z_z^{[1]} Z^{[s+1]} \\ &+ \left(\rho - \frac{\lambda}{12} + \frac{\mu}{2}\right) Z_z^{[1]} Z^{[s+2]} + \left(2\rho + \frac{\lambda}{3}\right) Z_{z^2}^{[1]} Z^{[1]} Z^{[s+1]}, \\ \nu =& \frac{\sum_{k=0}^m \left[\frac{k^{s+2}}{(s+2)!}\beta_k - \frac{k^{s+3}}{(s+3)!}\alpha_k - \left(\frac{2k^3-3k^2+k}{12}\lambda + \frac{k^2-k}{2}\mu\right)\alpha_k\right]}{\sum_{k=0}^m k\alpha_k}, \\ \rho =& \frac{\sum_{k=0}^m \left[\frac{k^2}{2}\beta_k - \frac{k^3}{6}\alpha_k\right]}{\sum_{k=0}^m k\alpha_k}\lambda; \end{aligned} \tag{4.3}$$

$$\begin{aligned} D =& \xi Z^{[s+4]} + \left\{\sigma - \frac{\rho}{2} + \chi - \frac{\mu}{12} + \frac{\nu}{2} - \eta\right\} Z_z^{[1]} Z^{[s+3]} \\ &+ \left\{\sigma - \frac{\lambda}{24} + \chi + \frac{\mu}{6} - \eta\right\} Z_z^{[1]} Z_z^{[1]} Z^{[s+2]} \\ &+ \left\{3\sigma - \rho - \frac{\lambda}{24} + 2\chi + \frac{\mu}{3} - 3\eta\right\} Z_{z^2}^{[1]} Z^{[1]} Z^{[s+2]} \\ &+ \left\{\sigma + \frac{\rho}{2} + \frac{\lambda}{24} - \epsilon\zeta\right\} Z_z^{[1]} Z_z^{[1]} Z_z^{[1]} Z^{[s+1]} \\ &+ \left\{2\sigma + \rho + \frac{\lambda}{12} - \eta - \epsilon\zeta\right\} Z_z^{[1]} Z_{z^2}^{[1]} Z^{[1]} Z^{[s+1]} \\ &+ \left\{3\sigma + \frac{\lambda}{8} - \eta - 2\epsilon\zeta\right\} Z_{z^2}^{[1]} Z^{[1]} \left(Z_z^{[1]} Z^{[s+1]}\right) \\ &+ \left\{3\sigma + \frac{\lambda}{8} - 2\eta - \epsilon\zeta\right\} Z_{z^2}^{[1]} Z^{[2]} Z^{[s+1]} \\ &+ \left\{3\sigma + \frac{\lambda}{8} - 2\eta - \epsilon\zeta\right\} Z_{z^3}^{[1]} \left(Z^{[1]}\right)^2 Z^{[s+1]}, \\ \xi =& \frac{\sum_{k=0}^m \left\{\frac{k^{s+3}}{(s+3)!}\beta_k - \left[\frac{k^{s+4}}{(s+4)!} + \frac{k^4-2k^3+k^2}{24}\lambda + \frac{2k^3-3k^2+k}{12}\mu + \frac{k^2-k}{2}\nu\right]\alpha_k\right\}}{\sum_{k=0}^m k\alpha_k}, \\ \sigma =& \frac{\sum_{k=0}^m \left[\frac{k^3}{6}\beta_k - \frac{k^4}{24}\alpha_k\right]}{\sum_{k=0}^m k\alpha_k}\lambda, \\ \chi =& \frac{\sum_{k=0}^m \left[\frac{k^2}{2}\beta_k - \frac{k^3}{6}\alpha_k\right]}{\sum_{k=0}^m k\alpha_k}\mu, \\ \eta =& \frac{\sum_{k=0}^m \frac{k^2-k}{2}\alpha_k}{\sum_{k=0}^m k\alpha_k}\rho, \\ \zeta =& \frac{\sum_{k=0}^m \frac{k^2-k}{2}\alpha_k}{\sum_{k=0}^m k\alpha_k}\lambda^2, \\ \epsilon =& \begin{cases} 1, & when\ s = 2; \\ 0, & when\ s \geq 3. \end{cases} \end{aligned} \tag{4.4}$$

We use the notation

$$Z^{[1]}_{z^3}\left(Z^{[1]}\right)^2 Z^{[s+1]} = \sum_{i,j,k=1}^{p} \frac{\partial^3\left[Z^{[1]}\right]}{\partial z_i \partial z_j \partial z_k}\left[Z^{[1]}\right]_{(i)}\left[Z^{[1]}\right]_{(j)}\left[Z^{[s+1]}\right]_{(k)}$$

where z_i is the i-th component of p-dim vector Z, and $\left[Z^{[1]}\right]_{(j)}$ stands for the j-th component of p-dim vector $Z^{[1]}$ (refer to [6]).

Remark 1. If $s \geq 3$, then $\sum_{k=0}^{m}\frac{k^2}{2}\beta_k = \sum_{k=0}^{m}\frac{k^3}{6}\alpha_k$, i.e., $\rho = 0$, $\chi = 0$, $\eta = 0$ and $\epsilon = 0$, then (4.3) and (4.4) become into

$$\begin{aligned} C =& \nu Z^{[s+3]} + \frac{\lambda}{6} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[s+1]} + \left(-\frac{\lambda}{12}+\frac{\mu}{2}\right) Z^{[1]}_{z} Z^{[s+2]} + \frac{\lambda}{3} Z^{[1]}_{z^2} Z^{[1]} Z^{[s+1]}, \\ \nu =& \frac{\sum_{k=0}^{m}\left[\frac{k^{s+2}}{(s+2)!}\beta_k - \frac{k^{s+3}}{(s+3)!}\alpha_k - \left(\frac{2k^3-3k^2+k}{12}\lambda + \frac{k^2-k}{2}\mu\right)\alpha_k\right]}{\sum_{k=0}^{m} k\alpha_k}; \end{aligned} \tag{4.3a}$$

$$\begin{aligned} D =& \xi Z^{[s+4]} + \left\{\sigma - \frac{\mu}{12} + \frac{\nu}{2}\right\} Z^{[1]}_{z} Z^{[s+3]} \\ &+ \left\{\sigma - \frac{\lambda}{24} + \frac{\mu}{6}\right\} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[s+2]} + \left\{3\sigma - \frac{\lambda}{24} + \frac{\mu}{3}\right\} Z^{[1]}_{z^2} Z^{[1]} Z^{[s+2]} \\ &+ \left\{\sigma + \frac{\lambda}{24}\right\}\left\{Z^{[1]}_{z} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[s+1]} + 2 Z^{[1]}_{z} Z^{[1]}_{z^2} Z^{[1]} Z^{[s+1]}\right. \\ &\qquad + 3 Z^{[1]}_{z^2} Z^{[1]}\left[Z^{[1]}_{z} Z^{[s+1]}\right] + 3 Z^{[1]}_{z^2} Z^{[2]} Z^{[s+1]} \\ &\qquad \left. + 3 Z^{[1]}_{z^3}\left(Z^{[1]}\right)^2 Z^{[s+1]}\right\}, \\ \xi =& \frac{\sum_{k=0}^{m}\left\{\frac{k^{s+3}}{(s+3)!}\beta_k - \left[\frac{k^{s+4}}{(s+4)!} + \frac{k^4-2k^3+k^2}{24}\lambda + \frac{2k^3-3k^2+k}{12}\mu + \frac{k^2-k}{2}\nu\right]\alpha_k\right\}}{\sum_{k=0}^{m} k\alpha_k}, \\ \sigma =& \frac{\sum_{k=0}^{m}\left[\frac{k^3}{6}\beta_k - \frac{k^4}{24}\alpha_k\right]}{\sum_{k=0}^{m} k\alpha_k}\lambda \end{aligned} \tag{4.4a}$$

respectively.

Especially, if $s \geq 4$, then $\sum_{k=0}^{m}\frac{k^3}{6}\beta_k = \sum_{k=0}^{m}\frac{k^4}{24}\alpha_k$, i.e., $\sigma = 0$, then (4.4a) becomes into

$$\begin{aligned} D =& \xi Z^{[s+4]} + \left\{\frac{\nu}{2} - \frac{\mu}{12}\right\} Z^{[1]}_{z} Z^{[s+3]} \\ &+ \left\{\frac{\mu}{6} - \frac{\lambda}{24}\right\} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[s+2]} + \left\{\frac{\mu}{3} - \frac{\lambda}{24}\right\} Z^{[1]}_{z^2} Z^{[1]} Z^{[s+2]} \\ &+ \frac{\lambda}{24}\left\{Z^{[1]}_{z} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[s+1]} + 2 Z^{[1]}_{z} Z^{[1]}_{z^2} Z^{[1]} Z^{[s+1]}\right. \\ &\qquad + 3 Z^{[1]}_{z^2} Z^{[1]}\left[Z^{[1]}_{z} Z^{[s+1]}\right] + 3 Z^{[1]}_{z^2} Z^{[2]} Z^{[s+1]} \\ &\qquad \left. + 3 Z^{[1]}_{z^3}\left(Z^{[1]}\right)^2 Z^{[s+1]}\right\}. \end{aligned} \tag{4.4b}$$

Remark 2. When system (1) is linear, i.e., $f(Z) = MZ$, M is a $p \times p$ matrix, then expansion (4) has a simpler form

$$G(Z) = \sum_{i=0}^{+\infty}\frac{\tau^i}{i!}M^i Z + \tau^{s+1}\lambda M^{s+1} Z + \tau^{s+2}\left(\frac{\lambda}{2}+\mu\right) M^{s+2} Z$$

$$+ \tau^{s+3}\left(\frac{\lambda}{12}+\frac{\mu}{2}+\nu+2\rho\right)M^{s+3}Z$$
$$+ \tau^{s+4}\left(\frac{\mu}{12}+\frac{\nu}{2}+\xi-2\eta-\epsilon\zeta+3\sigma+2\chi\right)M^{s+4}Z+O(\tau^{s+5}). \tag{4c}$$

Example 1. For the trapezoid scheme (with order $s=2$)

$$Z_1 - Z_0 = \frac{\tau}{2}\left[f(Z_1)+f(Z_0)\right], \tag{5}$$

from theorem 1 we have $\lambda=\frac{1}{12}$, $\mu=\frac{1}{24}$, $\nu=\frac{1}{80}$, $\rho=\frac{1}{144}$, $\xi=\frac{1}{360}$, $\sigma=\frac{1}{288}$, $\chi=\frac{1}{288}$, $\eta=\zeta=0$. And

$$\begin{aligned}
A =& \frac{1}{12}Z^{[3]},\\
B =& \frac{1}{24}Z^{[4]}+\frac{1}{24}Z^{[1]}_{z}Z^{[3]},\\
C =& \frac{1}{80}Z^{[5]}+\frac{1}{48}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[3]}+\frac{1}{48}Z^{[1]}_{z}Z^{[4]}+\frac{1}{24}Z^{[1]}_{z^2}Z^{[1]}Z^{[3]},\\
D =& \frac{1}{360}Z^{[6]}+\frac{1}{160}Z^{[1]}_{z}Z^{[5]}+\frac{1}{96}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[4]}\\
&+\frac{1}{48}Z^{[1]}_{z^2}Z^{[1]}Z^{[4]}+\frac{1}{96}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[3]}+\frac{1}{48}Z^{[1]}_{z}Z^{[1]}_{z^2}Z^{[1]}Z^{[3]}\\
&+\frac{1}{48}Z^{[1]}_{z^2}Z^{[1]}\left[Z^{[1]}_{z}Z^{[3]}\right]+\frac{1}{48}Z^{[1]}_{z^2}Z^{[2]}Z^{[3]}+\frac{1}{48}Z^{[1]}_{z^3}\left(Z^{[1]}\right)^2Z^{[3]}.
\end{aligned} \tag{6}$$

Example 2. For the leap-frog scheme (with order $s=2$)

$$Z_2 - Z_0 = 2\tau f(Z_1), \tag{7}$$

from theorem 1 we have $\lambda=-\frac{1}{6}$, $\mu=-\frac{1}{12}$, $\nu=-\frac{1}{120}$, $\rho=\frac{1}{36}$, $\xi=\frac{1}{360}$, $\sigma=\frac{1}{36}$, $\chi=\frac{1}{72}$, $\eta=\frac{1}{72}$, $\zeta=\frac{1}{72}$. And

$$\begin{aligned}
A =& -\frac{1}{6}Z^{[3]},\\
B =& -\frac{1}{12}Z^{[4]}-\frac{1}{12}Z^{[1]}_{z}Z^{[3]},\\
C =& -\frac{1}{120}Z^{[5]},\\
D =& \frac{1}{360}Z^{[6]}+\frac{1}{60}Z^{[1]}_{z}Z^{[5]}+\frac{1}{48}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[4]}\\
&+\frac{1}{48}Z^{[1]}_{z^2}Z^{[1]}Z^{[4]}+\frac{1}{48}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[3]}+\frac{1}{24}Z^{[1]}_{z}Z^{[1]}_{z^2}Z^{[1]}Z^{[3]}\\
&+\frac{1}{48}Z^{[1]}_{z^2}Z^{[1]}\left[Z^{[1]}_{z}Z^{[3]}\right]+\frac{1}{48}Z^{[1]}_{z^2}Z^{[2]}Z^{[3]}+\frac{1}{48}Z^{[1]}_{z^3}\left(Z^{[1]}\right)^2Z^{[3]}.
\end{aligned} \tag{8}$$

Example 3. For the Simpson scheme (with order $s=4$)

$$Z_2 - Z_0 = \frac{\tau}{3}\left[f(Z_2)+4f(Z_1)+f(Z_0)\right], \tag{9}$$

from theorem 1 we have $\lambda=\frac{1}{180}$, $\mu=\frac{1}{360}$, $\nu=\frac{1}{180\times7\times3}$, $\rho=0$, $\xi=-\frac{1}{180\times7\times8}$, $\sigma=\chi=\eta=\zeta=0$. And

$$A = \frac{1}{180}Z^{[5]},$$

$$
\begin{aligned}
B =& \frac{1}{360} Z^{[6]} + \frac{1}{360} Z_z^{[1]} Z^{[5]}, \\
C =& \frac{1}{180\times 7\times 3} Z^{[7]} + \frac{1}{180\times 6} Z_z^{[1]} Z_z^{[1]} Z^{[5]} + \frac{1}{180\times 6} Z_z^{[1]} Z^{[6]} + \frac{1}{180\times 3} Z_{z^2}^{[1]} Z^{[1]} Z^{[5]}, \\
D =& -\frac{1}{180\times 7\times 8} Z^{[8]} - \frac{1}{180\times 7\times 8} Z_z^{[1]} Z^{[7]} + \frac{1}{180\times 24} Z_z^{[1]} Z_z^{[1]} Z^{[6]} \\
& + \frac{1}{180\times 8} Z_{z^2}^{[1]} Z^{[1]} Z^{[6]} + \frac{1}{180\times 24} Z_z^{[1]} Z_z^{[1]} Z_z^{[1]} Z^{[5]} + \frac{1}{180\times 12} Z_z^{[1]} Z_{z^2}^{[1]} Z^{[1]} Z^{[5]} \\
& + \frac{1}{180\times 8} Z_{z^2}^{[1]} Z^{[1]} \left[Z_z^{[1]} Z^{[5]} \right] + \frac{1}{180\times 8} Z_{z^2}^{[1]} Z^{[2]} Z^{[5]} + \frac{1}{180\times 8} Z_{z^3}^{[1]} \left(Z^{[1]} \right)^2 Z^{[5]}.
\end{aligned}
\tag{10}
$$

Proof of Theorem 1. When we set

$$
G^k(Z) = \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} Z^{[i]} + \tau^{s+1} A_k(Z) + \tau^{s+2} B_k(Z) + \tau^{s+3} C_k(Z) + \tau^{s+4} D_k(Z) + O(\tau^{s+5}), \tag{11}
$$

then

$$
\begin{aligned}
& \sum_{i=0}^{+\infty} \frac{(k+1)^i \tau^i}{i!} Z^{[i]} + \tau^{s+1} A_{k+1}(Z) + \tau^{s+2} B_{k+1}(Z) \\
& \qquad + \tau^{s+3} C_{k+1}(Z) + \tau^{s+4} D_{k+1}(Z) + O(\tau^{s+5}) \\
& = G^{k+1}(Z) = G^k \left[G(Z) \right] \\
& = \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} \left[G(Z) \right]^{[i]} + \tau^{s+1} A_k \left[G(Z) \right] + \tau^{s+2} B_k \left[G(Z) \right] \\
& \quad + \tau^{s+3} C_k \left[G(Z) \right] + \tau^{s+4} D_k \left[G(Z) \right] + O(\tau^{s+5}) \\
& \equiv \widetilde{I} + \widetilde{II} + \widetilde{III} + \widetilde{IV} + \widetilde{V} + O(\tau^{s+5}),
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
\widetilde{I} =& \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right. \\
& \qquad \left. + \tau^{s+1} A_1(Z) + \tau^{s+2} B_1(Z) + \tau^{s+3} C_1(Z) + \tau^{s+4} D_1(Z) + O(\tau^{s+5}) \right]^{[i]} \\
=& \sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} + \tau^{s+1} A_1(Z) + \tau^{s+2} B_1(Z) + \tau^{s+3} C_1(Z) + \tau^{s+4} D_1(Z) + O(\tau^{s+5}) \\
& + \frac{k\tau}{1!} \left\{ Z^{[1]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right] + Z_z^{[1]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right] * \left(\tau^{s+1} A_1 + \tau^{s+2} B_1 + \tau^{s+3} C_1 \right) \right\} \\
& (\textit{here } s \geq 2 \textit{ is requested}) \\
& + \frac{k^2\tau^2}{2!} \left\{ Z^{[2]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right] + Z_z^{[2]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right] * \left(\tau^{s+1} A_1 + \tau^{s+2} B_1 \right) \right\} \\
& + \frac{k^3\tau^3}{3!} \left\{ Z^{[3]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right] + Z_z^{[3]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \right] * \left(\tau^{s+1} A_1 \right) \right\} \\
& + \cdots\cdots
\end{aligned}
\tag{12.1}
$$

$$
\begin{aligned}
=&\sum_{l=0}^{+\infty}\frac{(k+1)^l\tau^l}{l!}Z^{[l]}+\tau^{s+1}A_1+\tau^{s+2}\left\{B_1+kZ_z^{[1]}A_1\right\}\\
&+\tau^{s+3}\left\{C_1+kZ_z^{[1]}B_1+kZ_{z^2}^{[1]}Z^{[1]}A_1+\frac{k^2}{2}Z_z^{[2]}A_1\right\}\\
&+\tau^{s+4}\left\{D_1+kZ_z^{[1]}C_1+kZ_{z^2}^{[1]}Z^{[1]}B_1+\frac{k}{2}Z_{z^2}^{[1]}Z^{[2]}A_1+\frac{k}{2}Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2A_1\right.\\
&\qquad\left.+\frac{k^2}{2}Z_z^{[2]}B_1++\frac{k^2}{2}Z_{z^2}^{[2]}Z^{[1]}A_1+\frac{k^3}{6}Z_z^{[3]}A_1\right\}+O(\tau^{s+5});
\end{aligned}
$$

In the sequel, we set $W=A_1$ if $s=2$, $W=0$ otherwise.

$$
\begin{aligned}
\widetilde{II}=&\tau^{s+1}A_k\circ\left(Z+\tau Z^{[1]}+\frac{\tau^2}{2}Z^{[2]}+\frac{\tau^3}{6}Z^{[3]}+\tau^3\epsilon W\right)+O(\tau^{s+5})\\
&(\textit{here } s\geq 2 \textit{ is requested}) \qquad (12.2)\\
=&\tau^{s+1}A_k+\tau^{s+2}\left\{(A_k)_z\,Z^{[1]}\right\}+\tau^{s+3}\left\{\frac{1}{2}(A_k)_z\,Z^{[2]}+\frac{1}{2}(A_k)_{z^2}\left[Z^{[1]}\right]^2\right\}\\
&+\tau^{s+4}\left\{\frac{1}{6}(A_k)_z\,Z^{[3]}+\epsilon\,(A_k)_z\,W+\frac{1}{2}(A_k)_{z^2}\left[Z^{[1]}Z^{[2]}\right]+\frac{1}{6}(A_k)_{z^3}\left[Z^{[1]}\right]^3\right\}\\
&+O(\tau^{s+5});
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{III}=&\tau^{s+2}B_k\circ\left(Z+\tau Z^{[1]}+\frac{\tau^2}{2}Z^{[2]}\right)+O(\tau^{s+5})\\
=&\tau^{s+2}B_k+\tau^{s+3}\left\{(B_k)_z\,Z^{[1]}\right\} \qquad (12.3)\\
&+\tau^{s+4}\left\{\frac{1}{2}(B_k)_z\,Z^{[2]}+\frac{1}{2}(B_k)_{z^2}\left[Z^{[1]}\right]^2\right\}+O(\tau^{s+5});
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{IV}&=\tau^{s+3}C_k\circ\left(Z+\tau Z^{[1]}\right)+O(\tau^{s+5}) \qquad (12.4)\\
&=\tau^{s+3}C_k+\tau^{s+4}\left\{(C_k)_z\,Z^{[1]}\right\}+O(\tau^{s+5});
\end{aligned}
$$

$$
\widetilde{V}=\tau^{s+4}D_k+O(\tau^{s+5}). \qquad (12.5)
$$

We obtain from (12), (12.1)–(12.5)

$$
A_{k+1}=A_1+A_k; \qquad (13.1)
$$

$$
B_{k+1}=B_1+kZ_z^{[1]}A_1+(A_k)_z\,Z^{[1]}+B_k; \qquad (13.2)
$$

$$
\begin{aligned}
C_{k+1}=&C_1+kZ_z^{[1]}B_1+kZ_{z^2}^{[1]}\left[Z^{[1]}A_1\right]+\frac{k^2}{2}Z_z^{[2]}A_1+\frac{1}{2}(A_k)_z\,Z^{[2]}\\
&+\frac{1}{2}(A_k)_{z^2}\left[Z^{[1]}\right]^2+(B_k)_z\,Z^{[1]}+C_k; \qquad (13.3)
\end{aligned}
$$

$$
D_{k+1}=D_1+kZ_z^{[1]}C_1+kZ_{z^2}^{[1]}\left[Z^{[1]}B_1\right]
$$

$$+\frac{k}{2}Z^{[1]}_{z^2}\left[Z^{[2]}A_1\right]+\frac{k}{2}Z^{[1]}_{z^3}\left[\left(Z^{[1]}\right)^2A_1\right] \tag{13.4}$$
$$+\frac{k^2}{2}Z^{[2]}_{z}B_1+\frac{k^2}{2}Z^{[2]}_{z^2}\left[Z^{[1]}A_1\right]+\frac{k^3}{6}Z^{[3]}_{z}A_1+\frac{1}{6}\left(A_k\right)_zZ^{[3]}$$
$$+\epsilon\left(A_k\right)_zW+\frac{1}{2}\left(A_k\right)_{z^2}\left[Z^{[1]}Z^{[2]}\right]+\frac{1}{6}\left(A_k\right)_{z^3}\left[Z^{[1]}\right]^3$$
$$+\frac{1}{2}\left(B_k\right)_zZ^{[2]}+\frac{1}{2}\left(B_k\right)_{z^2}\left[Z^{[1]}\right]^2+\left(C_k\right)_zZ^{[1]}+D_k.$$

From (3) we have

$$\sum_{k=0}^{m}\alpha_k\left[\sum_{i=0}^{+\infty}\frac{k^i\tau^i}{i!}Z^{[j]}+\tau^{s+1}A_k(Z)+\tau^{s+2}B_k(Z)+\tau^{s+3}C_k(Z)+\tau^{s+4}D_k(Z)+O(\tau^{s+5})\right]$$
$$=\tau\sum_{k=0}^{m}\beta_kf\left(\sum_{i=0}^{+\infty}\frac{k^i\tau^i}{i!}Z^{[j]}+\tau^{s+1}A_k(Z)+\tau^{s+2}B_k(Z)+\tau^{s+3}C_k(Z)+O(\tau^{s+4})\right)$$
$$=\tau\sum_{k=0}^{m}\beta_k\left\{f\circ\left[\sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]}\right]+f_z\circ\left[\sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]}\right]*\left(\tau^{s+1}A_k+\tau^{s+2}B_k+\tau^{s+3}C_k\right)\right\}$$
$$+O(\tau^{s+5})\qquad (\textit{here } s\geq 2 \textit{ is requested}) \tag{14}$$
$$=\sum_{l=0}^{+\infty}\sum_{k=0}^{m}\beta_k\frac{k^l\tau^{l+1}}{l!}Z^{[l+1]}$$
$$+\tau^{s+2}\sum_{k=0}^{m}\beta_kZ^{[1]}_{z}A_k+\tau^{s+3}\sum_{k=0}^{m}\beta_k\left\{Z^{[1]}_{z}B_k+kZ^{[1]}_{z^2}Z^{[1]}A_k\right\}$$
$$+\tau^{s+4}\sum_{k=0}^{m}\beta_k\left\{Z^{[1]}_{z}C_k+kZ^{[1]}_{z^2}Z^{[1]}B_k+\frac{k^2}{2}Z^{[1]}_{z^2}Z^{[2]}A_k+\frac{k^2}{2}Z^{[1]}_{z^3}\left(Z^{[1]}\right)^2A_k\right\}+O(\tau^{s+5}),$$

comparing the coefficients of τ^{s+1}, τ^{s+2}, τ^{s+3} and τ^{s+4} respectively on both sides of (14) we obtain

$$\sum_{k=0}^{m}\alpha_kA_k=\sum_{k=0}^{m}\left\{\frac{k^s}{s!}\beta_k-\frac{k^{s+1}}{(s+1)!}\alpha_k\right\}Z^{[s+1]}; \tag{15.1}$$

$$\sum_{k=0}^{m}\alpha_kB_k=\sum_{k=0}^{m}\left\{\frac{k^{s+1}}{(s+1)!}\beta_k-\frac{k^{s+2}}{(s+2)!}\alpha_k\right\}Z^{[s+2]}+\sum_{k=0}^{m}\beta_kZ^{[1]}_{z}A_k; \tag{15.2}$$

$$\sum_{k=0}^{m}\alpha_kC_k=\sum_{k=0}^{m}\left\{\frac{k^{s+2}}{(s+2)!}\beta_k-\frac{k^{s+3}}{(s+3)!}\alpha_k\right\}Z^{[s+3]}$$
$$+\sum_{k=0}^{m}\beta_k\left\{Z^{[1]}_{z}B_k+kZ^{[1]}_{z^2}Z^{[1]}A_k\right\}; \tag{15.3}$$

$$\sum_{k=0}^{m}\alpha_kD_k=\sum_{k=0}^{m}\left\{\frac{k^{s+3}}{(s+3)!}\beta_k-\frac{k^{s+4}}{(s+4)!}\alpha_k\right\}Z^{[s+4]}$$
$$+\sum_{k=0}^{m}\beta_k\left\{Z^{[1]}_{z}C_k+kZ^{[1]}_{z^2}Z^{[1]}B_k\right. \tag{15.4}$$

$$+\frac{k^2}{2}Z^{[1]}_{z^2}Z^{[2]}A_k+\frac{k^2}{2}Z^{[1]}_{z^3}\left(Z^{[1]}\right)^2A_k\bigg\}.$$

From relations (13.1) and (15.1) we deduce directly

$$A_k = kA_1 \equiv kA, \tag{16}$$

and (4.1).

Substituting (16) into (13.2), we obtain

$$B_k = kB_1 + \frac{k^2-k}{2}\lambda\left(Z^{[1]}_z Z^{[s+1]} + Z^{[s+2]}\right), \tag{17}$$

substituting (17) and (16) into (15.2), we obtain

$$\begin{aligned}\left(\sum_{k=0}^{m} k\alpha_k\right)B_1 =& \sum_{k=0}^{m}\left\{\frac{k^{s+1}}{(s+1)!}\beta_k - \frac{k^{s+2}}{(s+2)!}\alpha_k - \frac{k^2-k}{2}\lambda\alpha_k\right\}Z^{[s+2]}\\ &+\sum_{k=0}^{m}\left\{k\lambda\beta_k - \frac{k^2-k}{2}\lambda\alpha_k\right\}Z^{[1]}_z Z^{[s+1]}.\end{aligned}$$

Since $s \geq 2$, $\sum_{k=0}^{m} k\lambda\beta_k = \sum_{k=0}^{m}\frac{k^2}{2}\lambda\alpha_k$, we have (4.2), and

$$B_k = \frac{k^2\lambda}{2}Z^{[1]}_z Z^{[s+1]} + \left(\frac{k^2-k}{2}\lambda + k\mu\right)Z^{[s+2]}. \tag{18}$$

Substituting (16) and (18) into (13.3), we have

$$\begin{aligned}C_{k+1} =& C_1 + \frac{k^2+k}{2}\lambda\left[Z^{[1]}_z Z^{[1]}_z Z^{[s+1]} + 2Z^{[1]}_{z^2}Z^{[1]}Z^{[s+1]}\right]\\ &+\left(\frac{k^2}{2}\lambda + k\mu\right)\left[Z^{[1]}_z Z^{[s+2]} + Z^{[s+3]}\right] + C_k,\end{aligned}$$

and then

$$\begin{aligned}C_k =& kC_1 + \frac{k^3-k}{6}\lambda\left[Z^{[1]}_z Z^{[1]}_z Z^{[s+1]} + 2Z^{[1]}_{z^2}Z^{[1]}Z^{[s+1]}\right]\\ &+\left(\frac{2k^3-3k^2+k}{12}\lambda + \frac{k^2-k}{2}\mu\right)\left[Z^{[1]}_z Z^{[s+2]} + Z^{[s+3]}\right].\end{aligned} \tag{19}$$

Substituting (19), (16) and (18) into (15.3), and utilizing $s \geq 2$, we obtain (4.3), and

$$\begin{aligned}C_k =& \left[\frac{2k^3-3k^2+k}{12}\lambda + \frac{k^2-k}{2}\mu + k\nu\right]Z^{[s+3]}\\ &+\left[\frac{k^3\lambda}{6} + k\rho\right]Z^{[1]}_z Z^{[1]}_z Z^{[s+1]} + \left[\frac{k^3\lambda}{3} + 2k\rho\right]Z^{[1]}_{z^2}Z^{[1]}Z^{[s+1]}\\ &+\left[\frac{2k^3-3k^2}{12}\lambda + \frac{k^2}{2}\mu + k\rho\right]Z^{[1]}_z Z^{[s+2]}.\end{aligned} \tag{20}$$

Substituting (16), (18) and (20) into (13.4), we obtain

$$D_{k+1} = D_1 + \left\{\frac{k^3}{6}\lambda + \frac{k^2}{2}\mu + k\nu + k\rho\right\}Z^{[1]}_z Z^{[s+3]}$$

$$
\begin{aligned}
&+\left\{\frac{2k^3-k}{12}\lambda+\frac{k^2+k}{2}\mu+2k\rho\right\}Z_z^{[1]}Z_z^{[1]}Z^{[s+2]}\\
&+\left\{\frac{2k^3+k^2}{4}\lambda+\left(k^2+k\right)\mu+3k\rho\right\}Z_{z^2}^{[1]}Z^{[1]}Z^{[s+2]}\\
&+\left\{\frac{2k^3+3k^2+2k}{12}\lambda+k\rho+k\epsilon\lambda^2\right\}Z_z^{[1]}Z_z^{[1]}Z_z^{[1]}Z^{[s+1]}\\
&+\left\{\frac{2k^3+3k^2+2k}{6}\lambda+3k\rho+k\epsilon\lambda^2\right\}Z_z^{[1]}Z_{z^2}^{[1]}Z^{[1]}Z^{[s+1]}\\
&+\left\{\frac{2k^3+3k^2+2k}{4}\lambda+k\rho+2k\epsilon\lambda^2\right\}Z_{z^2}^{[1]}Z^{[1]}\left[Z_z^{[1]}Z^{[s+1]}\right]\\
&+\left\{\frac{2k^3+3k^2+2k}{4}\lambda+2k\rho+k\epsilon\lambda^2\right\}Z_{z^2}^{[1]}Z^{[2]}Z^{[s+1]}\\
&+\left\{\frac{2k^3+3k^2+2k}{4}\lambda+2k\rho+k\epsilon\lambda^2\right\}Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2Z^{[s+1]}\\
&+\left\{\frac{k^3}{6}\lambda+\frac{k^2}{2}\mu+k\nu\right\}Z^{[s+4]}+D_k,
\end{aligned}
$$

and then

$$
\begin{aligned}
D_k =&kD_1+\left\{\frac{k^4-2k^3+k^2}{24}\lambda+\frac{2k^3-3k^2+k}{12}\mu+\frac{k^2-k}{2}\nu+\frac{k^2-k}{2}\rho\right\}Z_z^{[1]}Z^{[s+3]}\\
&+\left\{\frac{k^4-2k^3+k}{24}\lambda+\frac{k^3-k}{6}\mu+\left(k^2-k\right)\rho\right\}Z_z^{[1]}Z_z^{[1]}Z^{[s+2]}\\
&+\left\{\frac{3k^4-4k^3+k}{24}\lambda+\frac{k^3-k}{3}\mu+\frac{3\left(k^2-k\right)}{2}\rho\right\}Z_{z^2}^{[1]}Z^{[1]}Z^{[s+2]}\\
&+\left\{\frac{k^4-k}{24}\lambda+\frac{k^2-k}{2}\rho+\frac{k^2-k}{2}\epsilon\lambda^2\right\}Z_z^{[1]}Z_z^{[1]}Z_z^{[1]}Z^{[s+1]}\\
&+\left\{\frac{k^4-k}{12}\lambda+\frac{3\left(k^2-k\right)}{2}\rho+\frac{k^2-k}{2}\epsilon\lambda^2\right\}Z_z^{[1]}Z_{z^2}^{[1]}Z^{[1]}Z^{[s+1]}\\
&+\left\{\frac{k^4-k}{8}\lambda+\frac{k^2-k}{2}\rho+\left(k^2-k\right)\epsilon\lambda^2\right\}Z_{z^2}^{[1]}Z^{[1]}\left[Z_z^{[1]}Z^{[s+1]}\right]\\
&+\left\{\frac{k^4-k}{8}\lambda+\left(k^2-k\right)\rho+\frac{k^2-k}{2}\epsilon\lambda^2\right\}Z_{z^2}^{[1]}Z^{[2]}Z^{[s+1]}\\
&+\left\{\frac{k^4-k}{8}\lambda+\left(k^2-k\right)\rho+\frac{k^2-k}{2}\epsilon\lambda^2\right\}Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2Z^{[s+1]}\\
&+\left\{\frac{k^4-2k^3+k^2}{24}\lambda+\frac{2k^3-3k^2+k}{12}\mu+\frac{k^2-k}{2}\nu\right\}Z^{[s+4]}.
\end{aligned}
\tag{21.a}
$$

On the other hand, substituting (16), (18) and (20) into (15.4) we have

$$
\begin{aligned}
\sum_{k=0}^{m}\alpha_kD_k =&\sum_{k=0}^{m}\left\{\frac{k^{s+3}}{(s+3)!}\beta_k-\frac{k^{s+4}}{(s+4)!}\alpha_k\right\}Z^{[s+4]}\\
&+\sum_{k=0}^{m}\beta_k\left\{\left[\frac{2k^3-3k^2+k}{12}\lambda+\frac{k^2-k}{2}\mu+k\nu\right]Z_z^{[1]}Z^{[s+3]}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{k^3\lambda}{6}+k\rho\right] Z_z^{[1]} Z_z^{[1]} Z_z^{[1]} Z^{[s+1]} \\
&+\left[\frac{k^3\lambda}{3}+2k\rho\right] Z_z^{[1]} Z_{z^2}^{[1]} Z^{[1]} Z^{[s+1]} \\
&+\left[\frac{2k^3-3k^2}{12}\lambda+\frac{k^2}{2}\mu+k\rho\right] Z_z^{[1]} Z_z^{[1]} Z^{[s+2]} \\
&+\frac{k^3\lambda}{2} Z_{z^2}^{[1]} Z^{[1]}\left[Z_z^{[1]} Z^{[s+1]}\right] \\
&+\left[\frac{k^3-k^2}{2}\lambda+k^2\mu\right] Z_{z^2}^{[1]} Z^{[1]} Z^{[s+2]} \\
&+\frac{k^3\lambda}{2} Z_{z^2}^{[1]} Z^{[2]} Z^{[s+1]}+\frac{k^3\lambda}{2} Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2 Z^{[s+1]}\Bigg\}.
\end{aligned} \tag{21.b}
$$

Combining (21.a) and (21.b), and utilizing $s \geq 2$ we obtain (4.4). By now we have finished the proof.

Remark 3. Similarly, one can also get the expansions of the step-transition operators of some kinds of generalized linear multi-step methods (such as in [2], [6]), but the procedures would be much more complicated and tedious.

Remark 4. For some comments on Feng's definition of the step-transition operator for multi-step method, one can refer to McLachlan and Scovel [4].

2. Perturbation of Step-Transition Operator

Now let's consider how Z_m depends on the initial values of Z_k, $k = 0, 1, \cdots, m-1$ in equation (2) (refer to [4]):

Setting $Z_k = G^k(Z) + \epsilon_k$, $k = 0, 1, \cdots, m$, substituting this into (2) and using (3) we have

$$
\sum_{k=0}^{m} \alpha_k \epsilon_k = \tau \sum_{k=0}^{m} \beta_k \left[f\left(G^k + \epsilon_k\right) - f\left(G^k\right) \right]. \tag{22}
$$

Providing f is a smooth bounded operator $\|f_z\| \leq T$, then from (22) we obtain

$$
\sum_{k=0}^{m} |\alpha_k| \|\epsilon_k\| \leq \tau \sum_{k=0}^{m} |\beta_k| T \|\epsilon_k\|,
$$

so we have

Theorem 2. *If $Z_k = G^k(Z) + \epsilon_k$, $k = 0, 1, \cdots, m$, and f satisfies $\|f_z\| \leq T$, then from (2), (3) we obtain the following estimate:*

$$
\|\epsilon_m\| \leq \frac{\sum_{k=0}^{m-1} \left\{ |\alpha_k| + \tau |\beta_k| T \right\} \|\epsilon_k\|}{|\alpha_m| - \tau |\beta_m| T}. \tag{23}
$$

The proof of **Theorem 2** is easy.

Example 4. For the leap-frog scheme (7), if $f(Z) = MZ$ where M is a $p \times p$ matrix, then

$$
Z_2 - Z_0 = 2\tau M Z_1, \tag{24}
$$

or generally

$$
Z_{k+1} - Z_{k-1} = 2\tau M Z_k. \tag{25}
$$

According to (24), one can easily get the expression of the step-transition operator G:

$$G = \tau M + \sqrt{I + \tau^2 M^2}. \tag{26}$$

On the other hand, from (25), (24) we obtain

$$\begin{aligned} Z_k = &\left(\sqrt{I+\tau^2 M^2}\right)^{-1} \left(\tau M + \sqrt{I+\tau^2 M^2}\right)^k \frac{\left(-\tau M + \sqrt{I+\tau^2M^2}\right) Z_0 + Z_1}{2} \\ &+ \left(\sqrt{I+\tau^2 M^2}\right)^{-1} \left(\tau M - \sqrt{I+\tau^2 M^2}\right)^k \frac{\left(\tau M + \sqrt{I+\tau^2M^2}\right) Z_0 - Z_1}{2}. \end{aligned} \tag{27}$$

If $Z_1 = G(Z_0) + \delta$, then

$$\begin{aligned} Z_k - G^k(Z_0) &= \left(\sqrt{I+\tau^2M^2}\right)^{-1} \left\{G^k - (-G)^{-k}\right\} \frac{\delta}{2} \\ &= \sum_{\substack{0 \le i \le k \\ i \ odd}} \begin{pmatrix} k \\ i \end{pmatrix} (\tau M)^{k-i} \left(\sqrt{I+\tau^2M^2}\right)^{i-1} \delta \\ &= \sum_{0\le 2r+1 \le k} \begin{pmatrix} k \\ 2r+1 \end{pmatrix} (\tau M)^{k-1-2r} \left(I + \tau^2 M^2\right)^r \delta. \end{aligned} \tag{28}$$

When we set $\|M\| = w$, then

$$\begin{aligned} \|Z_k - G^k(Z_0)\| &\le \left(\sqrt{1+\tau^2w^2}\right)^{-1} \left(\tau w + \sqrt{1+\tau^2w^2}\right)^k \delta \\ &\le \left(1 + \tau w + \frac{1}{2}\tau^2 w^2\right)^k \delta. \end{aligned} \tag{29}$$

Example 5. For the Simpson scheme (9), if $f(Z) = MZ$ where M is a $p \times p$ matrix, then

$$Z_2 - Z_0 = \frac{\tau}{3} M \left(Z_2 + 4Z_1 + Z_0\right), \tag{30}$$

or generally

$$Z_{k+1} - Z_{k-1} = \frac{\tau}{3} M \left(Z_{k+1} + 4Z_k + Z_{k-1}\right). \tag{31}$$

According to (30), one can easily get the expression of the step-transition operator G:

$$G = \left(I - \frac{1}{3}\tau M\right)^{-1} \left\{\sqrt{I + \frac{1}{3}\tau^2 M^2} + \frac{2}{3}\tau M\right\}. \tag{32}$$

On the other hand, from (31), (30) we obtain

$$\begin{aligned} Z_k = &\frac{1}{2} \left(\sqrt{I + \frac{1}{3}\tau^2 M^2}\right)^{-1} \left(I - \frac{1}{3}\tau M\right)^{-k} \\ &\left(\sqrt{I + \frac{1}{3}\tau^2M^2} + \frac{2}{3}\tau M\right)^k \left\{\left(I - \frac{1}{3}\tau M\right) Z_1 + \left(\sqrt{I + \frac{1}{3}\tau^2M^2} - \frac{2}{3}\tau M\right) Z_0\right\} \end{aligned}$$

$$+\frac{1}{2}\left(\sqrt{I+\frac{1}{3}\tau^2M^2}\right)^{-1}\left(I-\frac{1}{3}\tau M\right)^{-k} \tag{33}$$
$$\left(-\sqrt{I+\frac{1}{3}\tau^2M^2}+\frac{2}{3}\tau M\right)^k\left\{-\left(I-\frac{1}{3}\tau M\right)Z_1+\left(\sqrt{I+\frac{1}{3}\tau^2M^2}+\frac{2}{3}\tau M\right)Z_0\right\}.$$

If $Z_1 = G(Z_0)+\delta$, then

$$\begin{aligned}Z_k-G^k(Z_0)=&\frac{1}{2}\left(\sqrt{I+\frac{1}{3}\tau^2M^2}\right)^{-1}\left(I-\frac{1}{3}\tau M\right)^{-(k-1)}\\&\left\{\left(\sqrt{I+\frac{1}{3}\tau^2M^2}+\frac{2}{3}\tau M\right)^k-\left(-\sqrt{I+\frac{1}{3}\tau^2M^2}+\frac{2}{3}\tau M\right)^k\right\}\delta \qquad (34)\\=&\left(I-\frac{1}{3}\tau M\right)^{-(k-1)}\sum_{\substack{0\le i\le k\\ i\ odd}}\begin{pmatrix}k\\ i\end{pmatrix}\left(\frac{2}{3}\tau M\right)^{k-i}\left(\sqrt{I+\frac{1}{3}\tau^2M^2}\right)^{i-1}\delta\\=&\left(I-\frac{1}{3}\tau M\right)^{-(k-1)}\sum_{0\le 2r+1\le k}\begin{pmatrix}k\\ 2r+1\end{pmatrix}\left(\frac{2}{3}\tau M\right)^{k-1-2r}\left(I+\frac{1}{3}\tau^2M^2\right)^r\delta.\end{aligned}$$

When we set $\|M\| = w$, then

$$\begin{aligned}\|Z_k-G^k(Z_0)\|\le&\left(\sqrt{1+\frac{1}{3}\tau^2w^2}\right)^{-1}\left(1-\frac{1}{3}\tau w\right)^{-(k-1)}\\&\left(\sqrt{1+\frac{1}{3}\tau^2w^2}+\frac{2}{3}\tau w\right)^k\delta \qquad (35)\\\le&\left\{\frac{1+\frac{2}{3}\tau w+\frac{1}{6}\tau^2w^2}{1-\frac{1}{3}\tau w}\right\}^k\delta.\end{aligned}$$

References

[1] K. Feng, The Step-Transition Operators for Multi-Step Methods of ODE's, (1990). Also in Collected Works of Feng Kang (II), National Defence Industry Press, Beijing, (1995), 274-283.

[2] E. Hairer, P. Leone, Order Barriers for Symplectic Multi-Value Methods, in Numerical Analysis 1997, Proceedings of the 17th Dundee Biennial Conference, June 24-27, 1997 (Edited by D.F. Griffiths, D.J. Higham and G.A. Watson), Pitman Research Notes in Mathematics Series Vol. **380** (1998), 133-149.

[3] U. Kirchgraber, Multi-Step Methods Are Essentially One-Step Methods, *Numer. Math.* **48** (1986), 85-90.

[4] R.I. McLachlan, J.C. Scovel, A Survey of Open Problems in Symplectic Integration, *Fields Inst. Communications*, **10** (1996), 151-180.

[5] D. Stoffer, General Linear Methods: Connection to One-Step Methods and Invariant Curves, *Numer. Math.,* **64** (1993), 395-407.

[6] Y.F. Tang, The Symplecticity of Multi-Step Methods, *Computers Math. Applic.,* **25**:3 (1993), 83-90.

[7] Y.F. Tang, On Conjugate Symplecticity of Multi-Step Methods, *J. Computa. Math.,* **18**:4 (2000), 431-438.