Journal of Computational Mathematics, Vol.20, No.5, 2002, 551–560.

SPECTRAL ANALYSIS OF THE FIRST-ORDER HERMITE CUBIC SPLINE COLLOCATION DIFFERENTIATION MATRICES[*1)]

Ji-ming Wu Long-jun Shen
(*Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, Beijing 100088, China*)

Abstract

It has been observed numerically in [1] that, under certain conditions, all eigenvalues of the first-order Hermite cubic spline collocation differentiation matrices with unsymmetrical collocation points lie in one of the half complex planes. In this paper, we provide a theoretical proof for this spectral result.

Key words: Spline collocation, Differentiation matrices, Spectral analysis.

1. Introduction

Hermite cubic spline collocation method has been extensively applied in the numerical solution of ODEs and PDEs due to its ease of implementation and high-order accuracy[2−5]. One of the main features of this method is that the approximate solution takes the form of Hermite cubic spline(or bicubic spline in 2D). When Hermite cubic spline is applied for discretizing the differential operator $u^{(k)}$, the so-called Hermite cubic spline collocation differentiation matrices arise. Eigenvalue analysis of these matrices plays an important part in the stability or convergence analysis of the corresponding collocation algorithms. It also has served as the theoretical foundation in developing several fast direct algorithms[6−9].

Spectral analysis of second-order Hermite cubic spline collocation differentiation matrices with Gauss collocation points (a kind of symmetric collocation points) was given analytically in [6] and consequently in [7,8] for eigenvectors. However, it has been found that Hermite cubic spline collocation differentiation matrices with unsymmetrical collocation points are more important in practice for stability and singularity considerations[5,11−15]. Based on the condensation technique[16], [1] studied the spectral properties of the second-order Hermite cubic spline collocation differentiation matrices with arbitrary collocation points. [1] also studied the spectral properties of the first-order Hermite cubic spline collocation differentiation matrices with symmetric collocation points. Meanwhile, for unsymmetrical collocation points, [1] observed numerically that all eigenvalues of the first-order Hermite cubic spline collocation differentiation matrices with Dirichlet boundary conditions or Neumann boundary conditions lie in one of the half complex planes. This spectral result is of most importance for stability considerations[1].

The primary purpose of this paper is to provide a theoretical proof for the aforementioned spectral result of the first-order Hermite cubic spline collocation differentiation matrices. Our argumentation is based on the condensation technique and Hurwitz's theorem. The rest of this paper is arranged as follows. In the second section, we will briefly introduce the Hermite cubic spline collocation method and generalize the aforementioned spectral result in two theorems. In the third section, we will prove these theorems theoretically.

* Received August 11, 2000.
1)The Project supported by A Grant from the Research Grants Council of the Hong Kong Spelial Administrative Region, China (Project No. CityU 1061/00p) the Foundation of Chinese Academy of Engineering Physics.

2. Hermite Cubic Spline Collocation Method

Assume that $\Pi_N = \{x_i\}_{i=1}^{N+1}$ is a uniform partition that divides interval $[0, 1]$ into N equal subintervals of length $h = 1/N$. The Hermite cubic approximation $v(x)$ is defined by

$$v(x) = \xi_1(s)u_i + h\xi_2(s)u'_i + \xi_3(s)u_{i+1} + h\xi_4(s)u'_{i+1}, \quad s \in [0,\ 1], \tag{1}$$

on each subinterval $[x_i,\ x_{i+1}]$, $i = 1,\ 2,\ \cdots,\ N$, where

$$\xi_1(s) = (1+2s)(1-s)^2,\ \xi_2(s) = s(1-s)^2,\ \xi_3(s) = s^2(3-2s),\ \xi_4(s) = s^2(s-1) \tag{2}$$

and $s = (x - x_i)/h$. The above definition implies that $u_i = v(x_i)$ and $u'_i = v'(x_i)$. Denote

$$\Omega = \{(x,\ y)\,|\, 0 < x < y < 1\} \subset R^2. \tag{3}$$

For any given $(\sigma_1,\ \sigma_2) \in \Omega$, let $\Pi_C = \{x^c_{i1},\ x^c_{i2}\}_{i=1}^N$ be the set of collocation points, where

$$x^c_{i1} = x_i + \sigma_1 h,\ x^c_{i2} = x_i + \sigma_2 h,\ i = 1,\ 2,\ \cdots,\ N. \tag{4}$$

Consider a collocation approximation to the kth-order differentiation operator $u^{(k)} (k = 0,\ 1, 2)$. Let $A_k (k = 0,\ 1,\ 2)$ denote the k-order collocation differentiation matrices, satisfying

$$(A_k v)_l = v^{(k)}(x^c_{ij}), \quad j = 1,\ 2; \quad i = 1,\ 2,\ \cdots,\ N. \tag{5}$$

Obviously the structure of A_k depends on the orderings of the collocation points and the unknowns for $v(x)$. Suppose $l = 2(i-1) + j$, and

$$\begin{aligned} &v = [\ hu'_1,\ u_2,\ hu'_2,\ \cdots,\ u_N,\ hu'_N,\ hu'_{N+1}]^T \ \mathit{or}\ [u_1,\ u_2,\ hu'_2,\ \cdots,\ u_N,\ hu'_N,\ u_{N+1}]^T, \\ &\Pi_C = \{x^c_{11},\ x^c_{12}, x^c_{21},\ x^c_{22},\ \cdots,\ x^c_{N1},\ x^c_{N2}\}. \end{aligned} \tag{6}$$

Then A_k with Dirichlet boundary conditions or Neumann boundary conditions has an almost block diagonal structure[6,3,1] and A_0 is nonsingular[1]. In this paper we only consider A_k with Dirichlet boundary conditions or Neumann boundary conditions, the corresponding spectral results for A_k with periodical boundary conditions can be found in [10].

Consider the following typical example[13,1]

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - p\frac{\partial u}{\partial x} + d(x,\ t). \tag{7}$$

If we use an Euler scheme for time discretization and the Hermite cubic collocation method in the x-direction, we shall get the following iterative scheme

$$\begin{cases} u^{n+1} = u^n + \tau(\varepsilon A_2 - pA_1)u^{(n)} + \tau\, d^n, \\ \qquad A_0 u^{(n)} = u^n, \end{cases} \tag{8}$$

where τ is the time stepsize. The stability depends on the eigenvalues distribution of the matrix $(\varepsilon A_2 - pA_1)A_0^{-1}$. On the one hand it is known that all eigenvlues of $A_2A_0^{-1}$ are real and nonpositive for any collocation points (see [1, 2, 6, 7, 8]). On the other hand, numerical tests show that the stability of (8) depends on the choice of collocation points when $0 < \varepsilon << p$ (see [1]). This indicates that $A_1A_0^{-1}$ plays a dominating part in the case of $0 < \varepsilon << p$. Furthermore, $A_1A_0^{-1}$ is the only factor to determine the stability of (8) when $\varepsilon = 0$. Thus it is very interesting and important to study the spectral properties of matrix $A_1A_0^{-1}$. In this respect, [1] presented a number of concrete formulae for the calculation of all eigenvalues of matrix $A_1A_0^{-1}$ with symmetric collocation points. As for $A_1A_0^{-1}$ with unsymmetrical collocation points, [1] observed numerically some interesting spectral results. Here we generalize these results in the following two theorems and in the next section we shall prove them theoretically.

Theorem 2.1. *Let* $eig\left(A_1A_0^{-1}\right)$ *denote the eigenvalue of* $A_1A_0^{-1}$*, then for* $A_1A_0^{-1}$ *with Dirichlet boundary conditions, there holds that*

$$Re\left[eig\left(A_1A_0^{-1}\right)\right] \begin{cases} <0, & if\ \sigma_1+\sigma_2>1\ and\ 2-3\sigma_1-3\sigma_2+6\sigma_1\sigma_2>0; \\ >0, & if\ \sigma_1+\sigma_2<1\ and\ 2-3\sigma_1-3\sigma_2+6\sigma_1\sigma_2>0. \end{cases} \tag{9}$$

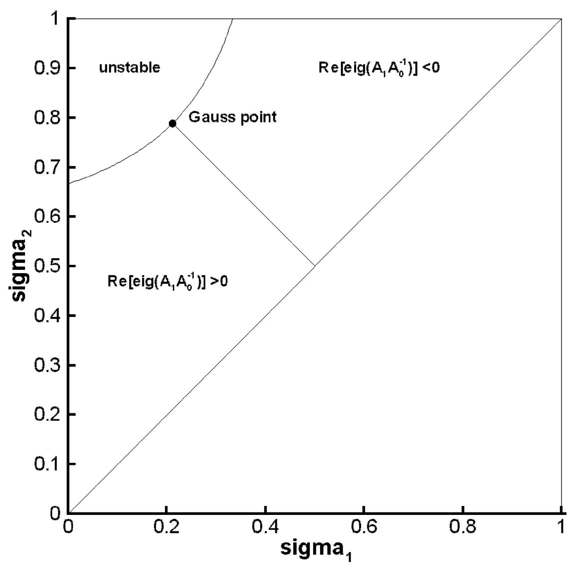

Figure 1. The spectral result for Dirichlet problem.

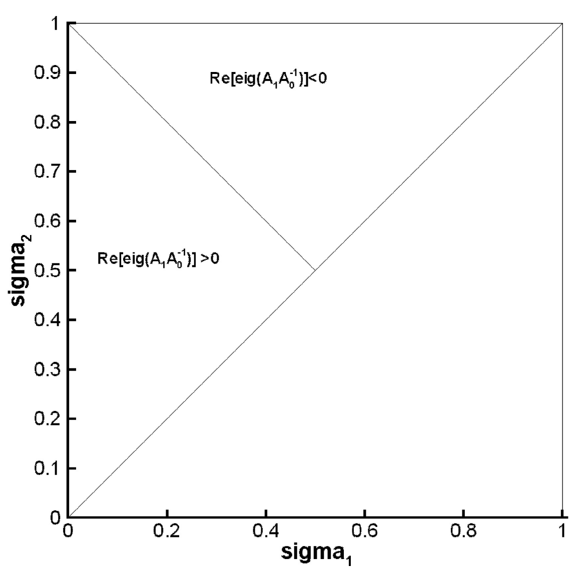

Figure 2. The spectral result for Neumann problem.

Theorem 2.2. *For* $A_1A_0^{-1}$ *with Neumann boundary conditions, it has a zero-eigenvalue and its nonzero eigenvalues satisfy*

$$Re\left[eig\left(A_1A_0^{-1}\right)\right]\ \begin{cases} <0, & if\ \sigma_1+\sigma_2>1; \\ >0, & if\ \sigma_1+\sigma_2<1. \end{cases} \tag{10}$$

The above spectral results are depicted in Figure 1 and Figure 2.

3. The Proofs of the Theorems

We begin our analysis by introducing the following lemma which is obtained in [1] by the condensation technique[16].

Lemma 3.1[1]**.** *Let $A_k(k=0,\ 1,\ 2)$ be matrices defined by (5) and (6), then there hold the following decompositions*

$$\det(A_k-\lambda A_0)=h^{-2k(N-1)}\det(C_k(\lambda))\cdot\det(tridiag(a_k(\lambda),\ b_k(\lambda),\ c_k(\lambda))) \tag{11}$$

$$\det(A_k-\lambda A_0)=h^{-2k(N-1)}\det(C_k^N(\lambda))\cdot\det(tridiag(a_k(\lambda),\ b_k(\lambda),\ c_k(\lambda))) \tag{12}$$

for Dirichlet boundary conditions and Neumann boundary conditions, respectively, where tridiag $(a_k(\lambda),\ b_k(\lambda), c_k(\lambda))$ denotes an $(N-1)\times(N-1)$ tridiagonal matrix and

$$a_k(\lambda)=-\det\begin{pmatrix} \xi_1^{(k)}(\sigma_1)-\lambda h^k\xi_1(\sigma_1) & \xi_2^{(k)}(\sigma_1)-\lambda h^k\xi_2(\sigma_1) \\ \xi_1^{(k)}(\sigma_2)-\lambda h^k\xi_1(\sigma_2) & \xi_2^{(k)}(\sigma_2)-\lambda h^k\xi_2(\sigma_2) \end{pmatrix}, \tag{13}$$

$$\begin{aligned} b_k(\lambda)=&\det\begin{pmatrix} \xi_2^{(k)}(\sigma_1)-\lambda h^k\xi_2(\sigma_1) & \xi_3^{(k)}(\sigma_1)-\lambda h^k\xi_3(\sigma_1) \\ \xi_2^{(k)}(\sigma_2)-\lambda h^k\xi_2(\sigma_2) & \xi_3^{(k)}(\sigma_2)-\lambda h^k\xi_3(\sigma_2) \end{pmatrix} \\ &-\det\begin{pmatrix} \xi_1^{(k)}(\sigma_1)-\lambda h^k\xi_1(\sigma_1) & \xi_4^{(k)}(\sigma_1)-\lambda h^k\xi_4(\sigma_1) \\ \xi_1^{(k)}(\sigma_2)-\lambda h^k\xi_1(\sigma_2) & \xi_4^{(k)}(\sigma_2)-\lambda h^k\xi_4(\sigma_2) \end{pmatrix}, \end{aligned} \tag{14}$$

$$c_k(\lambda)=-\det\begin{pmatrix} \xi_3^{(k)}(\sigma_1)-\lambda h^k\xi_3(\sigma_1) & \xi_4^{(k)}(\sigma_1)-\lambda h^k\xi_4(\sigma_1) \\ \xi_3^{(k)}(\sigma_2)-\lambda h^k\xi_3(\sigma_2) & \xi_4^{(k)}(\sigma_2)-\lambda h^k\xi_4(\sigma_2) \end{pmatrix}, \tag{15}$$

$$C_k(\lambda)=\begin{pmatrix} h^{-k}\xi_2^{(k)}(\sigma_1)-\lambda\xi_2(\sigma_1) & h^{-k}\xi_4^{(k)}(\sigma_1)-\lambda\xi_4(\sigma_1) \\ h^{-k}\xi_2^{(k)}(\sigma_2)-\lambda\xi_2(\sigma_2) & h^{-k}\xi_4^{(k)}(\sigma_2)-\lambda\xi_4(\sigma_2) \end{pmatrix}, \tag{16}$$

$$C_k^N(\lambda)=\begin{pmatrix} h^{-k}\xi_1^{(k)}(\sigma_1)-\lambda\xi_1(\sigma_1) & h^{-k}\xi_3^{(k)}(\sigma_1)-\lambda\xi_3(\sigma_1) \\ h^{-k}\xi_1^{(k)}(\sigma_2)-\lambda\xi_1(\sigma_2) & h^{-k}\xi_3^{(k)}(\sigma_2)-\lambda\xi_3(\sigma_2) \end{pmatrix}. \tag{17}$$

Lemma 3.2. *For any complex numbers α and β, it holds that*

$$\det\left(tridiag(1,\ \alpha+\beta,\ \alpha\beta)\right)=\sum_{j=0}^{N-1}\alpha^{N-1-j}\beta^j, \tag{18}$$

where $tridiag(1,\ \alpha+\beta,\ \alpha\beta)$ denotes an $(N-1)\times(N-1)$ tridiagonal matrix.

Lemma 3.2 can be easily proved by mathematical induction.

Lemma 3.3. *Suppose $a_k(\lambda),\ b_k(\lambda),\ c_k(\lambda)(k=0,\ 1,\ 2)$ are polynomials given by (2), and (13)–(15). Let $tridiag(a_k(\lambda),\ b_k(\lambda),\ c_k(\lambda))$ be an $(N-1)\times(N-1)$ tridiagonal matrix. Then*

$$\begin{aligned}&\{\lambda\in C|\ \det\left(tridiag(a_k(\lambda),\ b_k(\lambda),c_k(\lambda))\right)=0\,\}\\ &\subset\bigcup_{\rho\in[0,\ 1)}\left\{\lambda\in C|\ b_k^2(\lambda)-4\rho\, a_k(\lambda)c_k(\lambda)=0\,\right\}.\end{aligned}\tag{19}$$

Proof. For any root of $\det\left(tridiag(a_k(\lambda),\ b_k(\lambda),c_k(\lambda))\right)=0$, we suppose, without any harm to our argument, that $N>2$ and $a_k(\lambda)c_k(\lambda)\neq 0$. Now let $\alpha(\lambda)$ and $\beta(\lambda)$ be defined by

$$\alpha(\lambda)+\beta(\lambda)=\frac{b_k(\lambda)}{a_k(\lambda)},\ \alpha(\lambda)\beta(\lambda)=\frac{c_k(\lambda)}{a_k(\lambda)}.\tag{20}$$

Using Lemma 3.2, we have

$$\begin{aligned}\det\left(tridiag\left(a_k(\lambda),\ b_k(\lambda),\ c_k(\lambda)\right)\right)&=\left(a_k(\lambda)\right)^{N-1}\det\left(tridiag\left(1,\ \alpha(\lambda)+\beta(\lambda),\ \alpha(\lambda)\beta(\lambda)\right)\right)\\ &=\left(a_k(\lambda)\right)^{N-1}\sum_{j=0}^{N-1}\left(\alpha(\lambda)\right)^{N-1-j}\left(\beta(\lambda)\right)^j=0,\end{aligned}$$

which implies

$$\alpha(\lambda)=e^{\frac{2j\pi}{N}i}\beta(\lambda),\qquad j=1,\,2,\,\cdots,\,N-1\tag{21}$$

where $i=\sqrt{-1}$ is the imaginary unit. Eliminating $\alpha(\lambda)$ and $\beta(\lambda)$ from (20) and (21), we finally get

$$b_k^2(\lambda)-4a_k(\lambda)c_k(\lambda)\cos^2\frac{j\pi}{N}=0,\quad j=\ 1,\ 2,\ \cdots,\ N-1,\tag{22}$$

which leads to (19) and completes the proof of Lemma 3.3.

Lemma 3.4.(Hurwitz's theorem[17]) *Let*

$$f(\lambda)=\alpha_0+\alpha_1\lambda+\alpha_2\lambda^2+\cdots+\alpha_n\lambda^n\tag{23}$$

be a polynomial in λ with real coefficients and $\alpha_n\neq 0$. Denote

$$D_1(f)=\alpha_1,\quad D_2(f)=\det\begin{pmatrix}\alpha_1&\alpha_0\\ \alpha_3&\alpha_2\end{pmatrix},\quad D_3(f)=\det\begin{pmatrix}\alpha_1&\alpha_0&0\\ \alpha_3&\alpha_2&\alpha_1\\ \alpha_5&\alpha_4&\alpha_3\end{pmatrix}$$

$$D_4(f)=\det\begin{pmatrix}\alpha_1&\alpha_0&0&0\\ \alpha_3&\alpha_2&\alpha_1&\alpha_0\\ \alpha_5&\alpha_4&\alpha_3&\alpha_2\\ \alpha_7&\alpha_6&\alpha_5&\alpha_4\end{pmatrix},\ \cdots,$$

$$D_n(f)=\det\begin{pmatrix}\alpha_1&\alpha_0&0&\cdots&0\\ \alpha_3&\alpha_2&\alpha_1&\cdots&0\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ \alpha_{2n-1}&\alpha_{2n-2}&\alpha_{2n-3}&\cdots&\alpha_n\end{pmatrix},$$

where $\alpha_j=0$ if $j>n$. Then all roots of $f(\lambda)=0$ with $\alpha_0>0$ lie on the left side of the imaginary axis, i.e., $Re(\lambda)<0$, if and only if $D_j(f)>0,\ j=1,2,\cdots,n$.

Lemma 3.5. *If $\lambda\in C$ satisfies $b_1^2(\lambda)-4\rho\, a_1(\lambda)c_1(\lambda)=0,\ \rho\in[0,\ 1)$ is a constant and $\sigma_1+\sigma_2>1$, then $Re(\lambda)<0$.*

Proof. From (2) , (13), (14) and (15), we have

$$
\begin{aligned}
a_1(\lambda) &= -(1-\sigma_1)(1-\sigma_2)(\sigma_2-\sigma_1)[(1-\sigma_1)(1-\sigma_2)h^2\lambda^2 + 2(2-\sigma_1-\sigma_2)h\lambda + 6],\\
b_1(\lambda) &= (\sigma_2-\sigma_1)\{[2\sigma_1\sigma_2(1-\sigma_1)(1-\sigma_2) + \sigma_1(1-\sigma_1) + \sigma_2(1-\sigma_2)]h^2\lambda^2\\
&\quad -2(1-\sigma_1-\sigma_2)(1+\sigma_1+\sigma_2-2\sigma_1\sigma_2)h\lambda + 6(1+2\sigma_1\sigma_2-\sigma_1-\sigma_2)\},\\
c_1(\lambda) &= -\sigma_1\sigma_2(\sigma_2-\sigma_1)[\sigma_1\sigma_2h^2\lambda^2 - 2(\sigma_1+\sigma_2)h\lambda + 6].
\end{aligned}
\tag{24}
$$

For simplicity, we denote

$$
\tau_1 = \sigma_1\sigma_2 + (1-\sigma_1)(1-\sigma_2),\ \tau_2 = \sigma_1+\sigma_2-1. \tag{25}
$$

Then (24) can be rewritten as

$$
\begin{aligned}
a_1(\lambda) &= -\tfrac{1}{4}(\sigma_2-\sigma_1)(\tau_1-\tau_2)[(\tau_1-\tau_2)h^2\lambda^2 + 4(1-\tau_2)h\lambda + 12],\\
b_1(\lambda) &= \tfrac{1}{2}(\sigma_2-\sigma_1)[(\tau_1^2-3\tau_2^2+2\tau_1)h^2\lambda^2 + 4\tau_2(2-\tau_1)h\lambda + 12\tau_1],\\
c_1(\lambda) &= -\tfrac{1}{4}(\sigma_2-\sigma_1)(\tau_1+\tau_2)[(\tau_1+\tau_2)h^2\lambda^2 - 4(\tau_2+1)h\lambda + 12].
\end{aligned}
\tag{26}
$$

Noting that $0<\sigma_1<\sigma_2<1$ and $\sigma_1+\sigma_2>1$, we have

$$
0<\tau_2<\tau_1<1. \tag{27}
$$

For any $\rho \in [0,\ 1)$, set

$$
b_1^2(\lambda) - 4\rho\, a_1(\lambda)c_1(\lambda) = (\sigma_2-\sigma_1)^2 f_1(\lambda), \tag{28}
$$

where

$$
f_1(\lambda) = \alpha_0 + \alpha_1 h\lambda + \alpha_2 h^2\lambda^2 + \alpha_3 h^3\lambda^3 + \alpha_4 h^4\lambda^4 \quad . \tag{29}
$$

By using *Maple* software, we have

$$
\begin{aligned}
\alpha_0 &= 36[(1-\rho)\tau_1^2 + \rho\tau_2^2],\\
\alpha_1 &= 24\tau_2[\tau_1(2-\tau_1) + \rho(\tau_1^2-\tau_2^2)],\\
\alpha_2 &= 4\tau_2^2(2-\tau_1)^2 + 6\tau_1(\tau_1^2-3\tau_2^2+2\tau_1) - 2\rho(\tau_1^2-\tau_2^2)(2\tau_2^2+3\tau_1-2),\\
\alpha_3 &= 2\tau_2[(2-\tau_1)(\tau_1^2-3\tau_2^2+2\tau_1) + \rho(\tau_1-1)(\tau_1^2-\tau_2^2)],\\
\alpha_4 &= \tfrac{1}{4}[(\tau_1^2-3\tau_2^2+2\tau_1)^2 - \rho(\tau_1^2-\tau_2^2)^2].
\end{aligned}
$$

By Hurwitz's theorem, if we can prove $\alpha_0>0,\ \alpha_4\neq 0$ and $D_i(f_1)>0 \quad (i=1,\ 2,\ 3,\ 4)$, then the proof of Lemma 3.5 will be complete. From (27), it is easy to check that

$$
\alpha_0>0, \quad \alpha_4>0, \quad D_1(f_1) = \alpha_1 h > 0. \tag{30}
$$

Note $D_4(f_1) = \alpha_4 h^4 D_3(f_1)$. Therefore, what remains is to prove

$$
D_2(f_1) = \det\begin{pmatrix} h\alpha_1 & \alpha_0\\ h^3\alpha_3 & h^2\alpha_2 \end{pmatrix} > 0 \tag{31}
$$

and

$$D_3(f_1) = \det \begin{pmatrix} h\alpha_1 & \alpha_0 & 0 \\ h^3\alpha_3 & h^2\alpha_2 & h\alpha_1 \\ 0 & h^4\alpha_4 & h^3\alpha_3 \end{pmatrix} > 0. \tag{32}$$

By using *Maple* software, we get

$$D_2(f_1) = 6\tau_2 h^3[A_2(\tau_1,\tau_2)\rho^2 + B_2(\tau_1,\tau_2)\rho + C_2(\tau_1,\tau_2)], \tag{33}$$

where

$$\begin{aligned} &A_2(\tau_1,\tau_2) = 4(\tau_1^2-\tau_2^2)^2(1-3\tau_1-4\tau_2^2), \\ &B_2(\tau_1,\ \tau_2) = (\tau_1^2-\tau_2^2)[\tau_1(33-8\tau_1)\,(1-3\tau_1-4\tau_2^2)+8(\tau_1^2-\tau_2^2)\ +\tau_1(47+95\tau_1)], \\ &C_2(\tau_1,\tau_2) = 4\tau_1(2-\tau_1)[4\tau_2^2(2-\tau_1)^2+3\tau_1(\tau_1^2-3\tau_2^2+2\tau_1)]. \end{aligned} \tag{34}$$

It is obvious that

$$C_2(\tau_1,\ \tau_2) > 0. \tag{35}$$

Since

$$\begin{aligned} &A_2(\tau_1,\ \tau_2)+B_2(\tau_1,\ \tau_2)+C_2(\tau_1,\ \tau_2) \\ &= 4(32\tau_1^3+12\tau_1\tau_2^2+3\tau_2^4-69\tau_1^2\tau_2^2+30\tau_1\tau_2^4-4\tau_2^6) \\ &= 4[25\tau_1(\tau_1-\tau_2^2)^2+(7\tau_1^2+\tau_2^4)(\tau_1-\tau_2^2)+3\tau_2^4(1-\tau_2^2)+12\tau_1\tau_2^2(1-\tau_1)+4\tau_1\tau_2^4], \end{aligned}$$

it follows that

$$A_2(\tau_1,\ \tau_2)+B_2(\tau_1,\ \tau_2)+C_2(\tau_1,\ \tau_2) > 0. \tag{36}$$

Now, on the one hand, if

$$1-3\tau_1-4\tau_2^2 \le 0 \tag{37}$$

holds, then we have $A_2(\tau_1,\tau_2)\le 0$. Consequently, it follows from (33), (35) and (36) that

$$D_2(f_1) > 6\tau_2 h^3\rho\,[A_2(\tau_1,\ \tau_2)+B_2(\tau_1,\ \tau_2)+C_2(\tau_1,\ \tau_2)] > 0,$$

which implies (31). On the other hand, if (37) fails, then (34) yields

$$A_2(\tau_1,\ \tau_2) > 0, \quad B_2(\tau_1,\tau_2) > 0$$

and (31) follows just as well from (33) and (35). Thus we have proved (31).

Now we begin to prove (32). By using *Maple* software, we have

$$D_3(f_1) = 48\tau_2^2 h^6 g(\tau_1,\ \tau_2,\ \rho), \tag{38}$$

where

$$g(\tau_1,\ \tau_2,\ \rho) = A_3(\tau_1,\tau_2)\rho^3 + B_3(\tau_1,\tau_2)\rho^2\ + C_3(\tau_1,\tau_2)\rho + D_3(\tau_1,\tau_2), \tag{39}$$

$$A_3(\tau_1,\tau_2) = (\tau_1^2-\tau_2^2)^3(1-\tau_2^2)(4\tau_1-1), \tag{40}$$

$$B_3(\tau_1,\tau_2) = (\tau_1^2-\tau_2^2)^2[-12\tau_1(\tau_1^2-\tau_2^2) - 3(\tau_1-\tau_2^2)^2 - 12\tau_1(1-\tau_2^2)^2 \\ -10\tau_1^2\tau_2^2(1-\tau_1) - 2\tau_2^2(1-\tau_1)^2(2-\tau_1) \ -4\tau_1(1-\tau_1)(1+\tau_1-2\tau_2^2)], \tag{41}$$

$$D_3(\tau_1,\tau_2) = 4\tau_1\tau_2^2(2-\tau_1)^4(\tau_1^2-3\tau_2^2+2\tau_1), \tag{42}$$

and $C_3(\tau_1,\ \tau_2)$ is a polynomial in τ_1 and τ_2. It is not difficult to check that

$$B_3(\tau_1,\tau_2) < -12\tau_1(\tau_1^2-\tau_2^2)^3 < 0, \tag{43}$$

$$g(\tau_1,\ \tau_2,\ 0) = D_3(\tau_1,\tau_2) > 0. \tag{44}$$

Note that

$$3A_3(\tau_1,\tau_2) + B_3(\tau_1,\tau_2) < -3(\tau_1^2-\tau_2^2)^3(1-\tau_2^2+4\tau_1\tau_2^2) < 0 \tag{45}$$

and

$$g(\tau_1,\ \tau_2,\ 1) = \tau_1^2(\tau_1-\tau_2^2)^2[80(1-\tau_1)+21\tau_2^2] + 4\tau_1\tau_2^2(1-\tau_1)(\tau_1-\tau_2^2) \\ \times[7\tau_2^2+12(1-\tau_1)] + 8\tau_1\tau_2^6(\tau_1-\tau_2^2) + (15+4\tau_2^2)(\tau_1^2-\tau_2^2)\tau_2^4 \\ +15\tau_1^2\tau_2^4(1-2\tau_1+\tau_2^2) > 0. \tag{46}$$

Thus, on the one hand, if $A_3(\tau_1,\tau_2) \le 0$, from (43), (44) and (46), we have

$$D_3(f_1) > 48\tau_2^2h^6\rho\,[A_3(\tau_1,\tau_2) + B_3(\tau_1,\tau_2) \ + C_3(\tau_1,\tau_2) + D_3(\tau_1,\tau_2)] \\ = 48\tau_2^2h^6\rho\, g(\tau_1,\tau_2,\ 1) > 0.$$

On the other hand, if $A_3(\tau_1,\tau_2) > 0$,then it follows from (45) that

$$\frac{\partial^2 g}{\partial\rho^2} = 6A_3(\tau_1,\ \tau_2)\rho + 2B_3(\tau_1,\ \tau_2) < 6A_3(\tau_1,\ \tau_2) + 2B_3(\tau_1,\ \tau_2) < 0. \tag{47}$$

Consequently, (44), (46) and (47) yield

$$\min_{\rho\in[0,\ 1)} g(\tau_1,\ \tau_2,\ \rho) = \min\{g(\tau_1,\ \tau_2,\ 0),\ g(\tau_1,\ \tau_2,\ 1)\} > 0$$

which also leads to $D_3(f_1) > 0$. The proof of (32) is thus obtained which completes the proof of Lemma 3.5 .

By an argument similar to that of Lemma 3.5, we have

Lemma 3.6. *If* $\lambda \in C$ *satisfies* $b_1^2(\lambda) - 4\rho\, a_1(\lambda)c_1(\lambda) = 0$, $\rho \in [0,\ 1)$ *is a constant and* $\sigma_1+\sigma_2 < 1$, *then* $Re(\lambda) > 0$.

Lemma 3.3, 3.5 and 3.6 directly give

Lemma 3.7. *For the root of* $\det(tridiag(a_1(\lambda),\ b_1(\lambda), c_1(\lambda))) = 0$ *there holds that*

$$Re(\lambda) \begin{cases} < 0, & if\ \sigma_1+\sigma_2 > 1; \\ > 0, & if\ \sigma_1+\sigma_2 < 1. \end{cases} \tag{48}$$

The proof of Theorem 2.1 By Lemma 3.1, each eigenvalue of $A_1A_0^{-1}$ *with Dirichlet boundary conditions satisfies* $\det(tridiag(a_1(\lambda),\ b_1(\lambda), c_1(\lambda))) = 0$ *or*

$$\det(C_1(\lambda)) = 0. \tag{49}$$

By Lemma 3.7, $\det(tridiag(a_1(\lambda),\ b_1(\lambda), c_1(\lambda))) = 0$ *implies (9). If (49) holds, then from (2) and (16) we have*

$$\gamma_0 + \gamma_1 h\lambda + \gamma_2 h^2\lambda^2 = 0 \tag{50}$$

where

$$\begin{aligned} \gamma_0 &= 2 - 3\sigma_1 - 3\sigma_2 + 6\sigma_1\sigma_2, \\ \gamma_1 &= (\sigma_1 + \sigma_2 - 1)[\sigma_1(1-\sigma_2) + \sigma_2(1-\sigma_1)], \\ \gamma_2 &= \sigma_1\sigma_2(1-\sigma_1)(1-\sigma_2) > 0. \end{aligned} \tag{51}$$

Obviously, equation (50) has a positive root and a negative root if $\gamma_0 < 0$*. If* $\gamma_0 > 0$*, then*

$$Re(\lambda) = \begin{cases} -\frac{\gamma_1}{2h\gamma_2}, & if\ \gamma_1^2 - 4\gamma_2\gamma_0 \le 0; \\ \frac{-\gamma_1 \pm \sqrt{\gamma_1^2 - 4\gamma_2\gamma_0}}{2h\gamma_2}, & if\ \gamma_1^2 - 4\gamma_2\gamma_0 > 0. \end{cases} \tag{52}$$

Thus (9) follows from (51) and (52).

The proof of Theorem 2.2 From (17), we get

$$C_1^N(\lambda) = (\sigma_1 - \sigma_2)h^{-1}\lambda\{[(2\sigma_1 + \sigma_2)(1-\sigma_1) + (\sigma_1 + 2\sigma_2)(1-\sigma_2)]h\lambda + 6(\sigma_1 + \sigma_2 - 1)\}.$$

Thus, it follows from (12) that the eigenvalue of $A_1A_0^{-1}$ with Neumann boundary conditions satisfies $\det(tridiag(a_1(\lambda),\ b_1(\lambda), c_1(\lambda))) = 0$ or $\lambda = 0$, or $\lambda = \frac{6(1-\sigma_1-\sigma_2)h^{-1}}{(2\sigma_1+\sigma_2)(1-\sigma_1)+(\sigma_1+2\sigma_2)(1-\sigma_2)}$. Now, the proof of Theorem 2.2 can be easily obtained by using Lemma 3.7.

We note, from [1], that the spectral radius of $A_1A_0^{-1}$ with symmetrical collocation points is of order $O(N)$. We think that this is also true for $A_1A_0^{-1}$ with unsymmetrical collocation points, and it is not difficult to give a theoretical proof by making use of the results in this paper.

Acknowledgements. The authors thank Dr. W. Sun and the anonymous referees for their valuable suggestions and comments.

References

[1] W. Sun, Spectral analysis of Hermite cubic spline collocation systems, *SIAM J. Numer. Anal.*, **36**:6(1999), 1962-1975.

[2] J. H. Cerutti and S. V. Parter, Collocation methods for parabolic partial differential equations in one space dimension, *Numer. Math.*, **26**:3(1976), 227-254.

[3] W. R. Dyksen, R. E. Lynch, J. R. Rice and E. N. Houstis, The performance of the collocation and Galerkin methods with Hermite bicubics, *SIAM J. Numer. Anal.*, **21**:4(1984), 695-715.

[4] J. R. Rice and R. F. Boisvert, Solving Elliptic Problems Using Ellpack, Springer-Verlag, New York, 1985.

[5] U. M. Ascher, R. M. Mattheij and R. D. Russell, Numerical Solution of Boundary Value Problems for Ordinary Differential Equations, Prentice-Hall, Englewood Cliffs, NJ, 1988.

[6] W. R. Dyksen, Tensor product generalized ADI methods for separable elliptic problems, *SIAM J. Numer. Anal.*, **24**:1(1987), 59-76.

[7] W. Sun and N. G. Zamani, A fast algorithm for solving the tensor product collocation equations, *J. Franklin Inst.*, **326**(1989), 295-307.

[8] B. Bialecki, G. Fairweather, and K. R. Bennett, Fast direct solvers for piecewise Hermite bicubic orthogonal spline collocation equations, *SIAM J. Numer. Anal.*, **29**:1(1992), 156-173.

[9] W. Sun, A high-order direct method for solving Poisson's equation on a disc, *Numer. Math.*, **70**:4(1995), 501-506.

[10] R. D. Russell and W. Sun, Spline collocation differentiation matrices, *SIAM J. Numer. Anal.*, **34**:6(1997), 2274-2287.

[11] G. F. Pinder and A. Shapiro, A new collocation method for the solution of the convection- dominated transport equation, *Water Resources Research*, **15**(1979), 1177-1182.

[12] A. M. Shapiro and G. F .Pinder, Analysis of an upstream weighted collocation approximation to the transport equation, *J. Comp. Phys.*, **39**(1981), 46-71.

[13] M. B. Allen and G. F. Pinder, Collocation simulation of multiphase porous-medium flow, *J. Society of Petroleum Engineers*, Feb.(1983), 135-142.

[14] C. Ringhofer, On collocation schemes for quasilinear singularly perturbed boundary value problems, *SIAM J. Numer. Anal.*, **21**:5(1984), 864-882.

[15] U. Ascher and G. Bader, Stability of collocation at Gaussian points, *SIAM J. Numer.Anal.*, **23**:2(1986), 412-422.

[16] U. M. Ascher, J. Christiansen and R. D. Russell, Collocation software for boundary-value ODEs, *ACM Trans. Math. Software*, **7**(1981), 209-222.

[17] Encyclopedic Dictionary of Mathematics, by the Mathematical Society of Japan, The MIT Press, 1977, 39-40.