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FINITE ELEMENT METHODS FOR SOBOLEV EQUATIONS*[1)]

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Abstract

A new high-order time-stepping finite element method based upon the high-order numerical integration formula is formulated for Sobolev equations, whose computations consist of an iteration procedure coupled with a system of two elliptic equations. The optimal and superconvergence error estimates for this new method are derived both in space and in time. Also, a class of new error estimates of convergence and superconvergence for the time-continuous finite element method is demonstrated in which there are no time derivatives of the exact solution involved, such that these estimates can be bounded by the norms of the known data. Moreover, some useful a-posteriori error estimators are given on the basis of the superconvergence estimates.

Key words: Error estimates, finite element, Sobolev equation, numerical integration.

1. Introduction

Our purpose in this paper is to study the finite element method for the following Sobolev equation:

$$\begin{aligned} A(t)u_t + \quad & B(t)u = f(t), && \text{in} \quad \Omega \times J, \\ & u(\cdot, t) = 0, && \text{on} \quad \partial\Omega \times \bar{J}, \\ & u(\cdot, 0) = v, && x \in \Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset R^d$ $(d \geq 1)$ is an open bounded domain, $J = (0, T]$, $T > 0$, f and v are known smooth functions. We assume that the operator $A(t)$ is a strongly elliptic symmetric operator,

$$A(t) := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) \frac{\partial}{\partial x_j} \right) + a(x,t)I, \quad a(x,t) \geq 0,$$

and that $B(t)$ is an arbitrary second order elliptic operator,

$$B(t) := -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(b_{ij}(x,t) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^{d} b_i(x,t) \frac{\partial}{\partial x_i} + b(x,t)I,$$

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where I is the identity operator, a_{ij}, a, b_{ij}, b_i and b are smooth functions, and there exits $C_0 > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq C_0 \sum_{i=1}^{d} \xi_i^2, \qquad \forall \xi \in R^d, \quad (x,t) \in \Omega \times \bar{J}. \tag{1.1}$$

The problem (1.1) can arise from many physical processes. For the fomulation of (1.1) and the questions of existence, uniqueness and stability of the solution, we refer to [2, 3, 19] and the references cited in [6, 7, 8]. The numerical approximations to the solution of (1.1) have been investigated by many authors. Finite difference methods have been studied in [6, 10, 11], while Ewing [8] has considered several Galerkin approximations and obtained optimal error estimates for nonlinear boundary cases. Also, Arnold, Douglas and Thomée [1] and Nakao [17] have studied Galerkin approximations to the solution of (1.1) in a single space dimension with periodic boundary conditions. L^2 error estimates and superconvergence results are derived by these authors. Recently, the authors in [14, 15, 16] have used a so-called Ritz-Volterra type projection to study finite element approximations for nonlinear versions of the above problems and derived some optimal error estimates for Dirichlet and nonlinear boundary conditions. The L^p $(2 \leq p < \infty)$ norm error estimate can be found in [16] for linear equations.

In this paper we reformulate (1.1) as an integral equation of Volterra type, use the higher-order numerical integration formula to construct a higher-order time-stepping procedure and give some error estimates. The formulation of our numerical approximations is given in Section 2, and error estimates of convergence and superconvergence for the semi-discrete and the fully-discrete finite element methods are demonstrated in Sections 3, 4 and 5, respectively. The special feature of our error estimates in Sections 3 and 4 compared with the others [1, 6, 7, 8, 12-17] is that there are no time derivatives of the exact solution u of (1.1) involved in the analysis and the results, such that these estimates are bounded by the norms of the known data v and f.

2. Formulation of finite element methods

Let S_h be a family of finite element subspaces of $H_0^1(\Omega)$ with the following standard approximation properties: For some $l \geq 1$,

$$\inf_{\chi \in S_h} (||\chi - w|| + h||\chi - w||_1) \leq Ch^{r+1}||w||_{r+1}, \quad 1 \leq r \leq l, \quad w \in H^{r+1}(\Omega) \cap H_0^1(\Omega), \tag{2.1}$$

where $C > 0$ is a constant independent of h, and $||\cdot||_m$ is the norm in the Hilbert space $H^m(\Omega)$ with $||\cdot|| = ||\cdot||_0$, and $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm $||\cdot||_1$.

The time-continuous finite element approximation to the solution u of (1.1) can now be defined as a mapping $u_h(t) : \bar{J} \to S_h$ by

$$\begin{aligned} A(t; u_{h,t}, \chi) + \quad B(t; u_h, \chi) &= (f, \chi), \quad \chi \in S_h, \\ u_h(0) &= v_h \end{aligned} \tag{2.2}$$

where v_h is an appropriate approximation of v into S_h, $A(t;\cdot,\cdot)$ and $B(t;\cdot,\cdot)$ are the bilinear forms associated with the operators $A(t)$ and $B(t)$ on $H_0^1(\Omega) \times H_0^1(\Omega)$.

Before we define the fully-discrete method, let us define (see, for example, [16]) $A_h(t)$: $S_h \to S_h$ by

$$(A_h(t)\phi, \psi) = A(t; \phi, \psi), \qquad \forall \phi, \psi \in S_h \tag{2.3}$$

and $B_h(t) : S_h \to S_h$ by

$$(B_h(t)\phi, \psi) = B(t; \phi, \psi), \qquad \forall \phi, \psi \in S_h. \tag{2.4}$$

Also, we define the L^2-projection operator $P_h : L^2(\Omega) \to S_h$, for any $w \in L^2(\Omega)$, by

$$(P_h w - w, \chi) = 0, \qquad \forall \chi \in S_h. \tag{2.5}$$

Thus, using (2.3)-(2.5), we can rewrite (2.2) as

$$A_h(t)u_{h,t} + B_h(t)u_h = P_h f, \quad t > 0$$

or

$$u_{h,t} + A_h^{-1}(t)B_h(t)u_h = A_h^{-1}(t)P_h f, \quad t > 0. \tag{2.6}$$

Next, let $T_h : L^2(\Omega) \to S_h$ be the approximation operator of the operator $T := A^{-1}$ defined, for any $w \in L^2(\Omega)$, by

$$A(T_h w, \chi) = (w, \chi), \quad \forall \chi \in S_h.$$

Thus, from (2.3) and (2.5) we derive for an arbitrary $w \in L^2(\Omega)$ that

$$(P_h w, \chi) = (w, \chi) = A(T_h w, \chi) = (A_h T_h w, \chi), \quad \forall \chi \in S_h.$$

That is,

$$P_h = A_h T_h \text{ or } T_h = A_h^{-1} P_h, \text{ and } T_h = A_h^{-1} \text{ on } S_h. \tag{2.7}$$

And then, we obtain by using (2.7) and by integrating (2.6) with respect to t that

$$u_h(t) = v_h + \int_0^t T_h(s) f(s) ds - \int_0^t T_h(s) B_h(s) u_h(s) ds. \tag{2.8}$$

Remark 2.1. The integral equation (2.8) is the starting point for our error analysis and the formulation of our high-order time-stepping finite element method.

Now, let us consider a p-th order numerical integration formula. So, we let N denote a positive integer, $\Delta t = T/N$. Let $w_{n,j}$ be the weights such that for any $g(t) \in C^p(\bar{J})$ $(p \geq 1)$,

$$\int_0^{t_n} g(s) ds = \Delta t \sum_{j=1}^{n} w_{n,j} g(t_j) + E_n(g), \tag{2.7}$$

where $t_n = n\Delta t$ and the error $E_n(g)$ satisfies

$$|E_n(g)| \leq C(\Delta t)^p \max_{0 \leq t \leq t_n} \left| \frac{d^p g(t)}{dt^p} \right|, \tag{2.8}$$

for $n = 1, 2, \cdots, N$. We also assume that $w_{n,j}$ is non-negative and

$$\sum_{j=1}^{n} \Delta t w_{n,j} \leq C, \qquad w_{n,n} \leq C, \qquad n = 1, 2, \cdots, N. \tag{2.9}$$

We are now ready to define our time-stepping finite element approximaiton. Let $\{u_h^n\}_{n=0}^N$ be defined by

$$u_h^n = v_h + \Delta t \sum_{j=1}^{n} w_{n,j} T_h(t_j) f(t_j) - \Delta t \sum_{j=1}^{n} w_{n,j} T_h(t_j) B_h(t_j) u_h^j, \quad n = 1, 2, \cdots, N. \tag{2.10}$$

We first consider the algebric problem of how to use (2.12) to compute u_h^n. Let $w_{n,0} := 0$, then (2.12) can be rewritten as

$$\begin{aligned} u_h^n + \Delta t w_{n,n} T_h(t_n) B_h(t_n) u_h^n \quad &= \quad v_h + \Delta t \sum_{j=1}^{n} w_{n,j} T_h(t_j) f(t_j) \\ &\quad - \Delta t \sum_{j=0}^{n-1} w_{n,j} T_h(t_j) B_h(t_j) u_h^j, \quad n = 1, 2, \cdots, N. \end{aligned}$$

Multiplying (2.13) by $A_h(t_n) = T_h^{-1}(t_n)$, we obtain

$$A_h(t_n) u_h^n + \Delta t w_{n,n} B_h(t_n) u_h^n \quad = \quad A_h(t_n) v_h + \Delta t \sum_{j=1}^{n} w_{n,j} A_h(t_n) T_h(t_j) f(t_j)$$

$$\begin{aligned} & -\Delta t \sum_{j=0}^{n-1} w_{n,j} A_h(t_n) T_h(t_j) B_h(t_j) u_h^j \\ = \quad & A_h(t_n) v_h + \Delta t w_{n,n} f(t_n) \\ & +\Delta t \sum_{j=0}^{n-1} w_{n,j} A_h(t_n)(Z_j - W_j) \end{aligned}$$

where

$$\begin{aligned} Z_j &= T_h(t_j) f(t_j), \qquad j = 0, 1, \cdots, n-1, \\ W_j &= T_h(t_j) B_h(t_j) u_h^j, \qquad j = 0, 1, \cdots, n-1. \end{aligned}$$

Using (2.3) and (2.4) we can now write (2.14) as

$$\begin{aligned} A(t_n; u_h^n, \chi) + \Delta t w_{n,n} B(t_n; u_h^n, \chi) &= A(t_n; v_h, \chi) + \Delta t w_{n,n}(f(t_n), \chi) \\ & \quad +\Delta t \sum_{j=0}^{n-1} w_{n,j} A(t_n; Z_j - W_j, \chi), \quad \forall \chi \in S_h \end{aligned}$$

and (2.15) as

$$\begin{aligned} A(t_j; Z_j, \chi) &= (f(t_j), \chi), \qquad \forall \chi \in S_h, \qquad j = 0, 1, \cdots, n-1, \\ A(t_j; W_j, \chi) &= B(t_j; u_h^j, \chi), \qquad \forall \chi \in S_h, \qquad j = 0, 1, \cdots, n-1. \end{aligned}$$

Remark 2.2. If Z_j and W_j $(j = 0, 1, \cdots, n-1)$ are known, u_h^n can be computed through (2.16) because $A(t_n) + \Delta t w_{n,n} B(t_n)$ is also a positive definite elliptic operator with Δt sufficiently small since $A(t)$ is positive.

Numerical Procedure:

(a) Assume that $\{u_h^j\}, j = 0, 1, \cdots, n-1$, are known.

(b) Use (2.17) to compute $\{Z_j, W_j\}, j = 0, 1, \cdots, n-1$.

(c) Substitute $\{Z_j, W_j\}$ into (2.16) to compute u_h^n.

(d) Return to (a) until $n = N$.

Remark 2.3. Since $u_h^0 = v_h$ is known, the procedure (a)-(d) can be started.

Remark 2.4. At each time-step t_n, we need to solve three systems of elliptic equations to advance to the next time level, such that we must save each Z_j and W_j for $j = 0, 1, \cdots, n-1$. However, we gain high-order accuracy approximations in time.

Remark 2.5. If $A(t) = A$ is a time-independent operator, then we have

$$\begin{aligned} A(t_n; Z_j, \chi) &= A(t_j; Z_j, \chi), \quad \forall \chi \in S_h, \quad j = 0, 1, \cdots, \\ A(t_n; W_j, \chi) &= B(t_j; u_h^j, \chi), \quad \forall \chi \in S_h, \quad j = 0, 1, \cdots. \end{aligned}$$

So, (2.16) becomes

$$\begin{aligned} A(u_h^n, \chi) + \Delta t w_{n,n} B(t_n; u_h^n, \chi) &= A(v_h, \chi) + \Delta t \sum_{j=1}^{n} w_{n,j}(f(t_j), \chi) \\ & \quad -\Delta t \sum_{j=0}^{n-1} w_{n,j} B(t_j; u_h^j, \chi), \quad \forall \chi \in S_h. \end{aligned}$$

Thus, (2.18) is just an iteration scheme which requires to save u_h^j only.

Remark 2.6. If both A and B are time-independent, by using the trapezoidal rule for integration we have a Crank-Nicolson type scheme similar to those for Sobolev equations in [6, 7, 8, 14].

3. Error estimates for the semi-discrete scheme

We define $T(t) : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$ to be the solution operator for $A(t)$:

$$A(t; T(t)g, \phi) = (g, \phi), \quad \forall \phi \in H_0^1(\Omega), \tag{3.1}$$

which means that $T(t) = A^{-1}(t)$. Then we can write (1.1) as

$$u = v + \int_0^t T(s)f(s)ds - \int_0^t T(s)B(s)u(s)ds, \quad t \geq 0. \tag{3.2}$$

Hence, we have

Theorem 3.1. *Assume that $v \in H^r(\Omega) \cap H_0^1(\Omega)$, $f \in H^{r-2}(\Omega)$, $r \geq 0$, $t \in \bar{J}$, then there exists a unique $u(t) \in H^r(\Omega) \cap H_0^1(\Omega)$ such that (3.2) is satisfied and*

$$||u(t)||_r \leq C\left(||v||_r + \int_0^t ||f(s)||_{r-2}ds\right), \quad t \geq 0. \tag{3.3}$$

Proof. It is well-known [2, 18] that $||T(t)g||_r \leq C||g||_{r-2}$, and then $||T(t)B(t)g||_r \leq C||g||_r$ for $r \geq 0$. Therefore, the proof can be completed with an application of Gronwall's lemma [18]. Q.E.D.

Before we consider error estimates, it is convenient to list some lemmas that we shall need below.

Lemma 3.1. *Let $u(t)$ be the solution of (3.2), S_h be our finite element spaces defined in Section 2 and $||v - v_h||_0 \leq Ch^r||v||_r$. Then there exists a constant $C > 0$ such that for $-1 \leq q \leq r-1$ and $0 \leq k \leq r-1$,*

$$\begin{aligned} ||(T(t) - T_h(t))g||_{-q} &\leq Ch^{q+2+k}||g||_k, \\ ||(T(t) - T_h(t))u(t)||_{-q} &\leq Ch^{q+2+k}||u(t)||_k \\ &\leq Ch^{q+2+k}\left(||v||_k + \int_0^t ||f(s)||_{k-2}ds\right). \end{aligned}$$

Proof. The inequality (3.4) can be found in Thomée's book [18] and the inequality (3.5) is a consequence of (3.4) and (3.3). Q.E.D.

Remark 3.1. (3.4) and (3.5) are still valid if we replace $T(t)$ and $T_h(t)$ by $T^*(t)$ and $T_h^*(t)$ respectively, where $*$ denotes the adjoint operators of the corresponding operators [2, 18].

Lemma 3.2. *Let $u(t)$ and $u_h(t)$ be the solutions of (3.2) and (2.2) respectively, then there exists $C > 0$ such that for $-1 \leq q \leq r-1$,*

$$||T_h(t)(B(t)u(t) - B_h(t)u_h(t))||_1 \leq C||u(t) - u_h(t)||_1, \tag{3.4}$$

$$\begin{aligned} ||T_h(t)(B(t)u(t) - B_h(t)u_h(t))||_{-q} &\leq Ch^{q+1}||u(t) - u_h(t)||_1 \\ &\quad + C||u(t) - u_h(t)||_{-q}. \end{aligned} \tag{3.5}$$

Proof. As $A(t; T_h(t)\phi, \chi) = (\phi, \chi), \quad \forall \chi \in S_h$ and $\phi \in L^2(\Omega)$, we see from (1.2), (2.4) and the definition of the bilinear form $B(t; \cdot, \cdot)$ that

$$\begin{aligned} & C_0||T_h(t)(B(t)u(t) - B_h(t)u_h(t))||_1^2 \\ \leq\ & A(t; T_h(t)(B(t)u(t) - B_h(t)u_h(t)), T_h(t)(B(t)u(t) - B_h(t)u_h(t))) \\ =\ & (B(t)u(t) - B_h(t)u_h(t), T_h(t)(B(t)u(t) - B_h(t)u_h(t))) \\ =\ & B(t; u(t) - u_h(t), T_h(t)(B(t)u(t) - B_h(t)u_h(t))) \\ \leq\ & C||u(t) - u_h(t)||_1\ ||T_h(t)(B(t)u(t) - B_h(t)u_h(t))||_1. \end{aligned}$$

Thus, (3.6) is proved.

For (3.7), let $\phi \in H^q(\Omega)$, according to (2.4) and the definition of $B(t;\cdot,\cdot)$ we know from Lemma 3.1 that

$$\begin{aligned}
(T_h(t)(B(t)u(t) - B_h(t)u_h(t)), \phi) &= (B(t)u(t) - B_h(t)u_h(t), T_h^*(t)\phi) \\
&= B(t; u(t) - u_h(t), T_h^*(t)\phi) \\
&= B(t; u(t) - u_h(t)), (T_h^*(t) - T^*(t))\phi) \\
&\quad + (u(t) - u_h(t)), B^*(t)T^*(t)\phi) \\
&\leq C||u(t) - u_h(t)||_1 \; ||(T(t) - T_h(t))\phi||_1 \\
&\quad + ||u(t) - u_h(t)||_{-q} ||B^*(t)T^*(t)\phi||_q \\
&\leq C\left(h^{q+1}||u(t) - u_h(t)||_1 + ||u(t) - u_h(t)||_{-q}\right) ||\phi||_q,
\end{aligned}$$

from which (3.7) follows. Q.E.D.

Theorem 3.2. *Assume that $u(t)$ and $u_h(t)$ are the solutions of (3.2) and (2.2) respectively, and $v \in H^{r+1}(\Omega) \cap H_0^1(\Omega)$, $f \in H^{r-1}(\Omega)$, $r \geq 1$ and $||v - v_h||_1 \leq Ch^r ||v||_{r+1}$. Then we have*

$$||u(t) - u_h(t)||_1 \leq Ch^r \left(||v||_{r+1} + \int_0^t (||v(s)||_{r+1} + ||f(s)||_{r-1}) ds \right), \quad t \geq 0. \tag{3.6}$$

Proof. From (2.8) and (3.2) we see that

$$\begin{aligned}
u - u_h &= v - v_h + \int_0^t (T(s) - T_h(s)) f(s) ds \\
&\quad - \int_0^t (T(s)B(s)u(s) - T_h(s)B_h(s)u_h(s)) ds \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

Clearly, we have from Lemma 3.1 and our assumptions that

$$||I_1 + I_2||_1 \leq Ch^r \left(||v||_{r+1} + \int_0^t ||f(s)||_{r-1} ds \right). \tag{3.7}$$

But

$$I_3 = -\int_0^t (T(s) - T_h(s)) B(s)u(s) ds - \int_0^t T_h(s) \; (B(s)u(s) - B_h(s)u_h(s)) ds,$$

and thus, it follows from Lemmas 3.1 and 3.2 that

$$\begin{aligned}
||I_3||_1 &\leq Ch^r \int_0^t ||B(s)u(s)||_{r-1} ds + C\int_0^t ||u(s) - u_h(s)||_1 ds \\
&\leq Ch^r \int_0^t ||u(s)||_{r+1} ds + C\int_0^t ||u(s) - u_h(s)||_1 ds.
\end{aligned}$$

Now, we see from (3.11)-(3.12) and Theorem 3.1 that

$$||u(t) - u_h(t)||_1 \leq Ch^r \left(||v||_{r+1} + \int_0^t (||v(s)||_{r+1} + ||f(s)||_{r-1}) ds \right) + C\int_0^t ||u(s) - u_h(s)||_1 ds.$$

Then, applying Gronwall's lemma yields

$$||u(t) - u_h(t)||_1 \leq Ch^r \left(||v||_{r+1} + \int_0^t (||v(s)||_{r+1} + ||f(s)||_{r-1}) ds \right), \quad t \geq 0.$$

Hence, Theorem 3.2 is completed. Q.E.D.

Theorem 3.3. *Under the assumptions of Theorem 3.2, we have the following negative norm error estimates: for* $-1 \le q \le r-1$,

$$||u(t) - u_h(t)||_{-q} \le Ch^{r+1+q} \left(||v||_{r+1} + \int_0^t (||v(s)||_{r+1} + ||f(s)||_{r-1}) ds \right), \quad t \ge 0, \tag{3.8}$$

provided that $||v - v_h||_{-q} \le Ch^{q+r+1} ||v||_{r+1}$.

Proof. As in the proof of Theorem 3.2, we have from (3.10) and Lemma 3.1 that

$$||I_1||_{-q} \le Ch^{r+1+q} ||v||_{r+1}, \qquad ||I_2||_{-q} \le Ch^{r+1+q} \int_0^t ||f(s)||_{r-1} ds.$$

Similarly, it follows from Lemmas 3.1 and 3.2 that

$$\begin{aligned} ||I_3||_{-q} &\le \int_0^t ||(T(s) - T_h(s))B(s)u(s)||_{-q} ds \\ &\quad + \int_0^t ||T_h(s)\ (B(s)u(s) - B_h(s)u_h(s))||_{-q} ds \\ &\le Ch^{q+1+r} \int_0^t ||u(s)||_{r+1} ds \\ &\quad + Ch^{q+1} \int_0^t ||u(s) - u_h(s)||_1 ds + C \int_0^t ||u(s) - u_h(s)||_{-q} ds. \end{aligned}$$

And hence, we have from Theorems 3.1 and 3.2 that

$$\begin{aligned} ||u(t) - u_h(t)||_{-q} &\le Ch^{r+1+q} \left(||v||_{r+1} + \int_0^t (||v(s)||_{r+1} + ||f(s)||_{r-1}) ds \right) \\ &\quad + C \int_0^t ||u(s) - u_h(s)||_{-q} ds. \end{aligned}$$

Therefore, the proof is completed with an application of Gronwall's lemma. *Q.E.D.*

4. Global superconvergence for the semi-discrete scheme

In this section, we discuss superconvergence of the semi-discrete finite element method for the problem (1.1). Similar to Section 3, there are still no time derivatives of the exact solution u involved in our error analysis of this section. The strategy employed here is that we first examine the superclose accuracy between the interpolation of the exact solution and the finite element solution of (1.1) by means of integral identities, and then we utilize a suitable interpolation post-processing method to obtain global superconvergence approximations [12, 13].

First of all, let us introduce a concept. For this purpose, it is assumed that the domain Ω can be mapped onto a rectangular domain $\hat{\Omega}$ by a smooth and invertible transform Φ. Then, a rectangular partition $\hat{T_h}$ is imposed on $\hat{\Omega} := \Phi(\Omega)$, and the partition $T_h := \Phi^{-1}(\hat{T_h})$ over Ω is called a generalized rectangular mesh. For example, for a convex quadrilateral region Ω, its generalized rectangular mesh can be obtained by connecting the eqi-proportional points of the two pairs of opposite boundaries, since each quadrilateral element in this partition T_h is correspondingly mapped onto a rectangular element in the rectangular mesh $\hat{T_h}$ by using an invertible bilinear tranform Φ. We must point out that although our analysis and results here for superconvergence are valid for the generalized rectangular mesh, for simplicity, our attention is focused on finite element partition of Ω into rectangules. Moreover, it is also assumed that Ω is a polygon with boundaries paralell to the axes.

In order to obtain the superclose estimates between the interpolation of the exact solution and the finite element solution of (1.1), when the degree of the finite element space S_h is more than 1 we need to define a type of projection interpolation operators i_h^r, rather than the usual nodal Lagrange interpolation operators, of degree not exceeding r (≥ 2) in x and y. Let $e := [x_e - h_e, y_e - k_e] \in T_h$ be any rectangular element and l_i, p_i ($i = 1, 2, 3, 4$) its edges and vertices. Then, the bi-rth interpolation operator i_h^r is defined according to the following so called "vertex-edge-element" conditions [12]:

$$\begin{cases} i_h^r u \in Q_r(e), \\ i_h^r u(p_i) = u(p_i), \quad i = 1, 2, 3, 4, \\ \displaystyle\int_{l_i} (i_h^r u - u) v = 0, \quad \forall v \in P_{r-2}(l_i), \quad i = 1, 2, 3, 4, \\ \displaystyle\int_e (i_h^r u - u) v = 0, \quad \forall v \in Q_{r-2}(e), \end{cases}$$

where $P_{r-2}(l_i)$ and $Q_{r-2}(e)$ are the polynomial spaces of degree no more than $r-2$ on l_i and e, respectively. In our analysis later, the notation i_h^1 stands for the usual nodal Lagrange bilinear interpolation operator.

Introducing the two error functions

$$F_e := \frac{1}{2}[(y - y_e)^2 - k_e^2], \quad E_e := \frac{1}{2}[(x - x_e)^2 - h_e^2]$$

and employing the integral identity technique, we obtain the following lemma [12, 13].

Lemma 4.1. *In (1.1), assume that the coefficients b_{ij}, b_i, b of the second order elliptic operator $B(t)$ are all in $W^{1,\infty}(\Omega)$. Then, for all $\chi \in S_h$ we have the following estimate:*

$$B(t; u - i_h^r u, \chi) = \begin{cases} O(h^{r+1}) \|u\|_{r+2} \|\chi\|_1, & r \geq 1, \\ O(h^{r+2}) \|u\|_{r+2} \|\chi\|_2, & r \geq 2, \end{cases}$$

where $\|\chi\|_2 := \left(\sum_e \|\chi\|_{2,e}^2 \right)^{1/2}$.

In (2.2) we take v_h as the Ritz projection of v with respect to the operator $A(t)$, i.e.,

$$A(0; v - v_h, \chi) = 0, \quad \forall \chi \in S_h. \tag{4.1}$$

¿From (1.1) and (2.2) we derive the error equation,

$$A(t; (u - u_h)_t, \chi) + B(t; u - u_h, \chi) = 0, \quad \forall \chi \in S_h. \tag{4.2}$$

Thus, integrate (4.2) about variable t and use (4.1) to obtain

$$A(t; u(t) - u_h(t), \chi) + \int_0^t \hat{B}(s; u(s) - u_h(s), \chi) = 0, \quad \forall \chi \in S_h, \tag{4.3}$$

where $\hat{B}(t; u, v) := B(t; u, v) - A_t(t; u, v)$ is the bilinear form associated with the operator $\hat{B}(t) := B(t) - A_t(t)$, and $A_t(t)$ is the operator derived from $A(t)$ by differentiating its coefficients with respect to t. Now, we are ready to obtain our superclose theorems.

Theorem 4.1. *In (1.1), assume that $v \in H_0^1(\Omega) \cap H^{r+2}(\Omega)$, $f \in H^r(\Omega)$ and the coefficients of the operators $A(t)$ and $B(t)$ are sufficiently smooth. Then, we have the following superclose estimate:*

$$\|u_h - i_h^r u\|_1 \leq C h^{r+1} \left(\|v\|_{r+2} + \int_0^t (\|v(s)\|_{r+2} + \|f(s)\|_r) ds \right), \quad r \geq 1.$$

Proof. Let $\theta(t) := u_h(t) - i_h^r u(t)$. Then, it follows from (4.3) and Lemma 4.1 that

$$\begin{aligned} A(t;\theta(t),\chi) \quad &+ \int_0^t \hat{B}(s;\theta(s),\chi)ds \\ &= A(t;u(t)-i_h^r u(t),\chi) + \int_0^t \hat{B}(s;u(s)-i_h^r u(s),\chi)ds \\ &\leq Ch^{r+1}\|u\|_{r+2}\|\chi\|_1 + \int_0^t Ch^{r+1}\|u(s)\|_{r+2}ds\|\chi\|_1, \quad \forall \chi \in S_h. \end{aligned} \tag{4.4}$$

And then, we find from (4.4) that

$$\begin{aligned} C_0\|\theta(t)\|_1^2 \quad &\leq A(t;\theta(t),\theta(t)) \\ &= A(t;\theta(t),\theta(t)) + \int_0^t \hat{B}(s;\theta(s),\theta(t))ds - \int_0^t \hat{B}(s;\theta(s),\theta(t))ds \\ &\leq Ch^{r+1}\left(\|u\|_{r+2} + \int_0^t \|u(s)\|_{r+2}ds\right)\|\theta(t)\|_1 + C\int_0^t \|\theta(s)\|_1 ds\|\theta(t)\|_1 \\ &\leq Ch^{2(r+1)}\left(\|u\|_{r+2} + \int_0^t \|u(s)\|_{r+2}ds\right)^2 + C\left(\int_0^t \|\theta(s)\|_1 ds\right)^2 + \epsilon\|\theta(t)\|_1^2, \end{aligned}$$

with $\epsilon > 0$ sufficiently small; or

$$\|\theta\|_1 \leq Ch^{r+1}\left(\|u\|_{r+2} + \int_0^t \|u(s)\|_{r+2}ds\right) + C\int_0^t \|\theta(s)\|_1 ds.$$

Therefore, Gronwall's lemma and Theorem 3.1 imply

$$\|\theta\|_1 \leq Ch^{r+1}\left(\|v\|_{r+2} + \int_0^t (\|v(s)\|_{r+2} + \|f(s)\|_r)ds\right).$$

Q.E.D.

Theorem 4.2. *Under the conditions of Theorem 4.1, we have the following superclose estimate:*

$$\|u_h - i_h^r u\|_0 \leq Ch^{r+2}\left(\|v\|_{r+2} + \int_0^t (\|v(s)\|_{r+2} + \|f(s)\|_r)ds\right), \quad r \geq 2.$$

Proof. For arbitrary $\varphi \in L^2(\Omega)$, let $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of the following auxiliary variational problem:

$$A(t;\theta,\psi) = (\theta,\varphi),$$

where $\theta(t) := u_h(t) - i_h^r u(t)$. Then, we have the a-priori estimate

$$\|\psi\|_2 \leq C\|\varphi\|_0. \tag{4.5}$$

In addition, we also have

$$\begin{aligned} (\theta,\varphi) \quad &= A(t;\theta,\psi) \\ &= A(t;\theta,\psi-\chi) + A(t;\theta,\chi) + \int_0^t \hat{B}(s;\theta(s),\chi)ds \\ &\quad + \int_0^t \hat{B}(s;\theta(s),\psi-\chi)ds - \int_0^t \hat{B}(s;\theta(s),\psi)ds, \end{aligned} \tag{4.6}$$

where $\chi \in \hat{S}_h := \left\{\varphi \in H_0^1(\Omega) \cap H^2(\Omega) : \varphi|_e \in Q_r(e), e \in T_h\right\} \subset S_h$. $\hat{S}_h$ possesses the following approximation properties:

$$\inf_{\chi\in\hat{S}_h} \{\|u-\chi\|_0 + h\|u-\chi\|_1 + h^2\|u-\chi\|_2\} \leq Ch^l\|u\|_l, \quad u \in H_0^1(\Omega)\cap H^p(\Omega), \quad 2 \leq l \leq r+1, \tag{4.7}$$

which leads to

$$\|\chi\|_2 \le \|\chi - \psi\|_2 + \|\psi\|_2 \le C\|\psi\|_2. \tag{4.8}$$

It follows from (4.3) and Lemma 4.1 that

$$\begin{aligned} A(t;\theta,\chi) \quad &+ \int_0^t \hat{B}(s;\theta(s),\chi)ds \\ &= A(t;u(t) - i_h^r u(t),\chi) + \int_0^t \hat{B}(s;u(s) - i_h^r u(s),\chi)ds \\ &\le Ch^{r+2}\left(\|u\|_{r+2} + \int_0^t \|u(s)\|_{r+2}ds\right)\|\chi\|_2 \end{aligned}$$

which, together with (4.5)-(4.8), Green formula, Theorems 3.1 and 4.1, yields

$$\begin{aligned} (\theta,\varphi) \quad &\le C\|\theta\|_1\|\psi - \chi\|_1 + Ch^{r+2}\left(\|u\|_{r+2} + \int_0^t \|u(s)\|_{r+2}ds\right)\|\chi\|_2 \\ &+ C\int_0^t \|\theta(s)\|_1 ds\|\psi - \chi\|_1 + C\int_0^t \|\theta(s)\|_0 ds\|\psi\|_2 \\ &\le Ch^{r+2}\left(\|u\|_{r+2} + \int_0^t \|u(s)\|_{r+2}ds\right)\|\varphi\|_0 + C\int_0^t \|\theta(s)\|_0 ds\|\varphi\|_0 \end{aligned}$$

or

$$\|\theta\|_0 \le Ch^{r+2}\left(\|v\|_{r+2} + \int_0^t (\|v(s)\|_{r+2} + \|f(s)\|_r)ds\right) + C\int_0^t \|\theta(s)\|_0 ds.$$

Hence, Theorem 4.2 is completed according to Gronwall's lemma. *Q.E.D.*

In order to improve accuracy on a global scale, a reasonable post-processing method is proposed. For this end, we need to define another post-processing interpolation operator I_{2h}^{r+1} of degree at most $r+1$ in x and y. Thus, we assume that T_h has been obtained from T_{2h} with mesh size $2h$ by subdividing each element of T_{2h} into four congruent rectangles. Let $\tau := \bigcup_{i=1}^{4} e_i \in T_{2h}$ with $e_i \in T_h$. To express our idea clearly, we first consider the one-dimension case, where I_{2h}^{r+1} is determined by the following "vertex-interval" conditions:

$$\begin{cases} I_{2h}^{r+1}u(p_i) = u(p_i), \quad i = 1,2,3, \\ \displaystyle\int_{l_i} I_{2h}^{r+1}u = \int_{l_i} u, \quad i = 1,2, \\ \displaystyle\int_L I_{2h}^{r+1}uv = \int_L uv, \quad \forall v \in P_{r-2}(L). \end{cases}$$

Here, $L := l_1 \cup l_2 \in T_{2h}$, $l_i \in T_h$, and p_i $(i = 1,2,3)$ are the vertices of l_1 and l_2. Then, the operator I_{2h}^{r+1} in the two-dimension case is constructed by the tensor product of the two one-dimension interpolation operators $I_{2h,x}^{r+1}$ and $I_{2h,y}^{r+1}$ of degree not exceeding $r+1$ in x- and y-direction, respectively, as follows:

$$I_{2h}^{r+1} := I_{2h,x}^{r+1} \cdot I_{2h,y}^{r+1}.$$

Moreover, the following properties can be easily checked [12, 13]:

$$\begin{cases} I_{2h}^{r+1}i_h^r = I_{2h}^{r+1}, \\ \|I_{2h}^{r+1}v\|_q \le C\|v\|_q, \quad \forall v \in S_h, \quad q = 0,1, \\ \|I_{2h}^{r+1}u - u\|_q \le Ch^{r+2-q}\|u\|_{r+2}, \quad \forall u \in H^{r+2}(\Omega), \quad q = 0,1. \end{cases} \tag{4.9}$$

And then, we obtain the following global superconvergence theorem.

Theorem 4.3. *Under the conditions of Theorem 4.1, we have*

$$\|I_{2h}^{r+1}u_h - u\|_1 \le Ch^{r+1}\left(\|v\|_{r+2} + \int_0^t (\|v(s)\|_{r+2} + \|f(s)\|_r)ds\right), \quad r \ge 1, \tag{4.10}$$

$$\|I_{2h}^{r+1}u_h - u\|_0 \le Ch^{r+2}\left(\|v\|_{r+2} + \int_0^t (\|v(s)\|_{r+2} + \|f(s)\|_r)ds\right), \quad r \ge 2. \tag{4.11}$$

Proof. From one of the properties of the operator I_{2h}^{r+1} in (4.9) we find that

$$I_{2h}^{r+1}u_h - u = I_{2h}^{r+1}(u_h - i_h^r u) + (I_{2h}^{r+1}u - u).$$

Therefore, it follows from Theorem 4.1, (4.9) and Theorem 3.1 that

$$\begin{aligned}\|I_{2h}^{r+1}u_h - u\|_1 \quad &\le C\|u_h - i_h^r u\|_1 + Ch^{r+1}\|u\|_{r+2} \\ &\le Ch^{r+1}\left(\|v\|_{r+2} + \int_0^t (\|v(s)\|_{r+2} + \|f(s)\|_r)ds\right).\end{aligned}$$

(4.11) can also be obtained according to the same argument as that for (4.10). *Q.E.D.*

It is of great importance for a finite element method to have a computable a-posteriori error estimator by which we can assess the accuracy of finite element solution in applications. One way to construct error estimators is to employ certain superconvergence properties of the finite element solutions. In fact, we have

Theorem 4.4. *We have under the conditions of Theorem 4.1 that*

$$\|u - u_h\|_1 = \|I_{2h}^{r+1}u_h - u_h\|_1 + O(h^{r+1}), \quad r \ge 1, \tag{4.12}$$

$$\|u - u_h\|_0 = \|I_{2h}^{r+1}u_h - u_h\|_0 + O(h^{r+2}), \quad r \ge 2. \tag{4.13}$$

In addition, if there exist positive constants C_1, C_2 and small ϵ_1, $\epsilon_2 \in (0,1)$ such that

$$\|u - u_h\|_1 \ge C_1 h^{r+1-\epsilon_1}, \tag{4.14}$$

$$\|u - u_h\|_0 \ge C_1 h^{r+2-\epsilon_2}, \tag{4.15}$$

then there hold

$$\lim_{h\to 0} \frac{\|u - u_h\|_1}{\|I_{2h}^{r+1}u_h - u_h\|_1} = 1, \tag{4.16}$$

$$\lim_{h\to 0} \frac{\|u - u_h\|_0}{\|I_{2h}^{r+1}u_h - u_h\|_0} = 1. \tag{4.17}$$

Proof. It follows from Theorem 4.3 and

$$u - u_h = (I_{2h}^{r+1}u_h - u_h) + (u - I_{2h}^{r+1}u_h)$$

that

$$\|u - u_h\|_1 = \|I_{2h}^{r+1}u_h - u_h\|_1 + O(h^{r+1}).$$

Thus, by (4.14) we have

$$\frac{\|I_{2h}^{r+1}u_h - u_h\|_1}{\|u - u_h\|_1} + Ch^{\epsilon_1} \ge 1$$

or

$$\underline{\lim}_{h \to 0} \frac{\|I_{2h}^{r+1}u_h - u_h\|_1}{\|u - u_h\|_1} \ge 1. \tag{4.18}$$

Similarly, it follows from (4.14) and

$$\|I_{2h}^{r+1}u_h - u_h\|_1 = \|u - u_h\|_1 + O(h^{r+1})$$

that

$$\overline{\lim_{h\to 0}}\frac{\|I_{2h}^{r+1}u_h - u_h\|_1}{\|u-u_h\|_1} \le 1,$$

which, together with (4.18), leads to (4.16).

Analogously, we can obtain (4.13) from Theorem 4.3 and (4.17) from the condition (4.15). *Q.E.D.*

We know from (4.12) that the computable error estimate $\|I_{2h}^{r+1}u_h - u_h\|_1$ is the principal part of the finite element error $\|u-u_h\|_1$, and can be used as an a-posteriori error indicator to assess the accuracy of the finite element solution. Also, the condition (4.14) seems to be a reasonable assumption because $O(h^r)$ is the optimal convergence rate of the finite element solution in H^1-norm. The same comments are valid to (4.13) and (4.15).

5. Error estimates and global superconvergence for the fully-discrete scheme

Theorem 5.1. *Assume that* $v \in H^{r+1}(\Omega)\cap H_0^1(\Omega)$, $||v-v_h||_{-q} \le Ch^{r+1+q}||v||_{r+1}$, $D_t^k u \in L^\infty(J;H^{r+1}(\Omega))$ *and* $D_t^k f \in L^\infty(J;H^{r-1}(\Omega))$ $(r\ge 1)$ *with* $-1\le q\le r-1$ *and* $0\le k\le p$. *Then we have*

$$\max_{0\le j\le N}||e(t_j)||_{-q} \le C(h^{r+1+q}+(\Delta t)^p), \tag{5.1}$$

where $e(t_j)=u(t_j)-u_h^j$, $j=0,1,\cdots,N$.

Proof. We multiply (3.2) by $\phi\in H^q(\Omega)$, integrate over Ω and use numerical integration formula (2.9) to obtain

$$\begin{aligned}(u(t_n),\phi) \;=\;& (v,\phi)+\Delta t\sum_{j=1}^n w_{n,j}(T(t_j)f(t_j),\phi)\\ &-\Delta t\sum_{j=1}^n w_{n,j}(T(t_j)B(t_j)u(t_j),\phi)\\ &+E_n\left((T(t)f(t),\phi)-(T(t)B(t)u(t),\phi)\right).\end{aligned}$$

We assume for the time being that

$$\begin{aligned}&|E_n\left((T(t)f(t),\phi)-(T(t)B(t)u(t),\phi)\right)|\\ \le\;& C(\Delta t)^p\sum_{j=0}^p\Big(||D_t^j u||_{L^\infty(H^{-q}(\Omega))}+||D_t^j f||_{L^\infty(H^{-q-2}(\Omega))}\Big)||\phi||_q,\end{aligned}$$

and use (5.3) to show (5.1).

Now, we multiply (2.12) by $\phi\in H^q(\Omega)$ and integrate over Ω, and then subtract the resultant expression from (5.2) to obtain

$$\begin{aligned}(e(t_n),\phi) \;=\;& (v-v_h,\phi)+\Delta t\sum_{j=1}^n w_{n,j}\left((T(t_j)-T_h(t_j))f(t_j),\phi\right)\\ &-\Delta t\sum_{j=1}^n w_{n,j}(T(t_j)B(t_j)u(t_j)-T_h(t_j)B_h(t_j)u_h^j,\phi)\\ &+E_n\left((T(t)f(t),\phi)-(T(t)B(t)u(t),\phi)\right).\\ :=\;& J_1+J_2+J_3.\end{aligned}$$

It is easy to see from Lemma 3.1 that

$$\begin{aligned} |J_1| &\leq C\left(||v-v_h||_{-q}+\Delta t\sum_{j=1}^{n} w_{n,j}||(T(t_j)-T_h(t_j))f(t_j)||_{-q}\right)||\phi||_q \\ &\leq Ch^{r+1+q}\left(||v||_{r+1}+\Delta t\sum_{j=1}^{n} w_{n,j}||f(t_j)||_{r-1}\right)||\phi||_q \leq Ch^{q+r+1}||\phi||_q \end{aligned}$$

and from (5.3) that

$$|J_3| \leq C(\Delta t)^p||\phi||_q. \tag{5.2}$$

For J_2, if we rewrite it as

$$\begin{aligned} J_2 &= -\Delta t\sum_{j=1}^{n} w_{n,j}((T(t_j)-T_h(t_j))B(t_j)u(t_j),\phi) \\ &\quad -\Delta t\sum_{j=1}^{n} w_{n,j}(T_h(t_j)(B(t_j)u(t_j)-B_h(t_j)u_h^j),\phi) \\ &:= J_{21}+J_{22}. \end{aligned}$$

It follows from Lemma 3.1 and our assumptions that

$$|J_{21}| \leq Ch^{r+1+q}\sum_{j=1}^{n}\Delta t w_{n,j}||u(t_j)||_{r+1}||\phi||_q \leq Ch^{r+1+q}||\phi||_q$$

and from Lemma 3.2 and Theorem 3.2 that

$$\begin{aligned} |J_{22}| &\leq C\left(h^{q+1}\Delta t\sum_{j=1}^{n} w_{n,j}||e(t_j)||_1+\Delta t\sum_{j=1}^{n} w_{n,j}||e(t_j)||_{-q}\right)||\phi||_q \\ &\leq C\left(h^{q+1}(h^r+(\Delta t)^p)+\Delta t\sum_{j=1}^{n} w_{n,j}||e(t_j)||_{-q}\right)||\phi||_q, \end{aligned}$$

where we have used $||e(t_j)||_1 \leq C(h^r+(\Delta t)^p)$ which can be proved in a similar manner to that of Theorem 3.2. Thus, we know from combining the estimates for $J's$ that

$$|(e(t_n),\phi)| \leq C(h^{r+1+q}+h^{q+1}(\Delta t)^p+(\Delta t)^p)||\phi||_q + C\Delta t\sum_{j=0}^{n} w_{n,j}||e(t_j)||_{-q}||\phi||_q,$$

and then

$$||e(t_n)||_{-q} \leq C(h^{r+1+q}+h^{q+1}(\Delta t)^p+(\Delta t)^p)+C\Delta t\sum_{j=0}^{n} w_{n,j}||e(t_j)||_{-q},$$

where $w_{n,0}=0$. Hence, discrete Gronwall's lemma will yield the desired result. Q.E.D.

We now show (5.3) which is formulated as.

Lemma 5.1. *For any $\phi(t) \in H^q(\Omega)$, $-1 \leq q \leq r-1$, $t \in \bar{J}$, we have for some positive constant $C>0$ independent of Δt that*

$$|E_n((T(t)B(t)u,\phi))| \leq C(\Delta t)^p\left(\sum_{j=0}^{p}||u_t^{(j)}||_{L^\infty(J;H^{-q}(\Omega))}\right)||\phi||_q, \tag{5.3}$$

$$|E_n((T(t)f(t),\phi))| \leq C(\Delta t)^p\left(\sum_{j=0}^{p}||f_t^{(j)}||_{L^\infty(J;H^{-q-2}(\Omega))}\right)||\phi||_q, \tag{5.4}$$

where $g_t^{(i)} := \dfrac{d^i g}{dt^i}$ *for* $i = 0, 1, \cdots$.

Proof. We prove (5.8) only since (5.9) can be shown in a similar manner. From (2.10) we see that

$$\begin{aligned} |E_n((T(t)B(t)u(t), \phi))| &\leq C(\Delta t)^p \max_{0\leq t\leq t_n} \left|\frac{d^p}{dt^p}(T(t)B(t)u(t), \phi)\right| \\ &\leq C(\Delta t)^p \max_{0\leq t\leq t_n} \left|(\frac{d^p}{dt^p}(T(t)B(t)u(t)), \phi)\right|. \end{aligned}$$

Letting $w(t) = T(t)B(t)u(t)$, it follows that $A(t)w(t) = B(t)u(t)$. Thus, we obtain by differentiating it m times that

$$\sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} A^{(j)} w_t^{(m-j)} = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} B^{(j)} u_t^{(m-j)} \tag{5.5}$$

where

$$A^{(j)} = \frac{d^j}{dt^j} A(t), \qquad B^{(j)} = \frac{d^j}{dt^j} B(t), \qquad j = 0, 1, \cdots, \tag{5.6}$$

and then, we have

$$A(t) w_t^{(m)} = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} B^{(j)} u_t^{(m-j)} - \sum_{j=1}^{m} \begin{pmatrix} m \\ j \end{pmatrix} A^{(j)} w_t^{(m-j)}. \tag{5.7}$$

Therefore, we obtain by multiplying (5.13) by $T(t)$ that

$$||w_t^{(m)}||_{-q} \leq C \sum_{j=0}^{m} ||T(t) B^{(j)} u_t^{(m-j)}||_{-q} + C \sum_{j=1}^{m} ||T(t) A^{(j)} w_t^{(m-j)}||_{-q} \tag{5.8}$$

from which an induction argument implies

$$||w_t^{(p)}||_{-q} \leq C(p) \sum_{j=0}^{p} ||u_t^{(j)}||_{-q}, \quad p = 0, 1, \cdots. \tag{5.9}$$

Hence, we have

$$\left|(\frac{d^p}{dt^p}(T(t)B(t)u(t)), \phi)\right| \leq C(p) \sum_{j=0}^{p} ||u_t^{(j)}||_{-q} ||\phi||_q, \tag{5.10}$$

and then, (5.8) follows from (5.10). *Q.E.D.*

Next we demonstrate that the superclose estimates and the global superconvergence can also be obtained for the fully-discrete solution of the problem (1.1). Again, we take v_h in (2.12) as the Ritz projection of v with respect to the operator $A(t)$, i.e., we assume that (4.1) holds.

Integrate (2.2) about variable t to get

$$A(t; u_h, \chi) + \int_0^t \hat{B}(s; u_h(s), \chi) ds = A(0; v_h, \chi) + \int_0^t (f(s), \chi) ds, \quad \forall \chi \in S_h,$$

where $\hat{B}$ is the operator defined in Section 4. Thus, we obtain the equivalent form to (2.12),

$$\begin{aligned} A(t_n; u_h^n, \chi) \quad &+ \Delta t \sum_{j=1}^{n} w_{n,j} \hat{B}(t_j; u_h^j, \chi) \\ &= A(0; v_h, \chi) + \Delta t \sum_{j=1}^{n} w_{n,j} (f(t_j), \chi), \quad \forall \chi \in S_h, \quad n = 1, 2, \cdots, N. \end{aligned} \tag{5.17}$$

On the other hand, it follows from the bilinear form of (1.1) and integrating it with respect to t that

$$A(t_n;u(t_n),\chi)+\int_0^{t_n}\hat{B}(s;u(s),\chi)ds=A(0;v,\chi)+\int_0^{t_n}(f(s),\chi)ds,\quad \forall\chi\in S_h. \tag{5.18}$$

Therefore, subtracting (5.17) from (5.18) and utilizing (4.1) we find

$$\begin{aligned} A(t_n;u(t_n)-u_h^n,\chi) \quad &+\Delta t\sum_{j=1}^{n} w_{n,j}\hat{B}(t_j;u(t_j)-u_h^j,\chi) \\ &= E_n((f,\chi)-\hat{B}(s;u(s),\chi)),\quad \forall\chi\in S_h,\quad n=1,2,\cdots,N. \end{aligned} \tag{5.19}$$

Now we are in the position to state our global superconvergence.

Theorem 5.2. In (1.1), assume that $v\in H^{r+2}(\Omega)\cap H_0^1(\Omega)$, $D_t^k u\in L^\infty(J;H^{r+2}(\Omega))$ and $D_t^k f\in L^\infty(J;H^r(\Omega))$ with $0\le k\le p$ as well as the coefficients of the operators $A(t)$ and $B(t)$ are sufficiently smooth. Then we have

$$\max_{0\le j\le N}\|\theta(t_j)\|_1\le C\left(h^{r+1}+(\Delta t)^p\right),\qquad r\ge 1,$$

$$\max_{0\le j\le N}\|\theta(t_j)\|_0\le C\left(h^{r+2}+(\Delta t)^p\right),\qquad r\ge 2,$$

where $\theta(t_j):=u_h^j-i_h^r u(t_j)$.

Proof. Since the proofs are similar to those for Theorems 4.1 and 4.2, we omit the details. $Q.E.D.$

Like Section 4 we can derive superconvergence estimates from Theorem 5.2 by virtue of the interpolation post-processing method.

Theorem 5.3. We have under the conditions of Theorem 5.2 that

$$\max_{0\le j\le N}\|I_{2h}^{r+1}u_h^j-u(t_j)\|_1\le C\left(h^{r+1}+(\Delta t)^p\right),\quad r\ge 1,$$

$$\max_{0\le j\le N}\|I_{2h}^{r+1}u_h^j-u(t_j)\|_0\le C\left(h^{r+2}+(\Delta t)^p\right),\quad r\ge 2.$$

Similar to Section 4, from Theorem 5.3 we obtain the following a-posteriori error estimators:

$$\max_{0\le j\le N}\|u(t_j)-u_h^j\|_1=\max_{0\le j\le N}\|I_{2h}^{r+1}u_h^j-u_h^j\|_1+O\left(h^{r+1}+(\Delta t)^p\right),\quad r\ge 1,$$

$$\max_{0\le j\le N}\|u(t_j)-u_h^j\|_0=\max_{0\le j\le N}\|I_{2h}^{r+1}u_h^j-u_h^j\|_0+O\left(h^{r+2}+(\Delta t)^p\right),\quad r\ge 2.$$

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