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EXACT NONREFLECTING BOUNDARY CONDITIONS FOR AN ACOUSTIC PROBLEM IN THREE DIMENSIONS*[1)]

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Dedicated to the 80th birthday of Professor Zhou Yulin

Abstract

In this paper, nonreflecting artificial boundary conditions are considered for an acoustic problem in three dimensions. With the technique of Fourier decomposition under the orthogonal basis of spherical harmonics, three kinds of equivalent exact artificial boundary conditions are obtained on a spherical artificial boundary. A numerical test is presented to show the performance of the method.

Key words: Exterior problem, Nonreflecting artificial boundary condition, Acoustic equation, Three-dimensions.

1. Introduction

Numerical simulation has been of immense importance in the research fields of acoustics, such as sound propagation and sound scattering. Mathematically, these problems usually lead to some PDE defined on an unbounded domain. When one tries to devise a numerical scheme, a great difficulty occurs owing to the unboundedness of the physical domain. Since most of the numerical methods require a finite computational domain, a natural idea is to introduce some artificial boundary to limit the unbounded domain, and then set up some kind of boundary condition on the artificial boundary. This is just the basic idea of the so-called *artificial boundary method*. Generally, this boundary condition should be chosen carefully so that the problem restricted to the bounded domain is not only well-posed, but is also a highly accurate approximation of the original problem.

In the last several decades, mathematicians have made splendid progress in this method. Consider the example of wave-like equations. Here, Engquist and Majda [3] derived the artificial boundary conditions with the Padé approximation of the pseudodifferential operator on a line-type artificial boundary for the hyperbolic equation. Bayliss and Turkel [2] obtained a series of artificial boundary conditions based on asymptotic expansion of the solution for the same hyperbolic equation at large distance. Higdon [6] considered the two-dimensional hyperbolic equation in a rectangular computational domain. He designed a series of artificial boundary conditions which are perfectly absorbing for plane waves in some prescribed wave directions. All these artificial boundary conditions are local in both time and space.

Well-designed nonlocal artificial boundary conditions have the potential of being more accurate than any local one. Grote and Keller [5] once designed an exact artificial boundary condition on a spherical artificial boundary for a hyperbolic equation in three dimensions (the linear scalar acoustic equation belongs to this category). By decomposing the function into the sum of spherical harmonics, they obtained an exact boundary condition for each harmonic

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component by a suitable change of the unknowns. Then they combine all the components to get the final answer. Based on the Laplace transform, one can easily obtain the exact artificial boundary condition on a circular artificial boundary for the exterior problem of the scalar hyperbolic equation. But an integral kernel is involved in this boundary condition, which is the inverse Laplace transform of the logarithmic derivatives of a Hankel function. How to deal with this kernel becomes the key problem in its numerical implementation. This is just the problem considered in the paper of Alpert, Greengard and Hagstrom [1]. They use the technique of pole expansion to approximate the kernel by a series of exponential functions for any prescribed accuracy.

In this paper, we concentrate on the design of exact nonreflecting artificial boundary conditions for an acoustic problem. We propose a new method which is much different from the one of Grote and Keller [5].

2. Setup of the considered problem

Consider the setup shown in Figure 1. The shaded region with boundary Γ denotes some sound source, such as a vibrating drum. The generated wave propagates in the exterior domain $\hat{\Omega}$ outside the source boundary Γ. If the data on Γ has been detected, we want to solve the pressure field. In the real application, only a certain finite region close to the source, say Ω, is of "physical interest". We assume that Ω is bounded by Γ and a spherical surface B of radius R, which, in most of the literature, is called the artificial boundary. Finally , we define $D = \hat{\Omega}\backslash\bar{\Omega}$ to be the residual unbounded domain.

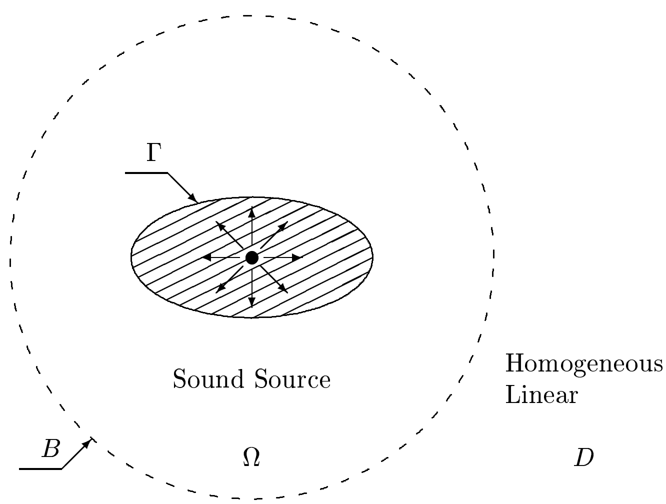

Figure 1: Setup of the considered sound propagation problem.

In order to apply the proposed method in the following, some regularity assumptions should be made on the unbounded domain D and the pressure field. In particular, we assume that the medium in D is homogeneous and behaves linearly; the pressure field is of sufficient smoothness and has zero initial value and null source on a open domain containing the closure of D. On the other hand, no such assumptions are necessary in Ω. The media can be even inhomogeneous and nonlinear. The reason for different assumptions, which will be clear at the end of the paper, is that we have to solve a problem analytically in D to build up a proper relation between the unknown and some of their derivatives on B, whereas the problem in Ω is only intended to be solved numerically after this relation is imposed on B. Any limitation to the generality of this setup is due to the limitations in the capability of the numerical scheme which we can adopt to solve the problem on the finite domain Ω. According to our assumption, in the unbounded domain D, the pressure field, which is denoted by u, satisfies

$$\frac{\partial^2 u}{\partial t^2} = \Delta u, \quad (\mathbf{x}, t) \in D \times (0, T], \tag{1}$$

$$\frac{\partial u}{\partial t}(\mathbf{x}, 0) = u(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in D. \tag{2}$$

For the sake of simplicity, we have assumed that the velocity of sound propagation is 1 in the above formulae. Since no boundary condition is imposed on B, this problem is not well-posed in $D \times [0, T]$. As stated on the above, our main goal is to find a *proper relation* on B, which can serve as a boundary condition for the problem in the finite computational domain Ω.

3. Nonreflecting artificial boundary conditions

Let us recall some results on the spherical harmonics in [7]. For any pair of integers (l, m) such that $l \geq 0$ and $|m| \leq l$, denote $Y_l^m(\mathbf{s})$ the m-th normalized spherical harmonic of order l, then all the spherical harmonics $\{Y_l^m(\mathbf{s}), l \geq 0, -l \leq m \leq l\}$ constitute an orthogonal basis of $L^2(S)$ where S denotes the unit spherical surface; moreover, if Δ_S denotes the Laplace-Beltrami operator, it holds

$$\Delta_S Y_l^m(\mathbf{s}) = -l(l+1) Y_l^m(\mathbf{s}), \quad \mathbf{s} \in S, -l \leq m \leq l. \tag{3}$$

The Laplace-Beltrami operator is associated with the Laplace operator Δ by

$$\Delta u(\mathbf{x}) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r}(r, \mathbf{s}) \right) + \frac{1}{r^2} \Delta_S u(r, \mathbf{s}) \tag{4}$$

where $r = |\mathbf{x}|$ and $\mathbf{x} = r\mathbf{s}$.

Now decompose u into the Fourier series under the orthogonal basis of spherical harmonics:

$$u(\mathbf{x}, t) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{l} u_m^l(r, t) Y_l^m(\mathbf{s}) \tag{5}$$

where

$$u_m^l(r, t) = \int_S u(r, \mathbf{s}, t) \bar{Y}_l^m(\mathbf{s}) d\mathbf{s}, \quad \frac{du_m^l}{dt}(r, 0) = u_m^l(r, 0) = 0.$$

Substituting expression (5) into (1)-(2) and using the relations (3)-(4), we obtain

$$\frac{\partial^2 u_m^l}{\partial t^2} = \frac{\partial^2 u_m^l}{\partial r^2} + \frac{2}{r} \frac{\partial u_m^l}{\partial r} - \frac{l(l+1)}{r^2} u_m^l, \quad (r, t) \in (1, +\infty) \times (0, T], \tag{6}$$

$$\frac{\partial u_m^l}{\partial t}(r, 0) = u_m^l(r, 0) = 0, \quad r \in (1, +\infty). \tag{7}$$

We will derive a relation on B by a constructive method. First, let us consider the following problem for any fixed $l \geq 0$:

$$\frac{\partial^2 G}{\partial t^2} = \frac{\partial^2 G}{\partial r^2} + \frac{2}{r} \frac{\partial G}{\partial r} - \frac{l(l+1)}{r^2} G, \quad (r, t) \in (R, +\infty) \times (0, T], \tag{8}$$

$$\frac{\partial G}{\partial t}(R, t) = 1, \quad t \in (0, T], \tag{9}$$

$$\frac{\partial G}{\partial t}(r, 0) = G(r, 0) = 0, \quad r \in (R, +\infty). \tag{10}$$

For any fixed $\mu > 0$, suppose $W(r)\sin\mu t$ satisfies equation (8). Then W satisfies

$$\frac{d^2W}{dr^2} + \frac{2}{r}\frac{dW}{dr} + \left(\mu^2 - \frac{l(l+1)}{r^2}\right)W = 0.$$

This is just the spherical Bessel equation and it admits two linearly independent solutions $W_1(\mu r)$ and $W_2(\mu r)$ defined by

$$W_1(r) = \sqrt{\frac{\pi}{2r}}J_{l+1/2}(r), \quad W_2(r) = \sqrt{\frac{\pi}{2r}}Y_{l+1/2}(r)$$

where J and Y are the first-kind and the second-kind Bessel functions.

Let

$$G^*(r,t) = \frac{2}{\pi}\int_0^{+\infty} \frac{\sin\mu t}{\mu^2}\frac{W_1(\mu r)W_2(\mu R) - W_1(\mu R)W_2(\mu r)}{W_1^2(\mu R) + W_2^2(\mu R)}d\mu.$$

It is straightforward to verify that G^* is well-defined and satisfies equation (8). Moreover

$$\begin{aligned}
\frac{\partial G^*}{\partial t}(R,t) &= 0, \quad t \in [0,T], \\
G^*(r,0) &= 0, \quad r \in [1,+\infty), \\
\frac{\partial G^*}{\partial t}(r,0) &= \frac{2}{\pi}\int_0^{+\infty}\frac{1}{\mu}\frac{W_1(\mu r)W_2(\mu R) - W_1(\mu R)W_2(\mu r)}{W_1^2(\mu R) + W_2^2(\mu R)}d\mu \\
&= \frac{2}{\pi}\sqrt{\frac{R}{r}}\int_0^{+\infty}\frac{1}{\mu}\frac{J_{l+1/2}(\mu r)Y_{l+1/2}(\mu R) - J_{l+1/2}(\mu R)Y_{l+1/2}(\mu r)}{J_{l+1/2}^2(\mu R) + Y_{l+1/2}^2(\mu R)}d\mu \\
&= -\left(\frac{R}{r}\right)^{l+1}, \quad r \in (1,+\infty).
\end{aligned}$$

The last equality comes from page 679 in [4].

Let

$$G(r,t) = G^*(r,t) + \left(\frac{R}{r}\right)^{l+1} t.$$

Then G is the solution of problem (8)-(10).

By Duhamel's principle, any solution of problem (6)-(7) satisfies

$$\begin{aligned}
u_m^l(r,t) &= \int_0^t \frac{\partial u_m^l(R,\tau)}{\partial \tau}\frac{\partial G(r,t-\tau)}{\partial t}d\tau \\
&= \left(\frac{R}{r}\right)^{l+1} u_m^l(R,t) + \int_0^t \frac{\partial^2 u_m^l(R,\tau)}{\partial \tau^2}G^*(r,t-\tau)d\tau;
\end{aligned}$$

thus

$$\frac{\partial u_m^l}{\partial r}(R,t) = -\frac{l+1}{R}u_m^l(R,t) + \int_0^t \frac{\partial^2 u_m^l(R,\tau)}{\partial \tau^2}\frac{\partial G^*(R,t-\tau)}{\partial r}d\tau.$$

Since

$$\frac{\partial G^*}{\partial r}(R,t) = -\frac{4}{\pi^2 R}\int_0^{+\infty}\frac{\sin\mu t}{\mu^2}\frac{1}{J_{l+1/2}^2(\mu R) + Y_{l+1/2}^2(\mu R)}d\mu,$$

and if we set

$$HZ_{l+1/2}(t) = \frac{4}{\pi^2}\int_0^{+\infty} \frac{\sin \mu t}{\mu^2} \frac{1}{J^2_{l+1/2}(\mu) + Y^2_{l+1/2}(\mu)} d\mu,$$

then

$$\frac{\partial G^*}{\partial r}(R,t) = -HZ_{l+1/2}\left(\frac{t}{R}\right)$$

and

$$\frac{\partial u^l_m}{\partial r}(R,t) = -\frac{l+1}{R}u^l_m(R,t) - \int_0^t \frac{\partial^2 u^l_m}{\partial \tau^2}(R,\tau) HZ_{l+1/2}\left(\frac{t-\tau}{R}\right) d\tau. \tag{11}$$

The functions $HZ_{l+1/2}(t)$ can be rewritten as

$$HZ_{l+1/2}(t) = \frac{4}{\pi^2}\int_0^{+\infty} \frac{\sin \mu t}{\mu^2} \left(\frac{1}{J^2_{l+1/2}(\mu) + Y^2_{l+1/2}(\mu)} - \frac{\pi}{2}\mu\right) d\mu + 1.$$

Thus $HZ_{l+1/2}(t)$ has continuous derivatives $HZ'_{l+1/2}(t)$ and $HZ''_{l+1/2}(t)$ and

$$HZ'_{l+1/2}(t) = \frac{4}{\pi^2}\int_0^{+\infty} \frac{\cos \mu t}{\mu} \left\{\frac{1}{J^2_{l+1/2}(\mu) + Y^2_{l+1/2}(\mu)} - \frac{\pi}{2}\mu\right\} d\mu, \tag{12}$$

$$HZ''_{l+1/2}(t) = -\frac{4}{\pi^2}\int_0^{+\infty} \sin \mu t \left\{\frac{1}{J^2_{l+1/2}(\mu) + Y^2_{l+1/2}(\mu)} - \frac{\pi}{2}\mu\right\} d\mu, \tag{13}$$

$$HZ_{l+1/2}(0^+) = 1, \tag{14}$$

$$HZ'_{l+1/2}(0^+) = -l. \tag{15}$$

The last equation is still a conjecture, but it can be verified numerically. Some of the plots of the functions $HZ_{l+1/2}$, $HZ'_{l+1/2}$ and $HZ''_{l+1/2}$ are shown in Figure 2.

Integrating by parts on the RHS of (11), we obtain

$$\frac{\partial u^l_m}{\partial r}(R,t) = -\frac{l+1}{R}u^l_m(R,t) - \int_0^t \frac{\partial^2 u^l_m}{\partial \tau^2}(R,\tau) HZ_{l+1/2}\left(\frac{t-\tau}{R}\right) d\tau, \tag{16}$$

$$\frac{\partial u^l_m}{\partial r}(R,t) = -\frac{l+1}{R}u^l_m(R,t) - \frac{\partial u^l_m}{\partial t}(R,t) - \frac{1}{R}\int_0^t \frac{\partial u^l_m}{\partial \tau}(R,\tau) HZ'_{l+1/2}\left(\frac{t-\tau}{R}\right) d\tau, \tag{17}$$

$$\frac{\partial u^l_m}{\partial r}(R,t) = -\frac{1}{R}u^l_m(R,t) - \frac{\partial u^l_m}{\partial t}(R,t) - \frac{1}{R^2}\int_0^t u^l_m(R,\tau) HZ''_{l+1/2}\left(\frac{t-\tau}{R}\right) d\tau. \tag{18}$$

The formulae (16)-(18) are exact for each Fourier component of the solution. Denote (16)-(18) as

$$\frac{\partial u^l_m}{\partial r}(R,t) = \mathcal{K}^i_l(u^l_m(R,t)), \quad i = 2,1,0.$$

They will be called the i-th kind artificial boundary condition for each Fourier component. By combing all these artificial boundary conditions we derive three equivalent exact artificial boundary conditions on the artificial boundary, namely,

$$\frac{\partial u}{\partial r}(R,\mathbf{s},t) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l} \mathcal{K}^i_l(u^l_m(R,t))Y^m_l(\mathbf{s}), \quad i = 0,1,2.$$

After one of these boundary conditions is imposed on B, the problem defined in Ω can be solved by a suitable numerical scheme.

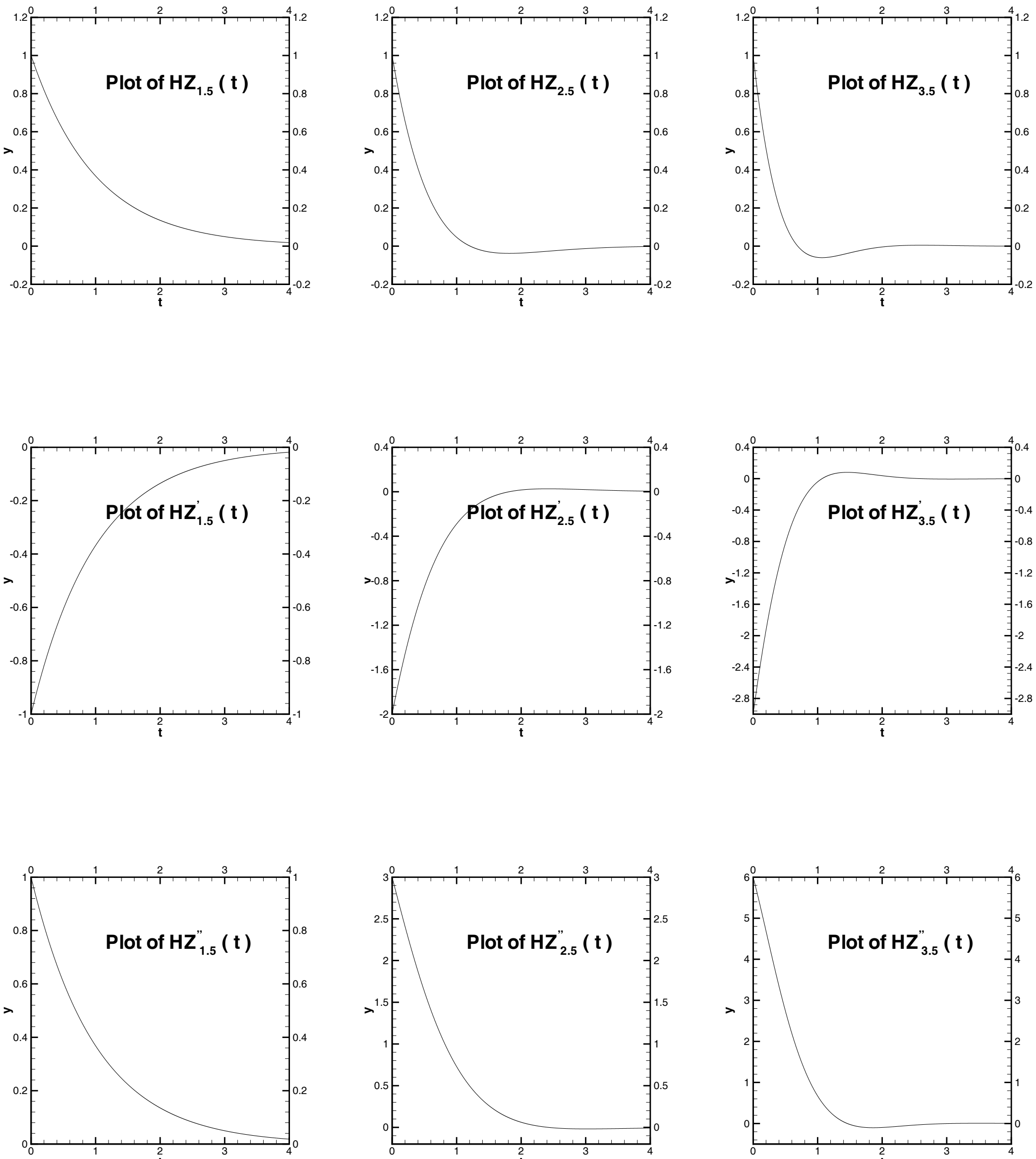

Figure 2: Plots of functions $HZ_{l+1/2}$, $HZ'_{l+1/2}$ and $HZ''_{l+1/2}$ for $l = 1, 2, 3$.

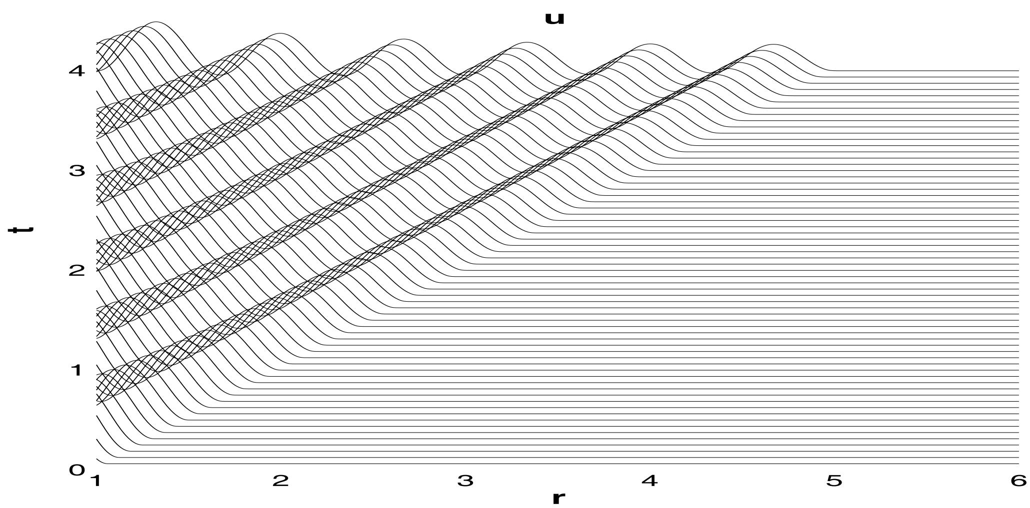

Figure 3: "Accurate solution" on the time interval $[0, 4]$

Remark. Since $HZ_{1/2}(t) \equiv 1$, the artificial boundary condition for $l = 0$ behaves quite simply like

$$\frac{\partial u_0^0}{\partial r}(R,t) = -\frac{1}{R}u_0^0(R,t) - \frac{\partial u_0^0}{\partial t}(R,t).$$

4. Numerical test

In this section, we present a simple example with only one Fourier component to test our artificial boundary conditions:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{l(l+1)}{r^2}u, \quad r > 1, 0 < t \leq 4, \\
&u(1,t) = 1 - \cos(3\pi t), \quad 0 < t \leq 4, \\
&\frac{\partial u}{\partial t}(r,0) = u(r,0) = 0, \quad r > 1.
\end{aligned}$$

Since it is hard to obtain the exact solution of this problem, we seek a highly-accurate numerical solution to play its role, namely, we take a reference solution to show how well our numerical solution approximates the "exact one". This solution is obtained by a standard second-order, centered difference discretization with $\Delta r = 1/2^{12}$ and $\Delta t/\Delta r = 0.5$. Figure 3 shows this solution on the domain $\{(r,t)|1 \leq r \leq 6, 0 \leq t \leq 4\}$ when $l = 1$.

Introducing an artificial boundary $\{(r,t)|r = 2, 0 < t \leq 4\}$, with any of the artificial boundary conditions imposed on this boundary, we get a problem only defined in the bounded domain $(1,2) \times (0,4]$. Obviously, its solution is the same as that of the original problem on this domain.

A linear finite element scheme with lumping technique is employed with $\Delta t/\Delta r = 0.5$ in the numerical implementation. Second-order centered differences are used to approximate the time derivatives. The integral term is approximated by some numerical quadrature scheme.

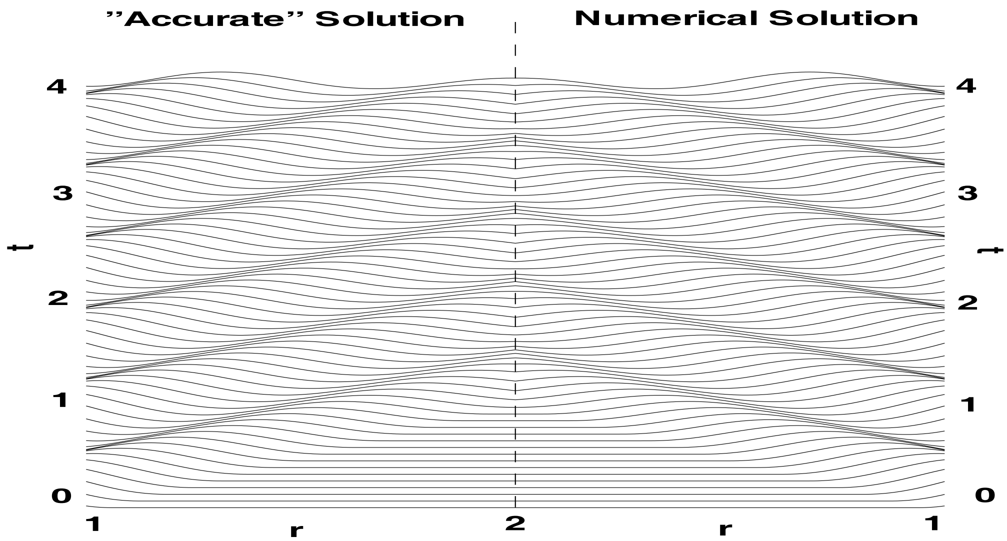

Figure 4: Comparative plot between "accurate solution" and numerical solution. The numerical solution is obtained with $\Delta r = 1/2^8$ when the second-kind ABC is employed.

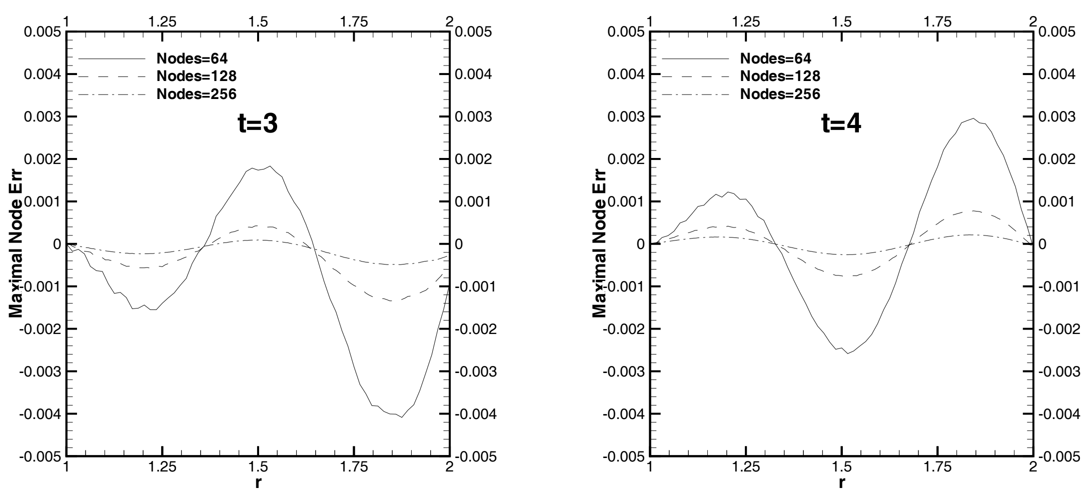

Figure 5: Maximal error plots on some time points. The second-kind ABC is employed.

Figure 4 compares the numerical solution with the "accurate solution". In this figure and in the following, ABC stands for *Artificial Boundary Condition.* Figure 5 shows the maximal error plots on some time points with different meshes. Both results are obtained with the second-kind artificial boundary condition.

Table 1, Table 2 and Table 3 show the maximal errors when zero-th, first and second-kind ABC are employed in the computation. It can be seen that for the first and second-kind artificial boundary conditions, the numerical errors degenerate with an almost-optimal convergence rate of 4 when the mesh is refined by a factor 2. As regards this particular test, it can also be observed that the first and second-kind boundary conditions are superior to the third-kind ones.

Table 1: Maximal error and the convergence rate when the zero-th kind ABC is employed

Max. Err. and Conv. Rate	Node=32	Node=64	Node=128	Node=256
t=2	1.23E-2	2.96E-3	7.44E-4	1.86E-4
	···	4.15	3.97	3.99
t=3	1.27E-2	2.94E-3	7.58E-4	1.86E-4
	···	4.34	3.88	4.07
t=4	1.20E-2	2.90E-3	7.24E-4	1.79E-4
	···	4.14	4.01	4.03

Table 2: Maximal error and the convergence rate when the first-kind ABC is employed

Max. Err. and Conv. Rate	Node=32	Node=64	Node=128	Node=256
t=2	1.19E-2	2.75E-3	7.30E-4	2.20E-4
	···	4.33	3.77	3.30
t=3	1.36E-2	3.33E-3	9.46E-4	2.76E-4
	···	4.08	3.52	3.41
t=4	1.24E-2	3.04E-3	7.81E-4	2.05E-4
	···	4.07	3.89	3.79

Table 3: Maximal error and the convergence rate when the second-kind ABC is employed

Max. Err. and Conv. Rate	Node=32	Node=64	Node=128	Node=256
t=2	1.26E-2	3.46E-3	1.24E-3	6.09E-4
	···	3.64	2.78	2.03
t=3	1.46E-2	4.08E-3	1.35E-3	4.89E-4
	···	3.58	3.01	2.77
t=4	1.18E-2	2.95E-3	7.77E-4	2.54E-4
	···	4.00	3.80	3.05

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