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THE UNCONDITIONAL CONVERGENT DIFFERENCE METHODS WITH INTRINSIC PARALLELISM FOR QUASILINEAR PARABOLIC SYSTEMS WITH TWO DIMENSIONS*1)

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Dedicated to the 80th birthday of Professor Zhou Yulin

Abstract

In the present work we are going to solve the boundary value problem for the quasilinear parabolic systems of partial differential equations with two space dimensions by the finite difference method with intrinsic parallelism. Some fundamental behaviors of general finite difference schemes with intrinsic parallelism for the mentioned problems are studied. By the method of a priori estimation of the discrete solutions of the nonlinear difference systems, and the interpolation formulas of the various norms of the discrete functions and the fixed–point technique in finite dimensional Euclidean space, the existence of the discrete vector solutions of the nonlinear difference system with intrinsic parallelism are proved. Moreover the convergence of the discrete vector solutions of these difference schemes to the unique generalized solution of the original quasilinear parabolic problem is proved.

Key words: Difference Scheme, Intrinsic Parallelism, Two Dimensional Quasilinear Parabolic System, Existence, Convergence.

1. Introduction

1. In [1]-[4] the finite difference methods with intrinsic parallelism for the multi-dimensional boundary value problems of the semilinear parabolic system are studied, where the difference approximations for the derivatives of second order are taken to be the various linear combinations of the two or four kinds of difference quotients. All of these general finite difference schemes having the intrinsic character of parallelism are proved to be stable and convergent conditionally, where some restriction conditions on time–step must be satisfied. Some special finite difference schemes with intrinsic parallelism for the linear parabolic problems have been discussed in [5] and [6]. These special difference schemes are proved to be stable and convergent unconditionally in discrete norms L^∞ and H^1, and the convergence order is $O(\tau+h)$ though the truncation error at the subdomain boundaries is $O(1)$. For the one-dimensional quasilinear parabolic systems we have also constructed some general difference schemes with intrinsic parallelism and proved that they are unconditional stable and convergent in [7].

2. Difference Schemes with Intrinsic Parallelism

2. Consider the boundary value problems for the two dimensional quasilinear parabolic systems of second order of the form

$$u_t = A(x,y,t,u)(u_{xx}+u_{yy}) + B(x,y,t,u)u_x + C(x,y,t,u)u_y + f(x,y,t,u) \qquad (1)$$

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where $(x,y)\in\Omega=(0,l_1)\times(0,l_2)$, $t\in(0,T]$, and $u(x,y,t)=(u_1(x,y,t),u_2(x,y,t),\cdots,u_m(x,y,t))$ is a m–dimensional vector unknown function $(m\geq 1)$; $A(x,y,t,u)$, $B(x,y,t,u)$ and $C(x,y,t,u)$ are given $m\times m$ matrix functions, and $f(x,t,u)$ is a given m–dimensional vector function and $u_x=\frac{\partial u}{\partial x}$, $u_y=\frac{\partial u}{\partial y}$, $u_{xx}=\frac{\partial^2 u}{\partial x^2}$, $u_{yy}=\frac{\partial^2 u}{\partial y^2}$ and $u_t=\frac{\partial u}{\partial t}$ are the corresponding m–dimensional vector derivatives of the m–dimensional unknown vector function $u(x,y,t)$.

Let us consider in the rectangular domain $Q_T=\bar{\Omega}\times[0,T]$ the boundary value problem for the system (1) with homogeneous boundary conditions

$$u(x,y,t)=0,\quad (x,y)\in\partial\Omega, 0<t\leq T, \tag{2}$$

and the initial condition

$$u(x,y,0)=\varphi(x,y),\quad (x,y)\in\bar{\Omega}. \tag{3}$$

Suppose that the following conditions are satisfied.

(I) $A(x,y,t,u)$, $B(x,y,t,u)$, $C(x,y,t,u)$and $f(x,y,t,u)$ are continuous functions with respect to $(x,y,t)\in Q_T$ and continuously differentiable with respect to $u\in R^m$; and there are constants $A_0>0$, $B_0>0, C_0>0$ and $C>0$ such that $|A(x,y,t,u)|\leq A_0$, $|B(x,y,t,u)|\leq B_0, |C(x,y,t,u)|\leq C_0$, and $|f(x,y,t,u)|\leq|f(x,y,t,0)|+C|u|$.

(II) There is a constant $\sigma_0>0$, such that, for any vector $\xi\in R^m$, and for $(x,y,t,u)\in Q_T\times R^m$,

$$(\xi, A(x,y,t,u)\xi)\geq\sigma_0|\xi|^2.$$

(III) The initial value m–dimensional vector function $\varphi(x,y)\in C^1(\bar{\Omega})$ and $\varphi(x,y)=0$ for $(x,y)\in\partial\Omega$.

3. Divide the domain $Q_T=\{0\leq x\leq l_1, 0\leq y\leq l_2, 0\leq t\leq T\}$ into small grids by the parallel planes $x=x_i$ $(i=0,1,\cdots,I)$, $y=y_j$ $(j=0,1,\cdots,J)$ and $t=t^n$ $(n=0,1,\cdots,N)$ with $x_i=ih_1$, $y_j=jh_2$ and $t^n=n\tau$, where $Ih_1=l_1, Jh_2=l_2$ and $N\tau=T$, I,J and N are integers and h_1,h_2 and τ are the steplengths of grids. Denote $v_\Delta=\{v_{ij}^n|i=0,1,\cdots,I;j=0,1,\cdots,J;n=0,1,\cdots,N\}$ the m–dimensional discrete vector function defined on the discrete rectangular domain $Q_\Delta=\{(x_i,y_j,t^n)|i=0,1,\cdots,I;j=0,1,\cdots,J;n=0,1,\cdots,N\}$ of the grid points.

Let us now construct the general difference schemes with intrinsic parallelism for the boundary value problem (1), (2) and (3):

$$\frac{v_{ij}^{n+1}-v_{ij}^n}{\tau}=A_{ij}^{n+1}\overset{*}{\Delta}v_{ij}^{n+1}+B_{ij}^{n+1}\bar{\delta}_x^1 v_{ij}^{n+1}+C_{ij}^{n+1}\bar{\delta}_y^1 v_{ij}^{n+1}+f_{ij}^{n+1}, \tag{$1)_\Delta$}$$
$$(i=1,2,\cdots,I-1;j=1,2,\cdots,J-1;n=0,1,\cdots,N-1),$$

where

$$\begin{aligned}\overset{*}{\Delta}v_{ij}^{n+1}&=\overset{*}{\delta}{}_x^2 v_{ij}^{n+1}+\overset{*}{\delta}{}_y^2 v_{ij}^{n+1}\\&=\frac{v_{i+1,j}^{n+\lambda_{ij}^n}-2v_{ij}^{n+1}+v_{i-1,j}^{n+\mu_{ij}^n}}{h_1^2}+\frac{v_{i,j+1}^{n+\bar{\lambda}_{ij}^n}-2v_{ij}^{n+1}+v_{i,j-1}^{n+\bar{\mu}_{ij}^n}}{h_2^2},\\A_{ij}^{n+1}&=A(x_i,y_j,t^{n+1},\bar{\delta}^0 v_{ij}^{n+1}),\\B_{ij}^{n+1}&=B(x_i,y_j,t^{n+1},\tilde{\delta}^0 v_{ij}^{n+1}),\\C_{ij}^{n+1}&=C(x_i,y_j,t^{n+1},\hat{\delta}^0 v_{ij}^{n+1}),\\f_{ij}^{n+1}&=f(x_i,y_j,t^{n+1},\breve{\delta}^0 v_{ij}^{n+1}).\end{aligned} \tag{4}$$

In this difference scheme, the expressions $\bar{\delta}^0 v_{ij}^{n+1}$, $\tilde{\delta}^0 v_{ij}^{n+1}$, $\hat{\delta}^0 v_{ij}^{n+1}$, $\breve{\delta}^0 v_{ij}^{n+1}$, and $\bar{\delta}_x^1 v_{ij}^{n+1}$, $\bar{\delta}_y^1 v_{ij}^{n+1}$ can be taken in the following manner. We can take

$$\begin{aligned}\bar{\delta}^0 v_{ij}^{n+1}\ =\ &\lambda_{ij}^n\alpha_{1ij}^n v_{i+1j}^{n+1}+\mu_{ij}^n\alpha_{2ij}^n v_{i-1j}^{n+1}+\bar{\lambda}_{ij}^n\alpha_{3ij}^n v_{ij+1}^{n+1}\\&+\bar{\mu}_{ij}^n\alpha_{4ij}^n v_{ij-1}^{n+1}+\alpha_{5ij}^n v_{ij}^{n+1}+\bar{\alpha}_{1ij}^n v_{i+1j}^n\\&+\bar{\alpha}_{2ij}^n v_{i-1j}^n+\bar{\alpha}_{3ij}^n v_{ij+1}^n+\bar{\alpha}_{4ij}^n v_{ij-1}^n+\bar{\alpha}_{5ij}^n v_{ij}^n\end{aligned} \tag{5}$$

such that the sum of coefficients equals to unit, that is

$$\lambda_{ij}^n\alpha_{1ij}^n+\mu_{ij}^n\alpha_{2ij}^n+\bar{\lambda}_{ij}^n\alpha_{3ij}^n+\bar{\mu}_{ij}^n\alpha_{4ij}^n+\alpha_{5ij}^n+\bar{\alpha}_{1ij}^n+\bar{\alpha}_{2ij}^n+\bar{\alpha}_{3ij}^n+\bar{\alpha}_{4ij}^n+\bar{\alpha}_{5ij}^n=1$$

and the sum of the absolute value of these coefficients is uniformly bounded by any given constant with respect to the indices i,j and n. The coefficients are dependent on the indices i,j and n, this means they are different for different layers and different grid points. This shows that the choice of the coefficients has great degree of freedom.

For the expressions $\bar{\delta}_x^1 v_{ij}^{n+1}$ and $\bar{\delta}_y^1 v_{ij}^{n+1}$, we can take for example as

$$\begin{aligned}\bar{\delta}_x^1 v_{ij}^{n+1} &= d_{1ij}^n\frac{v_{i+1j}^{n+\lambda_{ij}^n}-v_{ij}^{n+1}}{h_1}+d_{2ij}^n\frac{v_{ij}^{n+1}-v_{i-1j}^{n+\mu_{ij}^n}}{h_2}\\ &\quad+d_{3ij}^n\delta_x v_{ij}^n+d_{4ij}^n\delta_x v_{i-1j}^n,\end{aligned}\tag{6}$$

where

$$d_{1ij}^n+d_{2ij}^n+d_{3ij}^n+d_{4ij}^n=1$$

and the sum of absolute values of the coefficients is uniformly bounded by any given constant with respect to the indices i,j and n.

By the similar principal we have the expression for $\tilde{\delta}^0 v_{ij}^{n+1}$, $\hat{\delta}^0 v_{ij}^{n+1}$, $\breve{\delta}^0 v_{ij}^{n+1}$ and $\bar{\delta}_y^1 v_{ij}^{n+1}$ with an analogous behaviors.

The finite difference boundary conditions are of the form

$$v_{0j}^n=v_{Ij}^n=v_{i0}^n=v_{iJ}^n=0, \tag*{$(2)_\Delta$}$$

$$(i=0,1,\cdots,I;j=0,1,\cdots,J;n=0,1,\cdots,N).$$

The finite difference initial condition is of the form

$$v_{ij}^0=\varphi_{ij}, \tag*{$(3)_\Delta$}$$

$$(i=0,1,\cdots,I;j=0,1,\cdots,J),$$

where $\varphi_{ij}=\varphi(x_i,y_j)$ $(i=0,1,\cdots,I;j=0,1,\cdots,J)$.

4. Denote $h^*=\max(h_1,h_2)=h,\quad h_*=\min(h_1,h_2),\quad \Lambda\equiv\tau\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right)$. Introduce the assumption:

(IV) Suppose that h^*/h_* is uniformly bounded as h_1 and h_2 tend to zero.

In the following we shall use the symbols and notations in [8], and the following lemma will be useful (see [8]–[10]).

Lemma 1. For the discrete vector function $v_\Delta=\{v_{ij}^n|i=0,1,\cdots,I;j=0,1,\cdots,J;n=0,1,\cdots,N\}$ satisfying

$$v_{0j}^n=v_{Ij}^n=0,\ \ j=0,1,\cdots,J,$$

$$v_{i0}^n=v_{iJ}^n=0,\ \ i=0,1,\cdots,I,$$

we have

(i)

$$\|v_\Delta^n\|_2^2\le 2l_1^2\|\delta_x v_\Delta^n\|_2^2,\ \ \|v_\Delta^n\|_2^2\le 2l_2^2\|\delta_y v_\Delta^n\|_2^2;$$

(ii)

$$\|\delta_x^2 v_\Delta^n\|_2^2+\|\delta_y^2 v_\Delta^n\|_2^2\le 2\|\Delta v_\Delta^n\|_2^2;$$

(iii) for any $\varepsilon>0$, there are

$$\|v_\Delta^n\|_\infty^2\le\varepsilon\|\delta_x^2 v_\Delta^n\|_2^2+\frac{1}{\varepsilon}\|v_\Delta^n\|_2^2+\|\delta_x v_\Delta^n\|_2^2+\|\delta_y v_\Delta^n\|_2^2;$$

$$\|v_\Delta^n\|_\infty^2\le\varepsilon\|\delta_y^2 v_\Delta^n\|_2^2+\frac{1}{\varepsilon}\|v_\Delta^n\|_2^2+\|\delta_x v_\Delta^n\|_2^2+\|\delta_y v_\Delta^n\|_2^2.$$

3. Apriori Estimate and Existence

5. We are now going to prove the existence of the discrete vector solutions for the finite difference system $(1)_\Delta$–$(3)_\Delta$. Let us now at first turn to the *a priori* estimates of these solutions.

Making the scalar product of the vector $\overset{*}{\Delta}\ v_{ij}^{n+1} h_1 h_2 \tau$ with the vector finite difference equation $(1)_\Delta$ and summing up the resulting products for $i = 1, 2, \cdots, I-1; j = 1, 2, \cdots, J-1$, we have

$$\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\overset{*}{\Delta}\ v_{ij}^{n+1}, v_{ij}^{n+1} - v_{ij}^{n})h_1h_2 = \tau\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\overset{*}{\Delta}\ v_{ij}^{n+1}, A_{ij}^{n+1}\ \overset{*}{\Delta}\ v_{ij}^{n+1})h_1h_2$$
$$+\tau\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\overset{*}{\Delta}\ v_{ij}^{n+1}, B_{ij}^{n+1}\bar{\delta}_x^1 v_{ij}^{n+1} + C_{ij}^{n+1}\bar{\delta}_y^1 v_{ij}^{n+1} + f_{ij}^{n+1})h_1h_2. \tag{7}$$

For the left part of the above equality, we have

$$\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\overset{*}{\delta}\,{}_x^2 v_{ij}^{n+1}, v_{ij}^{n+1} - v_{ij}^{n})h_1h_2$$
$$= -\frac{1}{2}\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\left|\delta_x v_{ij}^{n+1}\right|^2 h_1h_2 + \frac{1}{2}\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\left|\delta_x v_{ij}^{n}\right|^2 h_1h_2$$
$$-\frac{1}{2h_1}\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\left[\left|v_{i+1j}^{n+1} - v_{i+1j}^{n}\right|^2 + \left|v_{ij}^{n+1} - v_{ij}^{n}\right|^2\right]h_2$$
$$-\frac{1}{h_1}\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}(1 - \mu_{i+1j}^{n} - \lambda_{ij}^{n})\left(v_{i+1j}^{n+1} - v_{i+1j}^{n}, v_{ij}^{n+1} - v_{ij}^{n}\right)h_2$$
$$= -\frac{1}{2}\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\left|\delta_x v_{ij}^{n+1}\right|^2 h_1h_2 + \frac{1}{2}\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\left|\delta_x v_{ij}^{n}\right|^2 h_1h_2$$
$$-\frac{1}{2h_1}\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\left(1 - \left|\tau_{i+\frac{1}{2},j}^{n}\right|\right)\left[\left|v_{i+1j}^{n+1} - v_{i+1j}^{n}\right|^2 + \left|v_{ij}^{n+1} - v_{ij}^{n}\right|^2\right]h_2$$
$$-\frac{1}{2h_1}\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\left|\tau_{i+\frac{1}{2},j}^{n}\right|\left|v_{i+1j}^{n+1} - v_{i+1j}^{n} + \left(sgn\tau_{i+\frac{1}{2},j}^{n}\right)\left(v_{ij}^{n+1} - v_{ij}^{n}\right)\right|^2 h_2,$$

where $\tau_{i+\frac{1}{2},j}^{n} = 1 - \mu_{i+1j}^{n} - \lambda_{ij}^{n}$; $sgn\tau_{i+\frac{1}{2},j}^{n} = 1$ if $\tau_{i+\frac{1}{2},j}^{n} \geq 0$, and $sgn\tau_{i+\frac{1}{2},j}^{n} = -1$ if $\tau_{i+\frac{1}{2},j}^{n} < 0$.

Then the equality (7) becomes

$$\left(\|\delta_x v_\Delta^{n+1}\|_2^2 + \|\delta_y v_\Delta^{n+1}\|_2^2\right) - \left(\|\delta_x v_\Delta^{n}\|_2^2 + \|\delta_y v_\Delta^{n}\|_2^2\right)$$
$$+\frac{1}{h_1^2}\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\left(1 - \left|\tau_{i+\frac{1}{2},j}^{n}\right|\right)\left[\left|v_{i+1j}^{n+1} - v_{i+1j}^{n}\right|^2 + \left|v_{ij}^{n+1} - v_{ij}^{n}\right|^2\right]h_1h_2$$
$$+\frac{1}{h_2^2}\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\left(1 - \left|\bar{\tau}_{i,j+\frac{1}{2}}^{n}\right|\right)\left[\left|v_{ij+1}^{n+1} - v_{ij+1}^{n}\right|^2 + \left|v_{ij}^{n+1} - v_{ij}^{n}\right|^2\right]h_1h_2$$
$$+\frac{1}{h_1^2}\sum_{i=0}^{I-1}\sum_{j=1}^{J-1}\left|\tau_{i+\frac{1}{2},j}^{n}\right|\left|v_{i+1j}^{n+1} - v_{i+1j}^{n} + \left(sgn\tau_{i+\frac{1}{2},j}^{n}\right)\left(v_{ij}^{n+1} - v_{ij}^{n}\right)\right|^2 h_2$$

$$+\frac{1}{h_2^2}\sum_{i=1}^{I-1}\sum_{j=0}^{J-1}\left|\bar{\tau}^n_{i,j+\frac{1}{2}}\right|\left|v^{n+1}_{ij+1}-v^n_{ij+1}+\left(sgn\bar{\tau}^n_{i,j+\frac{1}{2}}\right)\left(v^{n+1}_{ij}-v^n_{ij}\right)\right|^2 h_1$$

$$+2\tau\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\overset{*}{\Delta}v^{n+1}_{ij},A^{n+1}_{ij}\overset{*}{\Delta}v^{n+1}_{ij})h_1h_2$$

$$=-2\tau\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\overset{*}{\Delta}v^{n+1}_{ij},B^{n+1}_{ij}\bar{\delta}^1_x v^{n+1}_{ij}+C^{n+1}_{ij}\bar{\delta}^1_y v^{n+1}_{ij}+f^{n+1}_{ij})h_1h_2. \tag{8}$$

Then, from the equality (8) we can get

$$\left(\|\delta_x v^{n+1}_\Delta\|_2^2+\|\delta_y v^{n+1}_\Delta\|_2^2\right)-\left(\|\delta_x v^n_\Delta\|_2^2+\|\delta_y v^n_\Delta\|_2^2\right)$$

$$+2\tau\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\overset{*}{\Delta}v^{n+1}_{ij},A^{n+1}_{ij}\overset{*}{\Delta}v^{n+1}_{ij})h_1h_2$$

$$\le 2\tau\left|\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\overset{*}{\Delta}v^{n+1}_{ij},B^{n+1}_{ij}\bar{\delta}^1_x v^{n+1}_{ij}+C^{n+1}_{ij}\bar{\delta}^1_y v^{n+1}_{ij}+f^{n+1}_{ij})h_1h_2\right|. \tag{9}$$

Appling the Cauchy inequality to the right hand of the above inequality we obtain

$$\left|\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\overset{*}{\Delta}v^{n+1}_{ij},B^{n+1}_{ij}\bar{\delta}^1_x v^{n+1}_{ij}+C^{n+1}_{ij}\bar{\delta}^1_y v^{n+1}_{ij}+f^{n+1}_{ij}\right)h_1h_2\right|$$

$$\le\frac{1}{2}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\overset{*}{\Delta}v^{n+1}_{ij},A^n_{ij}\overset{*}{\Delta}v^{n+1}_{ij}\right)h_1h_2$$

$$+\frac{1}{2\sigma_0}\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|B^{n+1}_{ij}\bar{\delta}^1_x v^{n+1}_{ij}+C^{n+1}_{ij}\bar{\delta}^1_y v^{n+1}_{ij}+f^{n+1}_{ij}\right|^2 h_1h_2.$$

From $(1)_\Delta$ we have, for any $\varepsilon>0$

$$\varepsilon\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|\frac{v^{n+1}_{ij}-v^n_{ij}}{\tau}\right|^2 h_1h_2$$

$$\le 2\varepsilon\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|A^{n+1}_{ij}\overset{*}{\Delta}v^{n+1}_{ij}\right|^2 h_1h_2$$

$$+2\varepsilon\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|B^{n+1}_{ij}\bar{\delta}^1_x v^{n+1}_{ij}+C^{n+1}_{ij}\bar{\delta}^1_y v^{n+1}_{ij}+f^{n+1}_{ij}\right|^2 h_1h_2.$$

Further we have

$$\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|A^{n+1}_{ij}\overset{*}{\Delta}v^{n+1}_{ij}\right|^2 h_1h_2\le\Lambda_0\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left(\overset{*}{\Delta}v^{n+1}_{ij},A^{n+1}_{ij}\overset{*}{\Delta}v^{n+1}_{ij}\right)h_1h_2,$$

where the symbols $\rho(A)$, $\sigma(A)$ and Λ_0 are defined by

$$\rho(A)=\sup_{\xi\in R^m}\frac{|A\xi|}{|\xi|},\quad \sigma(A)=\inf_{\xi\in R^m}\frac{(\xi,A\xi)}{|\xi|^2},\quad \Lambda_0=\sup_{(x,y,t)\in Q_T,u\in R^m}\frac{\rho^2(A)}{\sigma(A)}.$$

Substituting these estimates into the right part of the inequality (9), we have

$$\left(\|\delta_x v^{n+1}_\Delta\|_2^2+\|\delta_y v^{n+1}_\Delta\|_2^2\right)-\left(\|\delta_x v^n_\Delta\|_2^2+\|\delta_y v^n_\Delta\|_2^2\right)$$

$$+2\tau\left(1-\varepsilon\Lambda_0\right)\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}(\overset{*}{\Delta}v_{ij}^{n+1},A_{ij}^{n+1}\overset{*}{\Delta}v_{ij}^{n+1})h_1h_2+\varepsilon\tau\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|\frac{v_{ij}^{n+1}-v_{ij}^n}{\tau}\right|^2h_1h_2$$

$$\leq 2\tau\left(\varepsilon+1\right)\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|B_{ij}^{n+1}\bar{\delta}_x^1v_{ij}^{n+1}+C_{ij}^{n+1}\bar{\delta}_y^1v_{ij}^{n+1}+f_{ij}^{n+1}\right|^2h_1h_2. \tag{10}$$

6. By taking $\varepsilon\Lambda_0\leq\frac{1}{2}$, we get

$$\left(\|\delta_xv_\Delta^{n+1}\|_2^2+\|\delta_yv_\Delta^{n+1}\|_2^2\right)-\left(\|\delta_xv_\Delta^n\|_2^2+\|\delta_yv_\Delta^n\|_2^2\right)$$
$$+\tau\sigma_0\left\|\overset{*}{\Delta}v_{h_1h_2}^{n+1}\right\|_2^2+\varepsilon\tau\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|\frac{v_{ij}^{n+1}-v_{ij}^n}{\tau}\right|^2h_1h_2$$
$$\leq C\tau\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|B_{ij}^{n+1}\bar{\delta}_x^1v_{ij}^{n+1}+C_{ij}^{n+1}\bar{\delta}_y^1v_{ij}^{n+1}+f_{ij}^{n+1}\right|^2h_1h_2, \tag{11}$$

where C depends only on Λ_0 and σ_0.

From the condition **(I)** and interpolation inequality (see Lemma 1), it follows that

$$\sum_{i=1}^{I-1}\sum_{j=1}^{J-1}\left|B_{ij}^{n+1}\bar{\delta}_x^1v_{ij}^{n+1}+C_{ij}^{n+1}\bar{\delta}_y^1v_{ij}^{n+1}+f_{ij}^{n+1}\right|^2h_1h_2$$
$$\leq C\left\{1+\|\delta_xv_\Delta^n\|_2^2+\|\delta_yv_\Delta^n\|_2^2+\left\|\delta_xv_\Delta^{n+1}\right\|_2^2+\left\|\delta_yv_\Delta^{n+1}\right\|_2^2+\tau\Lambda\left\|\frac{v_\Delta^{n+1}-v_\Delta^n}{\tau}\right\|_2^2\right\}.$$

Then, by combining these inequalities above, we obtain

$$\left(\|\delta_xv_\Delta^{n+1}\|_2^2+\|\delta_yv_\Delta^{n+1}\|_2^2\right)-\left(\|\delta_xv_\Delta^n\|_2^2+\|\delta_yv_\Delta^n\|_2^2\right)$$
$$+\tau\sigma_0\left\|\overset{*}{\Delta}v_\Delta^{n+1}\right\|_2^2+\varepsilon\tau\left\|\frac{v_\Delta^{n+1}-v_\Delta^n}{\tau}\right\|_2^2$$
$$\leq\tau C\left(1+\|\delta_xv_\Delta^{n+1}\|_2^2+\|\delta_yv_\Delta^{n+1}\|_2^2+\|\delta_xv_\Delta^n\|_2^2+\|\delta_yv_\Delta^n\|_2^2+\tau\Lambda\left\|\frac{v_\Delta^{n+1}-v_\Delta^n}{\tau}\right\|_2^2\right),$$

which gives, by letting $C\tau\Lambda\leq\frac{\varepsilon}{2}$,

$$\left(\|\delta_xv_\Delta^{n+1}\|_2^2+\|\delta_yv_\Delta^{n+1}\|_2^2\right)-\left(\|\delta_xv_\Delta^n\|_2^2+\|\delta_yv_\Delta^n\|_2^2\right)$$
$$+\tau\sigma_0\left\|\overset{*}{\Delta}v_\Delta^{n+1}\right\|_2^2+\frac{\varepsilon\tau}{2}\left\|\frac{v_\Delta^{n+1}-v_\Delta^n}{\tau}\right\|_2^2$$
$$\leq\tau C\left(1+\|\delta_xv_\Delta^{n+1}\|_2^2+\|\delta_yv_\Delta^{n+1}\|_2^2+\|\delta_xv_\Delta^n\|_2^2+\|\delta_yv_\Delta^n\|_2^2\right).$$

The above recurrence inequality implies the estimates that

$$\max_{n=0,1,\cdots,N}\left(\|\delta_xv_\Delta^n\|_2^2+\|\delta_yv_\Delta^n\|_2^2\right)$$
$$+\sum_{n=0}^{N-1}\left\|\overset{*}{\Delta}v_\Delta^{n+1}\right\|_2^2\tau+\sum_{n=0}^{N-1}\left\|\frac{v_\Delta^{n+1}-v_\Delta^n}{\tau}\right\|_2^2\tau\leq K. \tag{12}$$

Then we also have the estimates

$$\max_{n=0,1,\cdots,N}\|v_\Delta^n\|_2^2\leq K. \tag{13}$$

Obviously there holds

$$\overset{*}{\Delta}v_{ij}^{n+1}=\Delta v_{ij}^{n+1}-(1-\lambda_{ij}^n)\frac{\tau}{h_1^2}\cdot\frac{v_{i+1j}^{n+1}-v_{i+1j}^n}{\tau}-(1-\mu_{ij}^n)\frac{\tau}{h_1^2}\cdot\frac{v_{i-1j}^{n+1}-v_{i-1j}^n}{\tau}$$

$$-(1-\bar{\lambda}_{ij}^n)\frac{\tau}{h_2^2}\cdot\frac{v_{ij+1}^{n+1}-v_{ij+1}^n}{\tau}-(1-\bar{\mu}_{ij}^n)\frac{\tau}{h_2^2}\cdot\frac{v_{ij-1}^{n+1}-v_{ij-1}^n}{\tau},$$

where $\Delta v_{ij}^{n+1}=\delta_x^2 v_{ij}^{n+1}+\delta_y^2 v_{ij}^{n+1}$. Then it follows that, by using the estimates (12),

$$\left(\sum_{n=0}^{N-1}\left\|\Delta v_\Delta^{n+1}\right\|_2^2\tau\right)^{\frac{1}{2}}\le K. \tag{14}$$

7. By means of the fixed point theorem of continuous mapping in a finite dimensional space and the estimates (12)–(14), we obtain the following existence theorem for the general finite differnce scheme $(1)_\Delta$–$(3)_\Delta$ with intrinsic parallelism corresponding to the boundary value problem (2) and (3) for the quasilinear parabolic system (1).

Theorem 1. *Suppose that the conditions* **(I)**–**(IV)** *are fulfilled, and τ is small such that $\tau\Lambda\le\tau_0$, where τ_0 is a positive constant depending only on given data. Then the general finite difference scheme $(1)_\Delta$–$(3)_\Delta$ with intrinsic parallelism corresponding to the original problem (1), (2) and (3) has at least one discrete solution v_Δ.*

4. Convergence

8. In this section we are going to establish unconditional convergence of the solutions for finite difference scheme $(1)_\Delta$–$(3)_\Delta$ to the unique solution of (1)–(3) on the basis of the obtained estimates.

Define the piecewise constant functions

$$v_{h_1h_2}^\tau(x,y,t)=v_{ij}^n,\quad v_{th_1h_2}^\tau(x,y,t)=\frac{v_{ij}^{n+1}-v_{ij}^n}{\tau},$$
$$v_{xh_1h_2}^\tau(x,y,t)=\delta_x v_{ij}^n,\quad v_{yh_1h_2}^\tau(x,y,t)=\delta_y v_{ij}^n,$$

for $(x,y,t)\in Q_{ij}^n$ $(i=0,1,\cdots,I-1;j=0,1,\cdots,J-1;n=0,1,\cdots,N-1)$. And define

$$\Delta' v_{h_1h_2}^\tau(x,y,t)=\overset{*}{\Delta}v_{ij}^{n+1},\quad (x,y,t)\in Q_{ij}^n,$$
$$(i=1,2,\cdots,I-1;j=1,2,\cdots,J-1;n=0,1,\cdots,N-1)$$
$$\Delta' v_{h_1h_2}^\tau(x,y,t)=\overset{*}{\Delta}v_{1j}^{n+1},\quad (x,y,t)\in Q_{0j}^n,\ (j=1,2,\cdots,J-1;n=0,1,\cdots,N-1),$$
$$\Delta' v_{h_1h_2}^\tau(x,y,t)=\overset{*}{\Delta}v_{i1}^{n+1},\quad (x,y,t)\in Q_{i0}^n,\ \ (i=1,2,\cdots,I-1;n=0,1,\cdots,N-1).$$

Denote

$$P_{h_1h_2}^\tau=\left(\sum_{n=0}^{N-1}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\left((1-\lambda_{ij}^n)^2+(1-\mu_{ij}^n)^2\right)h_1h_2\tau\right)^{\frac{1}{2}}.$$

We introduce the following assumptions.

(V) $P_{h_1h_2}^\tau\to 0$ as $h_1\to 0$, $h_2\to 0$ and $\tau\to 0$.

(VI) The problem (1)–(3) has a unique solution in $L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H_0^1(\Omega))$.

Lemma 2. *Assume that* **(I)**–**(V)** *hold. Then there is a function $u(x,y,t)\in L^2(0,T;H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$ satisfying $\Delta u(x,y,t)\in L^2(Q_T)$, and as $h_1\to 0,h_2\to 0,\tau\to 0$ (for some subsequence), there hold*

(i) $v_{h_1h_2}^\tau(x,y,t),v_{xh_1h_2}^\tau(x,y,t),v_{yh_1h_2}^\tau(x,y,t)$ and $v_{th_1h_2}^\tau(x,y,t)$ converge weakly in $L^2(Q_T)$ to $u(x,y,t),u_x(x,y,t),u_y(x,y,t)$ and $u_t(x,y,t)$ respectively;

(ii) $v_{h_1h_2}^\tau(x,y,t)\to u(x,y,t)$ strongly in $L^2(Q_T)$ and a.e. in Q_T;

(iii) $\Delta' v_{h_1h_2}^\tau(x,y,t)\rightharpoonup\Delta u(x,y,t)$ weakly in $L^2(Q_T)$.

Remark. In fact, the function $u(x,y,t)$ obtained in Lemma 2 is in $L^\infty(0,T;\ H_0^1(\Omega))$ (see [9]). And $u(x,y,t)\in L^2(0,T;H^2(\Omega))$.

Proof of Lemma 2 From the estimates (12)–(13) it follows that (i) is true. To prove (ii) we construct the tri–linear function

$$\begin{aligned} v^{t\tau}_{h_1h_2}(x,y,t) =&\frac{1}{h_1h_2\tau}\left\{(x-x_i)(y-y_j)(t-t^n)v^{n+1}_{i+1j+1} + (x_{i+1}-x)(y-y_j)(t-t^n)v^{n+1}_{ij+1}\right.\\ &+(x-x_i)(y_{j+1}-y)(t-t^n)v^{n+1}_{i+1j} + (x_{i+1}-x)(y_{j+1}-y)(t-t^n)v^{n+1}_{ij}\\ &+(x-x_i)(y-y_j)(t^{n+1}-t)v^{n}_{i+1j+1} + (x_{i+1}-x)(y-y_j)(t^{n+1}-t)v^{n}_{ij+1}\\ &\left.+(x-x_i)(y_{j+1}-y)(t^{n+1}-t)v^{n}_{i+1j} + (x_{i+1}-x)(y_{j+1}-y)(t^{n+1}-t)v^{n}_{ij}\right\}, \end{aligned}$$

for $(x,y,t)\in \bar{Q}^n_{ij}$, $(i=0,1,\cdots,I-1; j=0,1,\cdots,J-1; n=0,1,\cdots,N-1)$. The direct calculation shows that $v^{t\tau}_{h_1h_2}(x,y,t)$ is uniformly bounded in $L^2(0,T;H^1_0(\Omega))\cap H^1(0,T;L^2(\Omega))$, and then $v^{t\tau}_{h_1h_2}(x,y,t)$ is precompact in $L^2(Q_T)$. Note that

$$\begin{aligned} \|v^{t\tau}_{h_1h_2}-v^{\tau}_{h_1h_2}\|_{L^2(Q_T)} &\le C\left(\tau\|v^{\tau}_{th_1h_2}\|_{L^2(Q_T)} + h_1\|v^{\tau}_{xh_1h_2}\|_{L^2(Q_T)} + h_2\|v^{\tau}_{yh_1h_2}\|_{L^2(Q_T)}\right)\\ &\le C'(\tau+h_1+h_2). \end{aligned}$$

It follows that (ii) holds.

It remains to prove (iii). Let $\Phi(x,y,t)\in C^\infty(Q_T)$ and $\Phi(x,y,t)=0$ near $\partial\Omega\times[0,T]$. Denote $\Phi^n_{ij}=\Phi(x_i,y_j,t^n)$. Define the piecewise constant functions $\Phi^\tau_{h_1h_2}(x,y,t)=\Phi^n_{ij}$, $\Delta'\Phi^\tau_{h_1h_2}(x,y,t) = \delta^2_x\Phi^n_{ij}+\delta^2_y\Phi^n_{ij}$, for $(x,y,t)\in Q^n_{ij}$. For small h_1 and h_2 there holds

$$\begin{aligned} &\int_{Q_T}\Delta' v^\tau_{h_1h_2}(x,y,t)\Phi^\tau_{h_1h_2}(x,y,t)dxdydt\\ =&\sum_{n=0}^{N-1}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\overset{*}{\Delta}\, v^{n+1}_{ij}\Phi^n_{ij}h_1h_2\tau = \sum_{n=0}^{N-1}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Delta v^{n+1}_{ij}\Phi^n_{ij}h_1h_2\tau\\ &-\sum_{n=0}^{N-1}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\frac{(1-\lambda^n_{ij})(v^{n+1}_{i+1j}-v^n_{i+1j})+(1-\mu^n_{ij})(v^{n+1}_{i-1j}-v^n_{i-1j})}{h_1^2}\Phi^n_{ij}h_1h_2\tau\\ &-\sum_{n=0}^{N-1}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\frac{(1-\bar\lambda^n_{ij})(v^{n+1}_{ij+1}-v^n_{ij+1})+(1-\bar\mu^n_{ij})(v^{n+1}_{ij-1}-v^n_{ij-1})}{h_2^2}\Phi^n_{ij}h_1h_2\tau\\ \equiv& I_1+I_2+I_3. \end{aligned}$$

It is easy to see that

$$I_1\to\int\int_{Q_T}u(x,y,t)\Delta\Phi(x,y,t)dxdydt,\ \text{ as }\ h_1\to0,h_2\to0,\tau\to0.$$

There holds

$$|I_2|\le 2\Lambda\max_{0\le n\le N-1}\|\Phi^{n+1}_\Delta\|_\infty\left(\sum_{n=0}^{N-1}\left\|\frac{v^{n+1}_\Delta-v^n_\Delta}{\tau}\right\|^2_2\tau\right)^{\frac12}P^\tau_{h_1h_2},$$

which gives

$$I_2\to0,\ \text{ as }\ h_1\to0,h_2\to0,\tau\to0.$$

Similarly we have

$$I_3\to0,\ \text{ as }\ h_1\to0,h_2\to0,\tau\to0.$$

Lemma 2 is proved.

9. Define the piecewise constant functions, for $(x,y,t)\in Q^n_{ij}$ $(i=0,1,\cdots,I-1; j=0,1,\cdots,J-1; n=0,1,\cdots,N-1)$,

$$\bar v^\tau_{h_1h_2}(x,y,t)=\bar\delta^0v^{n+1}_{ij},\ \tilde v^\tau_{h_1h_2}(x,y,t)=\tilde\delta^0v^{n+1}_{ij},\ \hat v^\tau_{h_1h_2}(x,y,t)=\hat\delta^0v^{n+1}_{ij},$$

$$\breve{v}^{\tau}_{h_1h_2}(x,y,t)=\breve{\delta}^0 v^{n+1}_{ij},\ \bar{v}^{\tau}_{xh_1h_2}(x,y,t)=\bar{\delta}^1_x v^{n+1}_{ij},\ \bar{v}^{\tau}_{yh_1h_2}(x,y,t)=\bar{\delta}^1_y v^{n+1}_{ij},$$

$$A^{\tau}_{h_1h_2}(x,y,t)=A^{n+1}_{ij},\ B^{\tau}_{h_1h_2}(x,y,t)=B^{n+1}_{ij},$$

$$C^{\tau}_{h_1h_2}(x,y,t)=C^{n+1}_{ij},\ f^{\tau}_{h_1h_2}(x,y,t)=f^{n+1}_{ij}.$$

Assume

(I)$'$ The $m\times m$ matrix function $A=A(x,y,t,u)$ is equi-continuous with respect to $(x,y,t)\in Q_T$ for $u\in R^m$, i.e., $\forall\varepsilon>0$, there is $\delta=\delta(\varepsilon)>0$ such that for any $(x',y',t')\in Q_T$ and $(x,y,t)\in Q_T$, when $|x'-x|+|y'-y|+|t'-t|<\delta$, there holds $|A(x',y',t',u)-A(x,y,t,u)|<\varepsilon$ for all $u\in R^m$. Moreover $A(x,y,t,u)$ is uniformly Lipschitz with respect to $u\in R^m$, i.e., there is a constant A_1 such that $|A(x,y,t,u_1)-A(x,y,t,u_2)|\le A_1|u_1-u_2|$ for all $u_1,u_2\in R^m$, and all $(x,y,t)\in Q_T$. Also $B(x,y,t,u)$, $C(x,y,t,u)$ and $f(x,y,t,u)$ are equi-continuous with respect to $(x,y,t)\in Q_T$ for $u\in R^m$, and uniformly Lipschitz with respect to $u\in R^m$.

Lemma 3. *Assume that the same conditions as those in Lemma 2 and* **(I)**$'$ *hold. When* $h_1\to 0,h_2\to 0,\tau\to 0$, *there are*

(i) $\bar{v}^{\tau}_{h_1h_2}(x,y,t)$, $\tilde{v}^{\tau}_{h_1h_2}(x,y,t)$, $\hat{v}^{\tau}_{h_1h_2}(x,y,t)$ *and* $\breve{v}^{\tau}_{h_1h_2}(x,y,t)$ *are all convergent to* $u(x,y,t)$ *strongly in* $L^2(Q_T)$ *and a.e. in* Q_T;

(ii) $\bar{v}^{\tau}_{xh_1h_2}(x,y,t)\rightharpoonup u_x(x,y,t)$ *and* $\bar{v}^{\tau}_{yh_1h_2}(x,y,t)\rightharpoonup u_y(x,y,t)$ *weakly in* $L^2(Q_T)$;

(iii) $A^{\tau}_{h_1h_2}(x,y,t)\to A(x,y,t,u(x,y,t))$, $B^{\tau}_{h_1h_2}(x,y,t)\to B(x,y,t,u(x,y,t))$, $C^{\tau}_{h_1h_2}(x,y,t)$ $\to C(x,y,t,u(x,y,t))$, $f^{\tau}_{h_1h_2}(x,y,t)\to f(x,y,t,u(x,y,t))$ *strongly in* $L^2(Q_T)$ *and a.e. in* Q_T.

Proof. Note that, by the definitions $\bar{v}^{\tau}_{h_1h_2}$ and $v^{\tau}_{h_1h_2}$ and the estimates (12)

$$\left\|\bar{v}^{\tau}_{h_1h_2}-v^{\tau}_{h_1h_2}\right\|^2_{L^2(Q_T)}\le C\left(h_1^2+h_2^2\right)\sum_{n=0}^{N-1}\left(\left\|\delta_x v^{n+1}_{\Delta}\right\|^2_2+\left\|\delta_y v^{n+1}_{\Delta}\right\|^2_2\right)\tau$$

$$+C\tau^2\sum_{n=0}^{N-1}\left\|\frac{v^{n+1}_{\Delta}-v^n_{\Delta}}{\tau}\right\|^2_2\tau\le C\left(h_1^2+h_2^2+\tau^2\right).$$

It follows (i) is true. From

$$\left\|\bar{v}^{\tau}_{xh_1h_2}\right\|_{L^2(Q_T)}\le C\left(\max_{0\le n\le N-1}\left\|\delta_x v^{n+1}_{\Delta}\right\|_2+\sum_{n=0}^{N-1}\left\|\frac{v^{n+1}_{\Delta}-v^n_{\Delta}}{\tau}\right\|^2_2\tau\right),$$

and

$$\left\|\bar{v}^{\tau}_{yh_1h_2}\right\|_{L^2(Q_T)}\le C\left(\max_{0\le n\le N-1}\left\|\delta_y v^{n+1}_{\Delta}\right\|_2+\sum_{n=0}^{N-1}\left\|\frac{v^{n+1}_{\Delta}-v^n_{\Delta}}{\tau}\right\|^2_2\tau\right),$$

we can get that (ii) holds. Now we prove (iii). By the assumption **(I)**$'$ there are

$$\begin{aligned}&\|A^{\tau}_{\Delta}(x,y,t)-A(x,y,t,u(x,y,t))\|^2_{L^2(Q_T)}\\ =&\sum_{n=0}^{N-1}\sum_{i=0}^{I-1}\sum_{j=0}^{J-1}\int\int_{Q^n_{ij}}\left|A(x_i,y_j,t^{n+1},\bar{\delta}^0 v^{n+1}_{ij})-A(x,y,t,u(x,y,t)))\right|^2dxdydt\\ \le&C\max_{|x'-x|\le h_1,|y'-y|\le h_2,|t'-t|\le\tau,v\in R^m}\left|A(x',y',t',v)-A(x,y',t,v)\right|^2\\ &+C\|\bar{v}^{\tau}_h(x,y,t)-u(x,y,t)\|^2_{L^2(Q_T)},\end{aligned}$$

which yields the first assertion of (iii). Similarly we can get the other assertions of (iii).

10. From

$$\int\int_{Q_T}[v^\tau_{th_1h_2}-A^\tau_{h_1h_2}\Delta' v^\tau_{h_1h_2}-B^\tau_{h_1h_2}\bar{v}^\tau_{xh_1h_2}-C^\tau_{h_1h_2}\bar{v}^\tau_{yh_1h_2}-f^\tau_{h_1h_2}]\Phi^\tau_{h_1h_2}dxdydt$$

$$=\sum_{n=0}^{N-1}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\left[\frac{v^{n+1}_{ij}-v^n_{ij}}{\tau}-A^{n+1}_{ij}\overset{*}{\Delta}v^{n+1}_{ij}-B^{n+1}_{ij}\bar{\delta}^1_x v^{n+1}_{ij}\right.$$

$$\left.-C^{n+1}_{ij}\bar{\delta}^1_y v^{n+1}_{ij}-f^{n+1}_{ij}\right]\Phi^n_{ij}h_1h_2\tau=0,$$

and letting $h_1\to 0, h_2\to 0, \tau\to 0$ (for some subsequences), we have, for any smooth test function $\Phi(x,y,t)$,

$$\int\int_{Q_T}[u_t(x,y,t)-A(x,y,t,u(x,y,t))\Delta u(x,y,t)-B(x,y,t,u(x,y,t))u_x(x,y,t)$$
$$-C(x,y,t,u(x,y,t))u_y(x,y,t)-f(x,y,t,u(x,y,t))]\,\Phi(x,y,t)dxdydt=0. \qquad (15)$$

Let $\Phi(x,y,t)$ be any smooth function vanishing near $t=T$. Define the piecewise constant function $\Phi^\tau_{th_1h_2}=\frac{\Phi^n_{ij}-\Phi^{n-1}_{ij}}{\tau}$ for $(x,y,t)\in Q^n_{ij}$, $(i=0,1,\cdots,I-1; j=0,1,\cdots,J-1; n=1,2,\cdots,N-1)$. Since

$$\int_{Q_T}v^\tau_{th_1h_2}\Phi^\tau_{h_1h_2}dxdydt=-\int_{Q_T\cap\{t\ge\tau\}}v^\tau_{h_1h_2}\Phi^\tau_{h_1h_2}dxdydt-\int_\Omega v^0_{h_1h_2}\Phi^0_{h_1h_2}dxdy,$$

where $v^0_{h_1h_2}=v^\tau_{h_1h_2}(x,y,0)$, $\Phi^0_{h_1h_2}=\Phi^\tau_{h_1h_2}(x,y,0)$, we can get

$$\int_\Omega\varphi(x,y)\Phi(x,y,0)dxdy+\int_{Q_T}\{u\Phi_t(x,y,t)-[A(x,y,t,u)\Delta u(x,y,t)$$
$$+B(x,y,t,u)u_x(x,y,t)+C(x,y,t,u)u_y(x,y,t)+f(x,y,t,u)]\,\Phi(x,y,t)\}\,dxdydt=0. \qquad (16)$$

11. Therefore we have proved that the m–dimensional vector function $u(x,y,t)\in L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H^1_0(\Omega))\cap H^1(0,T;L^2(\Omega))$ is just the generalized solution of the boundary problem with the homogeneous boundary condition (2) and the initial condition (3) for the quasilinear parabolic system (1). By means of the uniqueness of the generalized solution of the homogeneous boundary problem (1)–(3), we then can obtain the convergence theorem for the finite difference scheme $(1)_\Delta$–$(3)_\Delta$ with intrinsic parallelism as follows:

Theorem 2. *Under the conditions* **(I)–(VI)** *and* **(I)′**, *as the meshsteps* h_1, h_2 *and* τ *tend to zero, the m–dimensional discrete vector solution* $v_\Delta=v^\tau_{h_1h_2}=\{v^n_{ij}|i=0,1,\cdots,I;j=0,1,\cdots,J;n=0,1,\cdots,N\}$ *of the finite difference scheme* $(1)_\Delta$–$(3)_\Delta$ *with intrinsic parallelism converges (in the sense of Lemma 2) to the unique generalized solution* $u(x,y,t)$ *of the boundary problem (2) and (3) for the quasilinear parabolic system (1) of partial differential equations.*

5. Examples of Difference Schemes with Intrinsic Parallelism

12. Here we present some examples of difference schemes with intrinsic parallelism by taking specific parameters in the general schemes $(1)_\Delta$–$(3)_\Delta$. Some of them are proposed in [6]–[7], which are applicable to the numerical solution of parabolic system of equations appearing in massive scientific and engineering computing on parallel computers.

We take for the Laplace $\Delta u=u_{xx}+u_{yy}$ in nine different kinds of difference approximations, such as the center scheme $\mathcal{C}$:

$$\frac{u^{n+1}_{i+1j}-2u^{n+1}_{ij}+u^{n+1}_{i-1j}}{h_1^2}+\frac{u^{n+1}_{ij+1}-2u^{n+1}_{ij}+u^{n+1}_{ij-1}}{h_2^2};$$

the east scheme $\mathcal{E}$:

$$\frac{u^{n}_{i+1j}-2u^{n+1}_{ij}+u^{n+1}_{i-1j}}{h_1^2}+\frac{u^{n+1}_{ij+1}-2u^{n+1}_{ij}+u^{n+1}_{ij-1}}{h_2^2},$$

the west scheme $\mathcal{W}$:

$$\frac{u^{n+1}_{i+1j} - 2u^{n+1}_{ij} + u^{n}_{i-1j}}{h_1^2} + \frac{u^{n+1}_{ij+1} - 2u^{n+1}_{ij} + u^{n+1}_{ij-1}}{h_2^2};$$

the north scheme $\mathcal{N}$:

$$\frac{u^{n+1}_{i+1j} - 2u^{n+1}_{ij} + u^{n+1}_{i-1j}}{h_1^2} + \frac{u^{n}_{ij+1} - 2u^{n+1}_{ij} + u^{n+1}_{ij-1}}{h_2^2};$$

the south scheme $\mathcal{S}$:

$$\frac{u^{n+1}_{i+1j} - 2u^{n+1}_{ij} + u^{n+1}_{i-1j}}{h_1^2} + \frac{u^{n+1}_{ij+1} - 2u^{n+1}_{ij} + u^{n}_{ij-1}}{h_2^2};$$

the east–north scheme $\mathcal{EN}$:

$$\frac{u^{n}_{i+1j} - 2u^{n+1}_{ij} + u^{n+1}_{i-1j}}{h_1^2} + \frac{u^{n}_{ij+1} - 2u^{n+1}_{ij} + u^{n+1}_{ij-1}}{h_2^2};$$

the west–north scheme $\mathcal{WN}$:

$$\frac{u^{n+1}_{i+1j} - 2u^{n+1}_{ij} + u^{n}_{i-1j}}{h_1^2} + \frac{u^{n}_{ij+1} - 2u^{n+1}_{ij} + u^{n+1}_{ij-1}}{h_2^2};$$

the east–south scheme $\mathcal{ES}$:

$$\frac{u^{n}_{i+1j} - 2u^{n+1}_{ij} + u^{n+1}_{i-1j}}{h_1^2} + \frac{u^{n+1}_{ij+1} - 2u^{n+1}_{ij} + u^{n}_{ij-1}}{h_2^2},$$

the west–south scheme $\mathcal{WS}$:

$$\frac{u^{n+1}_{i+1j} - 2u^{n+1}_{ij} + u^{n}_{i-1j}}{h_1^2} + \frac{u^{n+1}_{ij+1} - 2u^{n+1}_{ij} + u^{n}_{ij-1}}{h_2^2}.$$

In general, for the interior points (x_i, y_j) of each subdomain we take $0 < \lambda^n_{ij} \le 1$ and $0 < \mu^n_{ij} \le 1$, and usually $\lambda^n_{ij} = 1$ and $\mu^n_{ij} = 1$, i.e., the center scheme $\mathcal{C}$; for the inner boundary points of each subdomain we use the other eight schmems. Then the scheme deviced is simple and easy to be implemented on the massive parallel computer. This scheme on a block of subdomains can be illustrated as

$$\begin{array}{ccccccc}
\mathcal{WN} & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N} & \mathcal{EN} \\
\mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} \\
\mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} \\
\mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} \\
\mathcal{WS} & \mathcal{S} & \mathcal{S} & \cdots & \mathcal{S} & \mathcal{S} & \mathcal{ES}
\end{array}$$

and in the vicinity of the corner of subdomain it is

$$\begin{array}{cccccccc}
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{S} & \mathcal{S} & \mathcal{SE} & \mathcal{WS} & \mathcal{S} & \mathcal{S} & \cdots \\
\cdots & \mathcal{N} & \mathcal{N} & \mathcal{NE} & \mathcal{WN} & \mathcal{N} & \mathcal{N} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots
\end{array}$$

By choosing the parameters in $(1)_\Delta$ suitably we can obtain other schemes with intrinsic parallelism, e.g., in the vicinity of the inner boundary of subdomains

$$\begin{array}{ccccccc}
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{C} & \mathcal{C} & \cdots
\end{array}$$

or

$$\begin{array}{ccccccc}
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots
\end{array}$$

or

$$\begin{array}{ccccccc}
\cdots & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{N} & \mathcal{N} & \cdots & \mathcal{N} & \mathcal{N} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \cdots
\end{array}$$

or

$$\begin{array}{ccccccc}
\cdots & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{S} & \mathcal{S} & \cdots & \mathcal{S} & \mathcal{S} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \cdots & \mathcal{C} & \mathcal{C} & \cdots
\end{array}$$

and in the vicinity of the corner of subdomain

$$\begin{array}{ccccccccc}
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{C} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{C} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{S} & \mathcal{S} & \mathcal{SE} & \mathcal{C} & \mathcal{WS} & \mathcal{S} & \mathcal{S} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{C} & \mathcal{C} & \mathcal{C} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{N} & \mathcal{N} & \mathcal{NE} & \mathcal{C} & \mathcal{WN} & \mathcal{N} & \mathcal{N} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{C} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots \\
\cdots & \mathcal{C} & \mathcal{C} & \mathcal{E} & \mathcal{C} & \mathcal{W} & \mathcal{C} & \mathcal{C} & \cdots
\end{array}$$

References

[1] Zhou Yu-lin, General Finite Difference Schemes with Intrinsic Parallelism for Multi–Dimensional Semilinear Parabolic Systems, *Beijing Mathematics*, **2**:2(1996), 1–32.

[2] Zhou Yu-lin, Shen Long-jun, Yuan Guang-wei, Some Practical Difference Schemes with Intrinsic Parallelism for Nonlinear Parabolic Systems, *Chinese J. Numer. Math. and Appl.*, **19**:3(1997), 46–57.

[3] Zhou Yu-lin, Yuan Guang-wei, Difference Method of General Schemes with Intrinsic Parallelism for Multi–Dimensional Semilinear Parabolic Systems with Bounded Measurable Coefficients, *Beijing Mathematics*, **3**:1(1997), 39–54.

[4] Guang-wei Yuan, Long-jun Shen, Yu-lin Zhou, Unconditional Stability of Alternating Difference Schemes with Intrinsic Parallelism for Two Dimensional Parabolic Systems, Numerical Methods for Partial Differential Equations, 15:625–636, 1999.

[5] Yuan Guang-wei, Shen Long-jun and Zhou Yu-lin, Unconditional Stability and Convergence of Parallel Difference Schemes for Parabolic Equations, 2001, Preprint.

[6] Yuan Guang-wei, Shen Long-jun and Zhou Yu-lin, Unconditional Stability and Convergence of Implicit Difference Domain Decomposition Procedures, 2001, Preprint.

[7] Zhou Yu-lin, Shen Long-jun, Yuan Guang-wei, The Unconditional Stable and Convergent Difference Methods with Intrinsic Parallelism for Quasilinear Parabolic Systems, 2002, Preprint.

[8] Zhou Yu-lin, Yuan Guang-wei, Difference Schemes for Nonlinear Parabolic Systems with Two and Three Dimensions, *Beijing Mathematics*, **1**:1(1995), 1–35.

[9] Zhou Yu-lin, Applications of Discrete Functional Analysis to Finite Difference Method, *International Academic Publishers, Beijing*, 1990.

[10] Yu-lin Zhou, Guang-wei Yuan, Some Discrete Interpolation Inequalities with Several Indices, *Beijing Mathematics*, **2**:2(1996), 94–108.