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ARTIFICIAL BOUNDARY CONDITIONS FOR "VORTEX IN CELL" METHOD*1)

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Dedicated to the 80th birthday of Professor Zhou Yulin

Abstract

This paper mainly designs artificial boundary conditions for "vortex in cell" method in solving two-dimensional incompressible inviscid fluid under two conditions: one is with periodical initial value in one direction and the other with compact supported initial value. To mimic the vortex motion, Euler equation is transformed into vorticity-stream function and the technique of vortex in cell is applied incorporating with the artificial boundary conditions.

Key Words: Incompressible inviscid flow, Vortex in cell method, Artificial boundary condition

1. Mathematical Model

The motion of two-dimensional incompressible inviscid fluid satisfies the following equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \frac{1}{\rho}\nabla P = \vec{f} \tag{1.1}$$

$$\nabla \boldsymbol{u} = 0, \tag{1.2}$$

where $\boldsymbol{u}(u,v)$ is the velocity vector of a particle at $\vec{x} = (x,y)$ and time t. $P(\vec{x},t)$ and $\rho(\vec{x},t)$ denote the pressure and the density of the fluid respectively. $\rho(x,t)$ is constant for incompressible fluid. Fone $\vec{f}$ acts on a unit fluid. ∇ is gradient operator. '$\cdot$' stands for the inner product. In (x,y)-plane $\nabla\Lambda = (\frac{\partial}{\partial y}, -\frac{\partial}{\partial x})$ is rotation operator. Here we assume $\vec{f} = \nabla\varphi$, where φ is a scalar function, and define vorticity ω as $\omega = -\nabla\Lambda\boldsymbol{u} = -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$. Inserting $\nabla\Lambda$ into Euler equation (1.1), one can obtain:

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{u} \cdot \nabla)\omega = 0. \tag{1.3}$$

Let $\vec{x}_0$ be an arbitrary point in the plane and define the stream function $\psi(\vec{x})$ as

$$\psi(\vec{x}) = \int_{\vec{x}_0}^{\vec{x}} v dx + u dy,$$

where the integration in (x,y) plane is independent of the integration path due to (**??**) and Green formula. Applying $\nabla\Lambda$ and $-\nabla\Lambda$ to $\psi(\vec{x})$ in succession, one can get the relationship of

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the stream function and vorticity, i. e., $-\Delta\psi = \omega$. Thus the incompressible inviscid fluid in vorticity-stream function is:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0 \tag{1.4}$$

$$-\Delta\psi = \omega \tag{1.5}$$

$$\frac{\partial \psi}{\partial y} = u \tag{1.6}$$

$$\frac{\partial \psi}{\partial x} = -v. \tag{1.7}$$

2. Technique of "vortex in cell"

Christansen [?] is the first one to use the technique of vortex in cell to simulate the motion of two-dimensional incompressible inviscid fluid. This paper adopts this method to simulate vortices interaction in (x, y)-plane. We discrete the fluid vorticity ω into sum of vortex points:

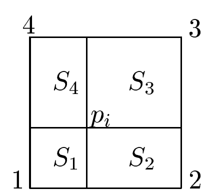
$$\omega(x, y) \approx \sum_{i=1}^{M} p_i \delta(x - x_i)\delta(y - y_i),$$

where M is the total number of vortex points, p_i is vorticity of the i-th point at (x_i, y_i). The velocity $\boldsymbol{u}_i = (u_i, v_i)$ of the i-th vortex point is given by

$$\frac{\partial x_i}{\partial t} = u_i \tag{2.1}$$

$$\frac{\partial y_i}{\partial t} = v_i. \tag{2.2}$$

Once we know the velocity (u_i, v_i) of the vortex point, then we can advance the vortex point by using (**??**)-(**??**). Vortex in cell method solves the Poisson's equation (**??**) on a uniform mesh by the usual five-point scheme. Thus to mimic the motion of vortex points, it is necessary to allocate the vortex point's vorticity to mesh points first and then, after solving the Poisson's equation (**??**) on uniform mesh, to redistribute mesh point's velocity to vortex points. Let s be the area of an mesh and s_1, s_2, s_3, s_4 be the four parts divided by a vortex point located in the mesh showed by the following figure. Then the vortex point allocates its vorticity p_i to the four surrounding mesh points as follows:



$$\omega_1 = \frac{s_3}{s}p_i \quad \omega_2 = \frac{s_4}{s}p_i \quad \omega_3 = \frac{s_1}{s}p_i \quad \omega_4 = \frac{s_2}{s}p_i.$$

All other vortex points located in the same mesh allocate their vorticity to the mesh points similarly. On the other hand, mesh points distribute their velocity to the vortex point p_i by the technique of vortex in cell. That is, assume $\boldsymbol{v}_l,\ l = 1, 2, 3, 4$ are the velocities of the four surrounding points of a mesh , then the velocity $\boldsymbol{u}_i$ of the vortex point p_i located in the mesh is defined by

$$\boldsymbol{u}_i = \frac{s_3}{s}\boldsymbol{v}_1 + \frac{s_4}{s}\boldsymbol{v}_2 + \frac{s_1}{s}\boldsymbol{v}_3 + \frac{s_2}{s}\boldsymbol{v}_4.$$

In order to solve the Poisson's equation (**??**) on uniform mesh by numerical method efficiently, it is necessary to design an artificial boundary condition for the Poisson's equation.

3. Artificial Boundary Condition for Periodical Initial Value

If the initial vortex function $\omega_0(x,y)$ is bounded, periodical in y and compact supported in x, then the following infinite conditions are needed:

$$\lim_{x\to+\infty}\psi(x,y,t)=\psi_{+\infty} \tag{3.1}$$

$$\lim_{x\to-\infty}\psi(x,y,t)=\psi_{-\infty}, \tag{3.2}$$

where $\psi_{-\infty}$ and $\psi_{+\infty}$ are some constants. It is known that $\omega(x,y,t)$ and $\psi(x,y,t)$ are periodical functions in y due to the uniqueness of the problem. Let L be period, then the solving region can be reduced from the whole plane to a strip region $\Omega_0=\boldsymbol{R}\times[-L/2,L/2]$. The periodical condition gives a boundary condition at $y=\pm L/2$ as $\psi(x,-L/2,t)=\psi(x,L/2,t)$. In order to solve the Poisson's equation in $\boldsymbol{R}\times[-L/2,L/2]$ numerically, it is necessary to provide some boundary conditions at $x=-X_l$ and $x=X_r$, where X_l and X_r are some positive constants. The traditional method is to set $\psi(-X_l,y,t)=\psi_{-\infty}$ and $\psi(X_r,y,t)=\psi_{+\infty}$, which is less precise. If more precise solution is needed, X_l and X_r must be set very large and this leads to the large cost. In this paper we derive a semidiscrete boundary condition for the Poisson's equation by using the method of line. Here we notice that Han and Bao [**?**] was the first one to apply the method of line to derive an artificial boundary condition for Poisson's equation in an infinity strip region. We first divide the region $[-L/2,L/2]$ into $2n$ uniform meshes with mesh size $h=L/2n$, and then discrete the Poisson Equation (**??**) in $\boldsymbol{R}\times[-L/2,L/2]$ into

$$-\frac{\partial^2\Psi(x)}{\partial x^2}+A\Psi(x)=W(x),\quad x\in\boldsymbol{R} \tag{3.3}$$

where

$$\Psi(x)=\begin{pmatrix}\psi_0(x)\\ \psi_1(x)\\ \vdots\\ \psi_{2n-1}\end{pmatrix},\quad W(x)=\begin{pmatrix}\omega_0(x)\\ \omega_1(x)\\ \vdots\\ \omega_{2n-1}\end{pmatrix}$$

and

$$A=\begin{pmatrix}2&-1&0&0&\cdots&0&-1\\ -1&2&-1&0&\cdots&0&0\\ 0&-1&2&-1&\cdots&0&0\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ -1&0&0&0&\cdots&-1&2\end{pmatrix}.$$

Here A is a $2n\times 2n$ matrix, $\psi_m(x)\approx\psi(x,mh)$ and $\omega_m(x)\approx\omega(x,mh)$ for $m=0,1,\ldots,2n-1$. Let $A=T\Lambda T^{-1}$, where Λ is a diagonal matrix with diagonal elements $\lambda_0,\lambda_1,\cdots,\lambda_{2n-1}$ which are the eigenvalues of matrix A and T is a matrix composed of the column eigenvectors of A. Then the differential system (**??**) can be written as

$$\frac{\partial^2 T^{-1}\Psi(x)}{\partial x^2}=\Lambda T^{-1}\Psi(x)-T^{-1}W(x). \tag{3.4}$$

Let

$$V(x)=T^{-1}\Psi(x)=\begin{pmatrix}v_0(x)\\ v_1(x)\\ \vdots\\ v_{2n-1}\end{pmatrix}\quad G(x)=-T^{-1}W(x)=\begin{pmatrix}g_0(x)\\ g_1(x)\\ \vdots\\ g_{2n-1}\end{pmatrix}$$

and suppose that the support of $\omega(\cdot, y)$ is contained in $(-R, R)$ for some positive R. Then the above system can be transformed into the following forms:

$$\frac{\partial^2 V(x)}{\partial x^2} = \Lambda V(x) + G(x) \quad \text{for} \quad x \in [-R, R] \tag{3.5}$$

$$\frac{\partial^2 V(x)}{\partial x^2} = \Lambda V(x) \quad \text{for} \quad x \in (-\infty, -R] \cup [R, +\infty) \tag{3.6}$$

It is easy to show that matrix A has unique zero eigenvalue and all other eigenvalues are positive and we denote $\lambda_0 = 0$. Solving the system (**??**), on account the infinity boundary conditions, gives

$$V(x) = C_+ e^{-\Lambda x} \quad \text{for} \quad x \in [R, \infty) \tag{3.7}$$

and

$$V(x) = C_- e^{\Lambda x} \quad \text{for} \quad x \in (-\infty, R], \tag{3.8}$$

where $e^{\pm\Lambda x}$ are diagonal matrices with diagonal elements $e^{\pm\lambda_m x}$ for $m = 0, 1, \ldots, 2n-1$ and $C_\pm$ are constant vectors $(c^0_\pm, c^1_\pm, \ldots, c^{2n-1}_\pm)$. In order to satisfy the infinity condition, $c^0_\pm$ is subject to constraints $T(c^0_\pm, 0, \ldots, 0)' = (\psi_{\pm\infty}, \ldots, \psi_{\pm\infty})$. Since $\lambda_0 = 0$ and its eigenvector is $(1, 1, \ldots, 1)$, the constraints are equivalent to $c^0_\pm = \psi_{\pm\infty}$. Therefore it follows from (**??**) and (**??**) that $V(x)$ has the following boundary conditions at $x = \pm R$:

$$v_0(\pm R) = \psi_{\pm\infty}, \quad v'_j(\pm R) = \mp\sqrt{\lambda_j} v(\pm R), \quad j = 1, 2, 3, \cdots 2n-1. \tag{3.9}$$

This is the semidiscrete artificial boundary condition and the ordinary differential systems (**??**) can be solved numerically in the finite region $[-R, R]$ by central difference method incorporating with (**??**).

In the following we simulate several examples on vortices interaction by using vortex in cell method incorporating with our semidiscrete artificial boundary condition (**??**). In order to test the accuracy of the artificial boundary condition we also compute some example by using the traditional constant boundary condition.

Example 1. Karman vortex streets [**?**]: two rows of vortex points with opposite vorticity are located at Z_{1k} and Z_{2k} on the complex plane are:

$$Z_{1k} = x_1 + i\,(y_1 + kL)\,, \quad Z_{2k} = x_2 + i\,(y_2 + kL)\,, \quad k = 0, \pm 1, \pm 2, \cdots$$

where Γ is the vorticity of each vortex point in the first row and $-\Gamma$ is the vorticity in the second row. The exact solution of $\psi(x, y)$ is:

$$\psi(x, y) = \frac{\Gamma}{2L}(x_1 - x_2) + \mathrm{Im}\left(\frac{\Gamma}{i2\pi}\ln\left(\frac{e^{-\frac{2\pi(x - x_1)}{L}} - 1}{e^{-\frac{2\pi(x - x_2)}{L}} - 1}\right)\right) \tag{3.10}$$

with $\psi(-\infty, y) = -\Gamma(x_1 - x_2)/2L$ and $\psi(+\infty, y) = +\Gamma(x_1 - x_2)/2L$.

Fig **??** and Fig **??** show the relative errors $|(\psi_\Delta - \psi)(\pm R, y)|/|\psi(\pm R, y)|$ for $-L/2 \le y \le L/2$, where ψ_Δ are numerical solutions with space steps $\Delta x = \Delta y = \Delta$ by using the artificial boundary condition and the constant boundary condition. From Fig **??** we see that when the artificial boundary region $[-R, R]$ is not so big (in Fig **??** $[-R, R] = [-1, 1]$, which just contains the support of ω), the relative error by using the artificial boundary condition (**??**) is much smaller than that by using the constant boundary condition. Only when $[-R, R]$ is big enough (in Fig **??** $[-R, R] = [-3, 3]$) the two relative errors are compatible. This clearly indicates that the artificial boundary condition is much more accurate and efficient than the tradition one.

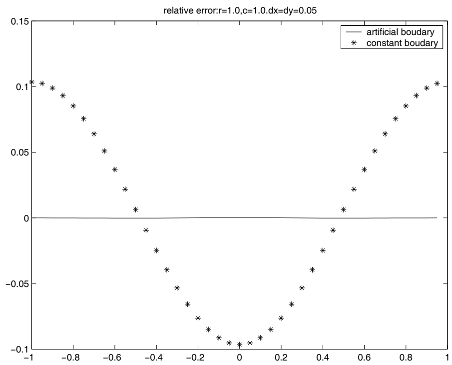


Figure 1: relative errors with parameters $L = 2.0$, $R = 1.0$ and $\Delta = 0.05$

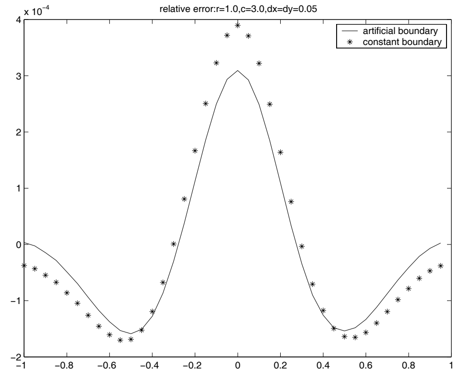


Figure 2: relative errors with parameters $L = 2.0$, $R = 3.0$ and $\Delta = 0.05$

Example 2. Two sheets of vortex patch: $\omega_0(x, y)$ is periodic in y with period L, which is defined on $\boldsymbol{R} \times [-L/2, L/2]$ by

$$\omega_0(x,y) = \begin{cases} 1 & (x,y) \in \{(x,y) | (x-x_0)^2 + y^2 \leq 0.01\} \\ -1 & (x,y) \in \{(x,y) | (x+x_0)^2 + y^2 \leq 0.01\} \\ 0 & \text{elsewhere.} \end{cases}$$

Here we consider two cases: one is $x_0 = 0.3$ and another is $x_0 = 0.11$. In the former case the two "far" separated sheets are never contact each other and in the later case the two "close" sheets are interact and coalesce. Fig. ??–?? and Fig. ??–?? show the motion of two sheets of vortex patch solved by vortex in cell method incorporating with the artificial boundary conditions.

4. Artificial Artificial Boundary Condition for Compact Supported Initial Value

In this subsection we consider a general initial value $\omega_0(x, y)$ compact supported in $\Omega = [-R, R] \times [-R, R]$ for some constant $R > 0$ and satisfies $\int_\Omega \omega_0(x, y) d\sigma = 0$. Thus $\psi(x, y)$ satisfies an infinity condition:

$$\lim_{x^2+y^2 \to \infty} \psi(x, y) = 0. \tag{4.1}$$

Traditional numerical method for solving Poisson equation (3.14) is to set $\psi(x, y) = 0$ at some finite artificial boundary. Here we use Laurent expansion method to derive a more precise artificial boundary condition for solving Poisson equation. The Laurent expansion method was used by Han and Wu to obtain an exact artificial boundary condition in [?]. Since $\omega(x, y, t) = 0$ in $\boldsymbol{R}^2 \setminus \overline{\Omega}$ and $-\Delta\psi = 0$, ψ is harmonic function in the same region. Thus we can define its conjugate function $\varphi(x, y)$. Let $\boldsymbol{x}_0 = (x_0, y_0)$ be a point in $\boldsymbol{R}^2 \setminus \overline{\Omega}$ and set $\varphi(x_0, y_0) = 0$ and then for $\forall\ \boldsymbol{x} = (x, y) \in \boldsymbol{R}^2 \setminus \overline{\Omega}$

$$\varphi(x, y) = \int_{\boldsymbol{x}_0}^{\boldsymbol{x}} \frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy$$

where the integration is along any plane curve connecting $\boldsymbol{x}_0$ to $\boldsymbol{x}$ in $\boldsymbol{R}^2 \setminus \overline{\Omega}$.

Let $z = x + iy$ and $F(z) = \varphi(x, y) + i\psi(x, y)$. Then $F(z)$ satisfies Cauchy-Riemann condition and is analytical in $\boldsymbol{R}^2 \setminus \overline{\Omega}$ with infinity as removable singular point. The Laurent Expansion of $F(z)$ at infinity is

$$F(z) = \sum_{n=0}^{+\infty} c_{-n} z^{-n}, \quad \text{where} \quad c_{-n} = \frac{1}{2\pi i} \oint_{\mathcal{L}} \frac{F(\xi)}{\xi^{1-n}} d\xi \tag{4.2}$$

and $\mathcal{L}$ is any closed curve surrounding Ω in $\boldsymbol{R}^2 \setminus \overline{\Omega}$. It is easy to see that $\mathrm{Im}\{c_0\} = 0$ due to the infinity condition (**??**). If we chose $\partial\Omega$ as an artificial boundary and let $\mathcal{L} = \partial\Omega$ and $z = x + iy \in \partial\Omega$, then the equation (**??**) is an exact artificial boundary condition for Poisson equation on Ω. In practical computation we have to truncate the series (**??**) to a finite sum:

$$F(z) = \sum_{n=0}^{N} c_{-n} z^{-n} \quad \text{for} \quad z \in \partial\Omega, \tag{4.3}$$

where N is a given positive integer and

$$c_{-n} = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{F(\xi)}{\xi^{1-n}} d\xi. \tag{4.4}$$

We call (**??**) the artificial boundary condition of N-th order.

In the following computation we chose $N = 2$ and then $\psi(x, y)$ can be calculated by using (**??**) as

$$\psi(x, y) = \mathrm{Im}(F(z)) = \mathrm{Im}\left(c_0 + c_{-1} z^{-1} + c_{-2} z^{-2}\right) \quad \text{for} \quad z \in \partial\Omega, \tag{4.5}$$

where $\mathrm{Im}(c_0) = 0$,

$$c_{-1} = \frac{1}{2\pi i} \int_{\partial\Omega} F(x + iy)(dx + idy) \quad \text{and} \quad c_{-2} = \frac{1}{2\pi i} \int_{\partial\Omega} F(x + iy)(x + iy)(dx + idy).$$

More precisely,

$$\begin{aligned} &\mathrm{Im}\left(c_{-1} z^{-1}\right) \\ &= \frac{1}{2\pi(x^2 + y^2)} \Bigg(x \int_{-R}^{R} \psi(R, y) dy + y \int_{-R}^{R} \psi(x, R) dx - x \int_{-R}^{R} \psi(-R, y) dy \\ &\qquad - y \int_{-R}^{R} \psi(x, -R) dx - y \int_{-R}^{R} \varphi(R, y) dy + x \int_{-R}^{R} \varphi(x, R) dx \\ &\qquad + y \int_{-R}^{R} \varphi(-R, y) dy - x \int_{-R}^{R} \varphi(x, -R) dx \Bigg) \end{aligned}$$

and $\mathrm{Im}\left(c_{-2} z^{-2}\right)$ has a similar expression. Here the integration of φ can be converted to integration of ψ through Cauchy-Riemann equation:

$$\int_{-R}^{R} \varphi(R, y) dy = \int_{-R}^{R} \left(\varphi(R, 0) + \int_0^y \frac{\partial \psi}{\partial \tau}(R, \tau) d\tau \right) dy = 2R\varphi(R, 0) - \int_{-R}^{R} \int_0^y \frac{\partial \psi}{\partial x}(R, \tau) d\tau dy,$$

$$\int_{-R}^{R} \varphi(x, R) dx = 2R\varphi(0, R) - \int_{-R}^{R} \int_0^y \frac{\partial \psi}{\partial y}(\tau, R) d\tau dx,$$

$$\int_{-R}^{R} \varphi(-R, y) dy = 2R\varphi(-R, 0) - \int_{-R}^{R} \int_0^y \frac{\partial \psi}{\partial x}(-R, \tau) d\tau dy$$

and

$$\int_{-R}^{R} \varphi(x,-R)dx = 2R\varphi(0,R) - \int_{-R}^{R}\int_{0}^{x} \frac{\partial \psi}{\partial y}(\tau,-R)d\tau dx.$$

Therefore the second order artificial boundary condition (**??**) gives a relationship between ψ and $\frac{\partial \psi}{\partial \boldsymbol{n}}$ along the artificial boundary $\partial\Omega$, which is a global artificial boundary condition.

We use the five-point difference scheme in the interior domain Ω incorporating with a discrete artificial boundary condition (**??**) on the artificial boundary $\partial\Omega$. Details of the discrete artificial boundary condition was reported in [**?**].

In the following we first test the accuracy of the artificial boundary condition numerically by solving a problem with two point vortices, and then simulate interaction of two vortex patches. In the numerical tests we use the vortex in cell method incorporating with the discrete artificial boundary (**??**) on $\partial\Omega$.

Example 1. Two point vortices:

$$\omega_0(\boldsymbol{x}) = \Gamma(\delta(\boldsymbol{x}-\boldsymbol{x}_1) - \delta(\boldsymbol{x}-\boldsymbol{x}_2)),$$

where $\boldsymbol{x}_1 = (0.3, 0.3)$, $\boldsymbol{x}_2 = (0.2, 0.2)$, $\Gamma = 0.01\pi$ and $\delta(\boldsymbol{x}) = \delta(x)\delta(y)$. In order to compare with the traditional boundary condition, three boundary conditions (the artificial boundary condition of first and second order and the zero boundary condition) are used in the numerical computation and absolute errors $|(\psi_\Delta - \psi)(x, \pm R)|$ for $x \in [-R, R]$ are shown in Fig **??**. The results clearly indicate that the artificial boundary condition is much better than the zero boundary condition.

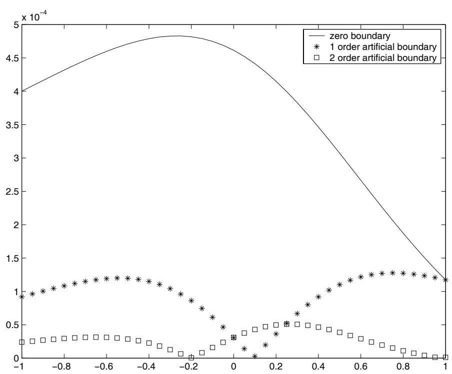


Figure 3: Absolute errors with parameters $R = 1.0$ and $\Delta = 0.05$

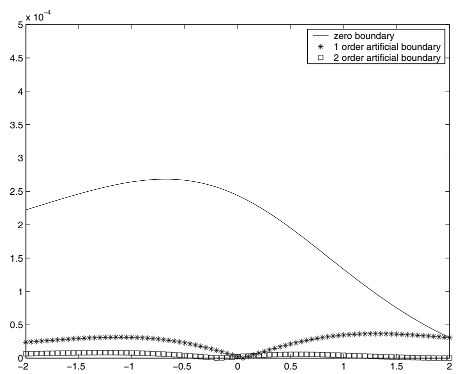


Figure 4: Absolute errors with parameters $R = 2.0$ and Δ=0.05

Example 2. Two vortex patches:

$$\omega_0(x,y) = \begin{cases} -0.5, & (x,y) \in \{(x,y)|(x-0.45)^2 + y^2 \leq 0.01\} \\ -0.5, & (x,y) \in \{(x,y)|(x+0.45)^2 + y^2 \leq 0.01\} \\ 0, & \text{otherwise.} \end{cases}$$

In order to satisfy the infinity condition (**??**), we add an extra point vortex $\omega_1(x,y) = 0.01\pi\delta(x)\delta(y)$ to the initial vorticity ω_0 and, after solving the Poisson equation, we subtract its stream function $\psi_1(x,y) = -0.005\ln(x^2+y^2)$ from the whole solution ψ. The numerical results are shown in Fig **??**–**??**.

Acknowledgement. The authors would like to thank Professor Han for useful discussions.

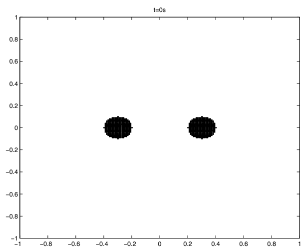


Figure 5: t=0

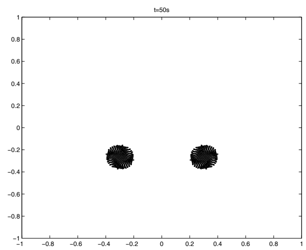


Figure 6: t=50

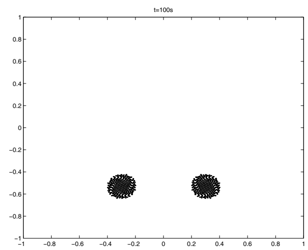


Figure 7: t=100

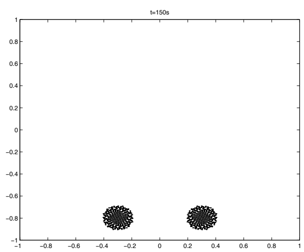


Figure 8: t=150

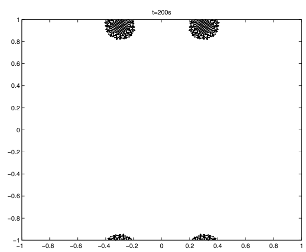


Figure 9: t=200

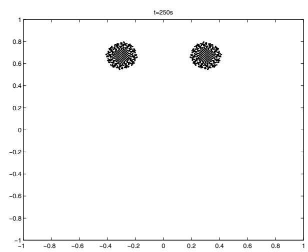


Figure 10: t=250

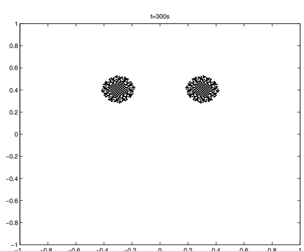


Figure 11: t=300

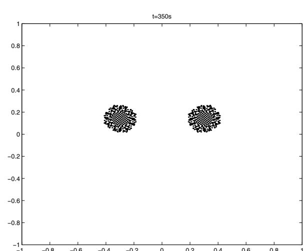


Figure 12: t=350

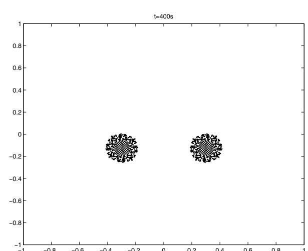


Figure 13: t=400

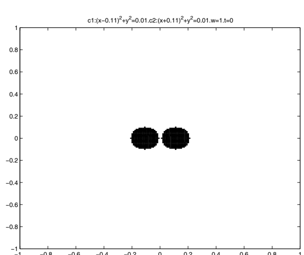


Figure 14: t=0

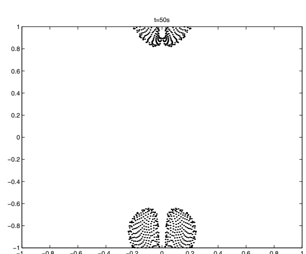


Figure 15: t=50

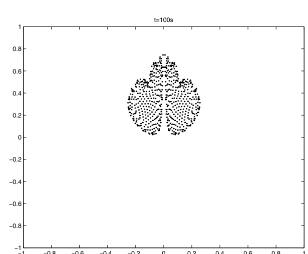


Figure 16: t=100

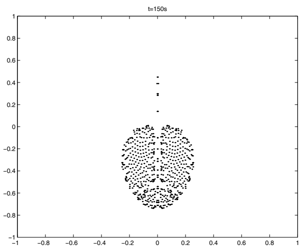


Figure 17: t=150

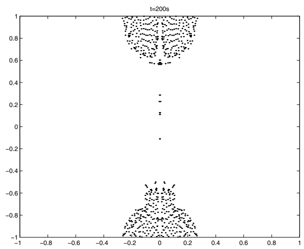


Figure 18: t=200

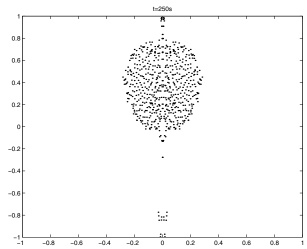


Figure 19: t=250

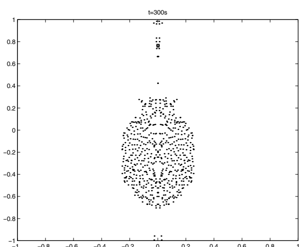


Figure 20: t=300

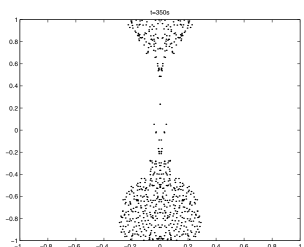


Figure 21: t=350

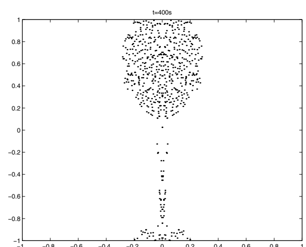


Figure 22: t=400

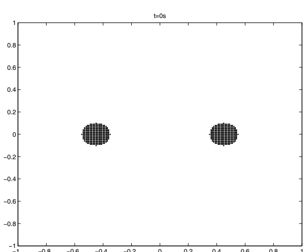


Figure 23: t=0

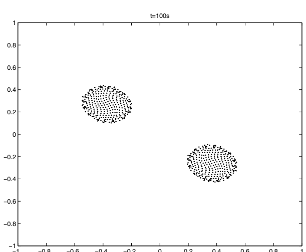


Figure 24: t=100

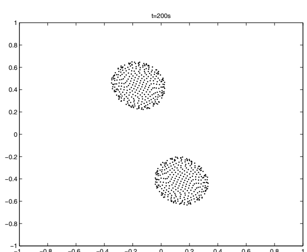


Figure 25: t=200

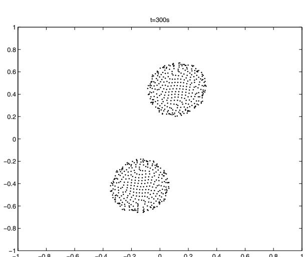


Figure 26: t=300

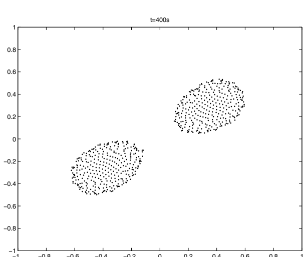


Figure 27: t=400

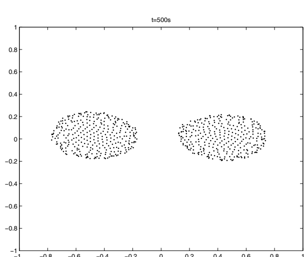


Figure 28: t=500

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