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MULTIVARIATE FOURIER SERIES OVER A CLASS OF NON TENSOR-PRODUCT PARTITION DOMAINS

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Dedicated to the 80th birthday of Professor Zhou Yulin

Abstract

This paper finds a way to extend the well-known Fourier methods, to so-called n+1 directions partition domains in n-dimension. In particular, in 2-D and 3-D cases, we study Fourier methods over 3-direction parallel hexagon partitions and 4-direction parallel parallelogram dodecahedron partitions, respectively. It has pointed that, the most concepts and results of Fourier methods on tensor-product case, such as periodicity,orthogonality of Fourier basis system, partial sum of Fourier series and its approximation behavior, can be moved on the new non tensor-product partition case.

Key words: Multivariate Fourier methods, Non tensor-product partitions, Multivariate Fourier series

1. Introduction

Fourier methods method play very important role in numerical approximation theory and its applications, e.g. see [1]. As we know, the original result has been studied in univariate case. Strictly, the tensor product approach is still staying in the one dimension level via decreasing dimension. How to generalize the approach into higher dimension, beyond box domains, is still an open problem. On an equilateral triangle case, Pinsky in 1980 [2] and 1985 [3] and Práger in 1998 [4] have studied eigen-decompositions of the Laplace operator as generalized Fourier transformation. Recently Sun [5]-[7] has constructed a partial foundation to define generalized Fourier transformation on an arbitrary triangular domain also via eigen-decomposition. It is well-known that a triangle in 2-D and a simplex in 3-D are natural non-box extensions of the interval $[0, 1]$ in 1-D, and the origin Fourier transformation is carried on the interval $[-1, 1]$. It seems there is no essential difference between intervals $[0, 1]$ and $[-1, 1]$ in 1-D, however, the situation is quite different in high dimension. What is more natural non-box extensions in 2-D and 3-D of the interval $[-1, 1]$ in 1-D? In this paper we point that a parallel hexagon and a parallel dodecahedron can be as a direct generalization in 2-D and 3-D of the symmetry interval $[-1, 1]$, respectively. In next sections at first we introduce 3-direction and 4-direction mesh in 2-D and 3-D, respectively. Then we define a parallel hexagon in 2-D, and a parallel hexagon prism and a parallel dodecahedron in 3-D as our three basic periodic domains. Finally we proposed an orthogonal basis system on related function space. We have proposed that the most concepts and results of Fourier methods on tensor-product case, such as periodicity, orthogonality of Fourier basis system, the related sine and cosine transformations, partial sum

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of Fourier series, discreting Fourier transformation (DFT), Fast Fourier transformation (FFT) and its approximation behavior, can be moved on the new non tensor-product partition case.

2. A basic function system on 3-direction partition

Given an origin point O and two plane vectors e_1 and e_2, we form a 3-direction 2-D partition as drawn in Fig. 1. To deal with symmetry along the three direction, we adapt a 3-direction coordinates instead of the usual two coordinates. Setting the origin point $O = (0, 0, 0)$, each partition line is represented by t_l=integer (l=1,2,3), and each 2-D point P is represented by

$$P = (t_1, t_2, t_3), \quad t_1 + t_2 + t_3 = 0, \tag{2.1}$$

and any function $f(P)$ defined on the plane can be written as

$$f(P) = f(t_1, t_2, t_3), \quad t_1 + t_2 + t_3 = 0$$

In particulary, P_k is called an integer node if and only if for an integer pair

$$P_k = (k_1, k_2, k_3), \quad k_1 + k_2 + k_3 = 0.$$

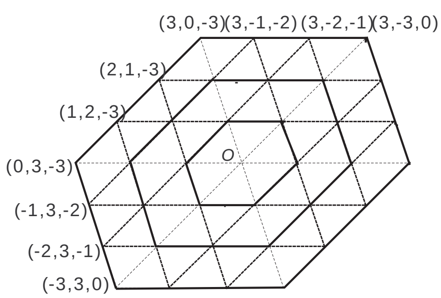


Fig.1: 3-direction partition

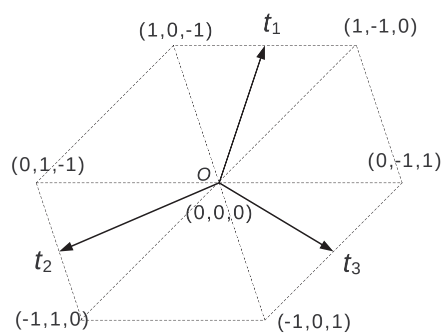


Fig.2:Parallel hexagon domain Ω

Definition 2.1. *A function $f(P)$, defined in the 3-direction coordinate, is called periodic with period $Q = (\tau_1, \tau_2, \tau_3), \tau_1 + \tau_2 + \tau_3 = 0$, if for all $P = (t_1, t_2, t_3), t_1 + t_2 + t_3 = 0$*

$$f(P + Q) = f(P)$$

We take the following parallel hexagon Ω, drawn in Fig.2, as our basic domain

$$\Omega = \{P | P = (t_1, t_2, t_3) \qquad t_1 + t_2 + t_3 = 0, \quad -1 \leq t_1, t_2, t_3 \leq 1\} \tag{2.2}$$

Lemma 2.1. *For any integer pair (n_1, n_2, n_3) with $n_1 + n_2 + n_3 = 0$, then*

$$n_1 - n_2 = n_2 - n_3 = n_3 - n_1 = \nu, \qquad (mod \;\; 3) \quad (\nu = -1, 0, 1)$$

and

$$n_1^2 + n_2^2 + n_3^2 = 2\nu^2 \qquad (mod \;\; 6), \quad (\nu = 0, 1)$$

Definition 2.2. *For a given integer pair j=(j_1, j_2, j_3) with $j_1 + j_2 + j_3 = 0$,let $\omega = e^{i\frac{2\pi}{3}}$, a complex function $g_j(P)$ on the three-direction coordinates is defined*

$$g_j(P) = \omega^{j_1 t_1 + j_2 t_2 + j_3 t_3}. \tag{2.3}$$

Lemma 2.2. *On any integer node $P=(n_1,n_2,n_3)$, function $g_j(P)$ can only take three possible values: 1, ω and $\bar{\omega}$, which are the three roots of $z^3=1$. Moreover, if $n_2-n_1=\nu(mod\ \ 3)$ and $j_1-j_2=\mu(mod\ \ 3)$ $(\nu,\mu=-1,0,1)$, then $g_j(P)=\omega^{\nu\mu}$.*

Theorem 2.1. *(Periodicity) For all nodes P, $g_j(P+Q)=g_j(P)$ if and only if Q is an integer vector $Q=(n_1,n_2,n_3), n_1+n_2+n_3=0, \quad and \quad 0<n_1^2+n_2^2+n_3^2=0 \quad (mod\ \ 6)$.*

<u>Proof:</u> Since $g_j(P+Q)=g_j(P)g_j(Q)$ and

$$g_j(Q)=\omega^{j_1n_1+j_2n_2+j_3n_3}=\omega^{(j_1-j_2)(n_2-n_1)}$$

hence, by using Lemma 2.1, $g_j(Q)=1$ holds if and only if $0<n_1^2+n_2^2+n_3^2=0 \quad (mod\ \ 6)$.

Corollary 2.1. *The basic hexagon: $\Omega=\{P|P=(t_1,t_2,t_3\} \quad t_1+t_2+t_3=0,-1\leq t_1,t_2,t_3\leq 1\}$ is a minimum periodic domain for all functions $g_j(P)$.*

We will denote the set of all continuous function $f(P)$ with periodic domain Ω by C_Ω.

Theorem 2.2. *(Normalization of integral on Ω) For $|j|=|j_1|+|j_2|+|j_3|$,*

$$\int_\Omega g_j(P)dP=c_\Omega\delta_{|j|,0}$$

where c_Ω – area of the basic hexagon Ω.

Proof. The results is trivial for $j=(0,0,0)$.In general case, we may decompose the integral into three terms, each integration domain is a parallelogram

$$I_j=\int_\Omega g_j(P)dP=I_j^{[1]}+I_j^{[2]}+I_j^{[3]}$$

where

$$I_j^{[1]}=\int_0^1 e^{i\frac{2\pi}{3}(j_2-j_1)t_2}dt2\int_{-1}^0 e^{i\frac{2\pi}{3}(j_3-j_1)t_3}dt3,$$

$$I_j^{[2]}=\int_0^1 e^{i\frac{2\pi}{3}(j_3-j_2)t_3}dt3\int_{-1}^0 e^{i\frac{2\pi}{3}(j_1-j_2)t_1}dt1,\quad I_j^{[3]}=\int_0^1 e^{i\frac{2\pi}{3}(j_1-j_3)t_1}dt2\int_{-1}^0 e^{i\frac{2\pi}{3}(j_2-j_3)t_2}dt2$$

First we assume $j_3-j_2\neq 0$, then $j_2-j_1\neq 0$ and $j_1-j_3\neq 0$ hold simultaneously by Lemma 2.1,

$$I_j^{[1]}=\frac{(e^{\frac{2i\pi}{3}(j_2-j_1)}-1)}{\frac{2i\pi}{3}(j_2-j_1)}\frac{(e^{\frac{2i\pi}{3}(j_1-j_3)}-1)}{\frac{2i\pi}{3}(j_1-j_3)},$$

$$I_j^{[2]}=\frac{(e^{\frac{2i\pi}{3}(j_3-j_2)}-1)}{\frac{2i\pi}{3}(j_3-j_2)}\frac{(e^{\frac{2i\pi}{3}(j_2-j_1)-1})}{\frac{2i\pi}{3}(j_2-j_1)},\quad I_j^{[3]}=\frac{(e^{\frac{2i\pi}{3}(j_1-j_3)}-1)}{\frac{2i\pi}{3}(j_1-j_3)}\frac{(e^{\frac{2i\pi}{3}(j_3-j_2)}-1)}{\frac{2i\pi}{3}(j_3-j_2)}.$$

Since $e^{\frac{2i\pi}{3}(j_2-j_1)}=e^{\frac{2i\pi}{3}(j_3-j_2)}=e^{\frac{2i\pi}{3}(j_1-j_3)}$, hence $I_j=I_j^{[1]}+I_j^{[2]}+I_j^{[3]}=0$.

If $j_1=-2j_2=-2j_3\neq 0$, for an example, then $e^{\frac{2i\pi}{3}(j_2-j_1)}=e^{2\pi ij_2}=1$, $I_j^{[1]}=I_j^{[2]}=I_j^{[3]}=0$, the integral I_j still vanishes. Thus, the proof is completed.

Corollary 2.2.

$$<g_j,g_k>_\Omega=c_\Omega\delta_{|j-k|,0} \tag{2.4}$$

where inner product is defined as

$$<f,g>_\Omega=\int_\Omega f(P)\bar{g}(P)dp$$

Theorem 2.3. *(Completeness and orthogonality) For all integer triple* $j = (j_1, j_2, j_3)$

$$g_j(P) = e^{i\frac{2\pi}{3}(j_1 t_1 + j_2 t_2 + j_3 t_3)}$$

with $t_1 + t_2 + t_3 = 0$ *forms an orthogonal basis system, in the sense (2.4), in the space* $C_\Omega(R^2)$.

The orthogonality has been proved from (2.4), the completeness can be proved by positive operator theory in next section via partial sum of the Fourier series or directly by well-known Stone theorem, e.g. see [1]. Thus, we may define so-called the best approximation as follows

Definition 2.3.

$$E_n[f] = \min_{a_j, 0 \leq |j| \leq 2n} \max_{P \in \Omega} |f(P) - \sum_{|j|=0}^{2n} a_j g_j(P)| \tag{2.5}$$

3. Bivariate Fourier series and error estimates

Definition 3.1. *For a function* $f(P) \in L(\Omega)$*, the related generalized Fourier series (GFS) are defined as*

$$f(P) \sim \sum_{|j|=0}^{\infty} \gamma_j \quad g_j(P), \quad \gamma_j = \frac{1}{c_\Omega} < f, g_j >_\Omega \tag{3.1}$$

where $|j| = |j_1| + |j_2| + |j_3|$.

Following lemmas are useful later.

Lemma 3.1. *If* $t_1 + t_2 + t_3 = 0$*, then*

$$\sin 2t_1 + \sin 2t_2 + \sin 2t_3 = -4 \sin t_1 \sin t_2 \sin t_3$$

and

$$\cos 2t_1 + \cos 2t_2 + \cos 2t_3 = 4 \cos t_1 \cos t_2 \cos t_3 - 1$$

Lemma 3.2. $\sum_{|j|=0}^{2n} g_j(P) =$

$$\frac{\sin \frac{(n+1)\pi(t_3-t_2)}{3} \sin \frac{(n+1)\pi(t_2-t_1)}{3} \sin \frac{(n+1)\pi(t_1-t_3)}{3} - \sin \frac{n\pi(t_3-t_2)}{3} \sin \frac{n\pi(t_2-t_1)}{3} \sin \frac{n\pi(t_1-t_3)}{3}}{\sin \frac{\pi(t_3-t_2)}{3} \sin \frac{\pi(t_2-t_1)}{3} \sin \frac{\pi(t_1-t_3)}{3}}$$

Proof. Decomposing the sum into several parts, we have

$$\sum_{|j|=0}^{2n} g_j(P) = 1 + \{ \sum_{j_2=-n}^{-1} \sum_{j_1=0}^{n} + \sum_{j_3=-n}^{-1} \sum_{j_2=0}^{n} + \sum_{j_1=-n}^{-1} \sum_{j_3=0}^{n} \} g_j(P) \tag{3.2}$$

where

$$\sum_{j_2=-n}^{-1} \sum_{j_1=0}^{n} g_j(P) = \sum_{j_2=-n}^{-1} \omega^{j_2(t_2-t_3)} \sum_{j_1=0}^{n} \omega^{j_1(t_1-t_3)} = \frac{1-\omega^{-n(t_2-t_3)}}{\omega^{(t_2-t_3)}-1} \frac{\omega^{(n+1)(t_1-t_3)}-1}{\omega^{(t_1-t_3)}-1}$$

$$= \frac{\omega^{\frac{n+1}{2}(t_3-t_2)} \sin \frac{n\pi}{3}(t_2-t_3)}{\sin \frac{\pi}{3}(t_2-t_3)} \frac{\omega^{\frac{n}{2}(t_1-t_3)} \sin \frac{(n+1)\pi}{3}(t_1-t_3)}{\sin \frac{\pi}{3}(t_1-t_3)},$$

$$\sum_{j_3=-n}^{-1}\sum_{j_2=0}^{n} g_j(P) = \frac{1-\omega^{-n(t_3-t_1)}}{\omega^{(t_3-t_1)}-1}\frac{\omega^{(n+1)(t_2-t_1)}-1}{\omega^{(t_2-t_1)}-1},$$

$$\sum_{j_1=-n}^{-1}\sum_{j_3=0}^{n} g_j(P) = \frac{1-\omega^{-n(t_1-t_2)}}{\omega^{(t_1-t_2)}-1}\frac{\omega^{(n+1)(t_3-t_2)}-1}{\omega^{(t_3-t_2)}-1}.$$

By using Lemma 3.1 a straightforward computation leads to Lemma 3.2.

Differing with univariate case, there are several ways to define partial sum for the above bivariate Fourier series.

Definition 3.2. *1-st and 2-nd order partial sum of Fourier series (3.1) can be defined by*

$$S_n = \sum_{|j|=0}^{2n} \gamma_j \quad g_j(P), \quad S_n^{[2]} = \frac{1}{n}\sum_{m=0}^{n-1} S_m. \tag{3.3}$$

or

$$\bar{S}_n = \sum_{j_1=-n}^{n}\sum_{j_2=-n}^{n} \gamma_j \quad g_j(P), \quad \bar{S}_n^{[2]} = \frac{1}{n}\sum_{m=0}^{n-1} \bar{S}_m. \tag{3.4}$$

respectively.

Similar with 1-D case, based on Lemma 3.1 and 3.2 a straightforward computation leads these partial sum to be rewritten in terms of integration form.

Theorem 3.1.

$$S_n[f](P) = \frac{1}{c_\Omega}\int_\Omega f(P-Q)G_n(Q)dQ, \quad S_n^{[2]}[f](P) = \frac{1}{c_\Omega}\int_\Omega f(P-Q)G_n^{[2]}(Q)dQ \tag{3.5}$$

and

$$\bar{S}_n[f](P) = \frac{1}{c_\Omega}\int_\Omega f(P-Q)\bar{G}_n(Q)dQ, \quad \bar{S}_n^{[2]}[f](P) = \frac{1}{c_\Omega}\int_\Omega f(P-Q)\bar{G}_n^{[2]}(Q)dQ \tag{3.6}$$

where $G_n(P) =$

$$\frac{\sin\frac{(n+1)\pi(t_3-t_2)}{3}\sin\frac{(n+1)\pi(t_2-t_1)}{3}\sin\frac{(n+1)\pi(t_1-t_3)}{3} - \sin\frac{n\pi(t_3-t_2)}{3}\sin\frac{n\pi(t_2-t_1)}{3}\sin\frac{n\pi(t_1-t_3)}{3}}{\sin\frac{\pi(t_3-t_2)}{3}\sin\frac{\pi(t_2-t_1)}{3}\sin\frac{\pi(t_1-t_3)}{3}} \tag{3.7}$$

$$G_n^{[2]}(P) = \frac{1}{n}\frac{\sin\frac{n\pi}{3}(t_3-t_2)\sin\frac{n\pi}{3}(t_2-t_1)\sin\frac{n\pi}{3}(t_1-t_3)}{\sin\frac{\pi}{3}(t_3-t_2)\sin\frac{\pi}{3}(t_2-t_1)\sin\frac{\pi}{3}(t_1-t_3)}. \tag{3.8}$$

and

$$\bar{G}_n(P) = \frac{\sin\frac{(n+\frac{1}{2})\pi}{3}(t_3-t_2)}{\sin\frac{\pi}{3}(t_3-t_2)}\frac{\sin\frac{(n+\frac{1}{2})\pi}{3}(t_3-t_1)}{\sin\frac{\pi}{3}(t_3-t_1)}, \tag{3.9}$$

$$\bar{G}_n^{[2]}(P) = \frac{1}{n}[\frac{\sin\frac{n\pi}{3}(t_3-t_2)}{\sin\frac{\pi}{3}(t_3-t_2)}\frac{\sin\frac{n\pi}{3}(t_3-t_1)}{\sin\frac{\pi}{3}(t_3-t_1)}]^2 > 0. \tag{3.10}$$

Now based on above formulas of partial sum, as an application, we turn to study convergence of bivariate Fourier series. First, we point that

Theorem 3.2. *If* $f(P) \in C_\Omega(R^2)$, *then its second order partial sum* $\bar{G}_n^{[2]}[f]$ *converges to* f *itself uniformly.*

Proof. Note as a linear operator, $\bar{G}_n^{[2]}$ is positive. By using well-known positive operator theory", it is sufficient and easy to check the result is true for the following nine functions:

$$f(P)=1,\cos\frac{2\pi}{3}(t_3-t_2),\sin\frac{2\pi}{3}(t_3-t_2),\cos\frac{2\pi}{3}(t_3-t_1),\sin\frac{2\pi}{3}(t_3-t_1),$$
$$\cos\frac{2\pi}{3}(t_1-t_2),\sin\frac{2\pi}{3}(t_1-t_2),\cos\frac{2\pi}{3}(t_1+t_2-2t_3),\sin\frac{2\pi}{3}(t_1+t_2-2t_3).$$

Theorem 3.3. *For $f(P)\in C_\Omega(R^2)$, if its all coefficients γ_j of the Generalized Fourier series $\sum_{|j|=0}^{\infty}\gamma_j g_j(P)$ are zeros, the function $f(P)$ must equal to zero itself.*

Thus we have also proved the completeness theorem in last subsection. Moreover, we get

Theorem 3.4. *If $f(P)\in C_\Omega^1$, then its generalized Fourier series (3.1) converges uniformly.*

Finally, we list some results on error estimate of finite bivariate Fourier series. These are natural extensions of the univariate case.

Theorem 3.5. *If a bounded function $|f(P)|\le M$ is periodic over the basic domain Ω, then there is an upper estimation of its finite Fourier series*

$$|S_n|=|\sum_{|j|=0}^{2n}\gamma_j g_j(P)|\le C_1M(\ln n)^2,\quad |\bar{S}_n|=|\sum_{j_1=-n}^{n}\sum_{j_2=-n}^{n}\gamma_j g_j(P)|\le C_2M(\ln n)^2 \tag{3.11}$$

where C_1 and C_2 are constants.

Proof: We only give the proof of the second part. In fact, from (3.6) and (3.9),

$$|\bar{S}_n|\le CM\int_\Omega|\bar{G}_n(Q)|dQ\le C_2M(\ln n)^2,$$

the right inequality is caused by using the following lemma:

Lemma 3.3.

$$\int_\Omega\frac{|\sin\frac{n\pi}{3}(t_3-t_2)|}{\sin\frac{\pi}{3}(t_3-t_2)}\frac{|\sin\frac{n\pi}{3}(t_3-t_1)|}{\sin\frac{\pi}{3}(t_3-t_1)}dP\le C(\ln n)^2. \tag{3.12}$$

Furthermore, similar with univariate case, based on integration by parts we have following error estimations. The detailed proof here is omitted.

Theorem 3.6. *Let $f\in C_\Omega^{2k}$ and $|f^{(k,k)}|\le M_k$, then following two error estimates hold*

$$|S_n[f]-f|\le C_1M_k(\ln n)^2(n^{-k})^2,\quad |\bar{S}_n[f]-f|\le C_2M_k(\ln n)^2(n^{-k})^2. \tag{3.13}$$

where C_1 and C_2 are two constants.

Note that the related best approximation estimation in (2.5)

$$E_n[f]\le CM_k(n^{-k})^2. \tag{3.14}$$

The above facts indicate that for the same smooth functions, in rough speaking, the generalized Fourier series (GFS) has the same approximation rate to the best uniform approximation, except the factor $(\ln n)^2$.

4. Generalized Sine and Cosine in 3-direction mesh

As natural extension of 1-D $\sin z=\frac{1}{2i}(e^{iz}-e^{-iz})$ and $\cos z=\frac{1}{2}(e^{iz}+e^{-iz})$, now we define

Definition 4.1. $TSin_j(P) :=$

$$\frac{1}{2i}\{g_{j_1,j_2,j_3}(P) + g_{j_2,j_3,j_1}(P) + g_{j_3,j_1,j_2}(P) - g_{-j_1,-j_3,-j_2}(P)$$
$$-g_{-j_2,-j_1,-j_3}(P) - g_{-j_3,-j_2,-j_1}(P)\} \tag{4.1}$$

Definition 4.2. $TCos_j(P) :=$

$$\frac{1}{2}\{g_{j_1,j_2,j_3}(P) + g_{j_2,j_3,j_1}(P) + g_{j_3,j_1,j_2}(P) + g_{-j_1,-j_3,-j_2}(P)$$
$$+g_{-j_2,-j_1,-j_3}(P) + g_{-j_3,-j_2,-j_1}(P)\} \tag{4.2}$$

Based on two facts $j_1 + j_2 + j_3 = 0$ and $t_1 + t_2 + t_3 = 0$, there are several equivalent forms for functions $TSin$ and $TCos$. For instance,

$$TSin_j(P) = e^{i\frac{\pi}{3}(j_2-j_3)(t_2-t_3)} \sin j_1\pi t_1 + e^{i\frac{\pi}{3}(j_2-j_3)(t_3-t_1)} \sin j_1\pi t_2 + e^{i\frac{\pi}{3}(j_2-j_3)(t_1-t_2)} \sin j_1\pi t_3. \tag{4.3}$$

$$TCos_j(P) = e^{i\frac{\pi}{3}(j_2-j_3)(t_2-t_3)} \cos j_1\pi t_1 + e^{i\frac{\pi}{3}(j_2-j_3)(t_3-t_1)} \cos j_1\pi t_2 + e^{i\frac{\pi}{3}(j_2-j_3)(t_1-t_2)} \cos j_1\pi t_3. \tag{4.4}$$

Theorem 4.1. *Tsin function vanishes on all integer net lines:*

$$TSin_j(P) = 0, \quad if \quad t_l = integer, \quad l = 1, 2, \quad or \quad 3$$

Theorem 4.2. *Let Γ be a direction connecting the vertex and the middle-point of corresponding side, then for all integer net lines*

$$\frac{\partial TCos_j(P)}{\partial \Gamma} = 0, \quad if \quad t_l = integer, \quad l = 1, 2, \quad or \quad 3.$$

Note that $\sin z\pi = 0$ and $\frac{d}{dz}\cos z\pi = 0$ if $z = integer$. Hence, it is reasonable to call $Tsin$ and $Tcos$ as generalized sine and cosine functions in the 3-direction partition, respectively.

Let Ω_T be a sub-triangle domain in the 3-direction partition, we consider the following eigenvalue problem

$$\mathcal{L}u = \lambda u, \qquad \mathcal{L} = -(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2})^2 - (\frac{\partial}{\partial t_2} - \frac{\partial}{\partial t_3})^2 - (\frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_1})^2 \tag{4.5}$$

with zero Dirichlet boundary $u|_{\partial\Omega_T} = 0$ or Neumann condition $\frac{\partial u}{\partial \Gamma}|_{\partial\Omega_T} = 0$.

By the way, we point that $\mathcal{L} = -\frac{2}{3}h^2\Delta$ in equilateral triangle case.

It is easy to verify Tsin and Tcos functions form eigen-functions of the related zero Dirichlet and Neumann boundary problems, respectively.The corresponding eigenvalues equal to

$$\lambda_j = (\frac{2\pi}{3})^2((j_1 - j_2)^2 + (j_2 - j_3)^2 + (j_3 - j_1)^2) \tag{4.6}$$

Below we give two examples to show the above generalized sine and cosine function can be applied in spectral methods.

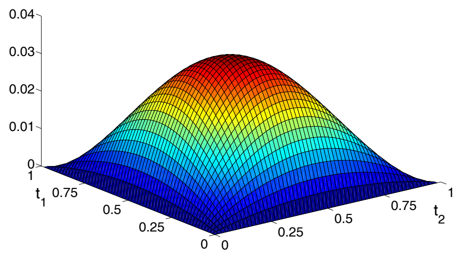
Example 1: (Dirichlet boundary) Suppose the true solution is $u_1(t_1, t_2, t_3) = t_1t_2t_3$, the related approximation spectral solution becomes

$$u_{1,N} = \sum_{j=1}^{N} \frac{1}{2j^3\pi^3}\{\sin 2j\pi t_1 + \sin 2j\pi t_2 + \sin 2j\pi t_3\}$$

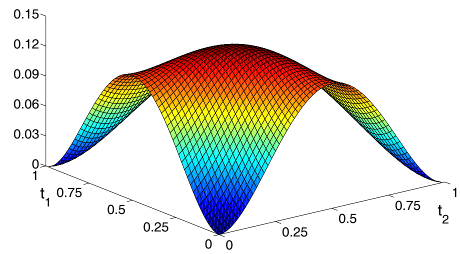
Taking $N = 4, 8, 16, 32, 64$, the maximum truncation error between the true solution and the approximated solution is listed in Table 1.

Table 4.1: Test 1

N	4	8	16	32	64
$\|u-u_N\|_\infty$	$5.83e-4$	$1.65e-4$	$3.66e-5$	$9.55e-6$	$2.08e-6$



$(u_{1,N})$



$(u_{2,N})$

Example 2:(Neumann boundary) For a given true solution $u_2(t_1,t_2,t_3)=t_1^2(1-t_1)^2+t_2^2(1-t_2)^2+t_3^2(1-t_3)^2$ the related approximation spectral solution is

$$u_{2,N}=\frac{1}{10}-\sum_{j=1}^{N}\frac{3}{j^4\pi^4}\{\cos 2j\pi t_1+\cos 2j\pi t_2+\cos 2j\pi t_3\}$$

Taking $N=4,8,16,32,64$, the maximum truncation error between the true solution and the approximated solution is listed in Table 2.

Table 4.2: Test 2

N	4	8	16	32	64
$\|u-u_N\|_\infty$	$3.30e-4$	$4.98e-5$	$6.84e-6$	$8.97e-7$	$1.15e-7$

The numerical tests in Table 1 and Table 2 match analysis results in Theorem 3.6.

5. 3-D Fourier series over parallel hexagonal prism and dodecahedron partitions

In 3-D case we first consider a mixing partition: a partial tensor product of 1-direction t_0 and 3-direction mesh (t_1,t_2,t_3). The corresponding 3-D basic domain is a parallel hexagon prism

$$\Omega_p=\{P|P=(t_0,t_1,t_2,t_3)\quad t_1+t_2+t_3=0,-1\le t_0,t_1,t_2,t_3\le 1\}. \tag{5.1}$$

And the related basic function system becomes

$$g_j(P)=e^{i\pi j_0 t_0}*e^{i\frac{2\pi}{3}(j_1t_1+j_2t_2+j_3t_3)} \tag{5.2}$$

where 4-d index

$$j=(j_0,j_1,j_2,j_3)\qquad with\qquad j_1+j_2+j_3=0.$$

In more general 3-D case, we need to construct 4-direction partition as follows. Given a set of three linear independent vectors e_1,e_2,e_3, let $e_4=-e_1-e_2-e_3$, we may obtain six normals of six planes among the four directions as follows

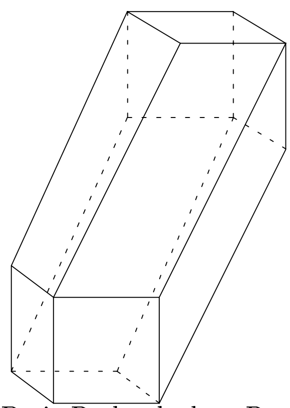
$$n_1 = e_2 \times e_3; \quad n_2 = e_3 \times e_1; \quad n_3 = e_1 \times e_2;$$

$$n_4 = e_3 \times e_4 = n_1 - n_2; n_5 = e_1 \times e_4 = n_2 - n_3; n_6 = e_2 \times e_4 = n_3 - n_1; \tag{5.3}$$

For any 3-D point P, we set up a 6-D coordinate system

$$t_1 = (P, n_1); t_2 = (P, n_2); t_3 = (P, n_3); t_4 = (P, n_4); t_5 = (P, n_5); t_6 = (P, n_6) \tag{5.4}$$

with $t_4 = t_1 - t_2; t_5 = t_2 - t_3; t_6 = t_3 - t_1$.



(Fig. A-1: Basic Dodecahedron Domain)

As our basic domain, we take following parallel dodecahedron, see Fig. A1-A2, which contains 14 vertex, 12 parallelogram and 24 edges.

$$\Omega_d = \{P | P = (t_1, t_2, t_3, t_4, t_5, t_6), -1 \le t_\nu \le 1, (1 \le \nu \le 6),$$

$$t_4 = t_1 - t_2; t_5 = t_2 - t_3; t_6 = t_3 - t_1\}. \tag{5.5}$$

(Fig. A-2: Dodecahedron Domain)

(Fig. B: Parallel Dodecahedron Partition)

Thus we have formed a dodecahedron partition in Fig. B. The related basic function system is defined as

$$g_j(P) = e^{i \frac{\pi}{2} j \cdot P} \tag{5.6}$$

where

$$j = \{j_1, j_2, j_3, j_4, j_5, j_6\}, \quad with \quad j_4 = j_1 - j_2; j_5 = j_2 - j_3; j_6 = j_3 - j_1,$$

and

$$j \cdot P = \sum_{\nu=1}^{6} j_\nu t_\nu = (3j_1 - j_2 - j_3)t_1 + (3j_2 - j_3 - j_1)t_2 + (3j_3 - j_1 - j_2)t_3.$$

Lemma 5.1. *For non zero index* j

$$\int_{\Omega_d} g_j(P)dP = 0. \tag{5.7}$$

To prove Lemma 5.1, as done in (3.2), now we decompose the integral into four terms, each integration domain is a parallelepiped.

Most results in bivariate case can be moved in this 3-D case, such as periodicity, orthogonality and completeness of the basic function system. We list a basic theorem here. Proofs are omitted for saving pages.

Theorem 5.1. *(Basic theorem) For all integer triple* $j = (j_1, j_2, j_3)$, *and*

$$\bar{j}_1 = 3j_1 - j_2 - j_3, \bar{j}_2 = -j_1 + 3j_2 - j_3, \bar{j}_3 = -j_1 - j_2 + 3j_3$$

the function family

$$g_j(P) = e^{i\frac{\pi}{2}(\bar{j}_1 t_1 + \bar{j}_2 t_2 + \bar{j}_3 t_3)} \tag{5.8}$$

forms an orthogonal basis system in the space $C_\Omega(R^3)$ *such that*

$$< g_j, g_k >_{\Omega_d} := \int_{\Omega_d} g_j(P)\bar{g}_k(P)dPdP = 0 = c_{\Omega_d}\delta_{|j-k|,0} \tag{5.9}$$

where c_{Ω_d} *is volume of the basic dodecahedron, defined in (5.5).*

Just as in 2-D case, the basic function (5.2) and (5.6) can be applied as 3-D shape functions for finite element methods.

Finally, it is worth to point that the above results in 3-D can be entended to general high dimension without essential difficnlty.

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