Journal of Computational Mathematics, Vol.21, No.2, 2003, 113–124.

EXPLICIT BOUNDS OF EIGENVALUES FOR STIFFNESS MATRICES BY QUADRATIC HIERARCHICAL BASIS METHOD*1)

Sang Dong KIM†
(*Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea*)
Byeong Chun SHIN‡
(*Department of Mathematics, Chonnam National University, Gwangju 500-757, Korea*)

Abstract

The bounds for the eigenvalues of the stiffness matrices in the finite element discretization corresponding to $Lu := -u''$ with zero boundary conditions by quadratic hierarchical basis are shown explicitly. The condition number of the resulting system behaves like $O(\frac{1}{h})$ where h is the mesh size. We also analyze a main diagonal preconditioner of the stiffness matrix which reduces the condition number of the preconditioned system to $O(1)$.

Key words: hierarchical basis, multilevel

1. Introduction

The main object in this paper is to investigate the explicit bounds of the eigenvalues for the stiffness matrix B_j arisen from the finite element method using piecewise quadratic hierarchical multilevel basis instead of the usual piecewise quadratic nodal basis for the one dimensional elliptic operator $Lu := -u''$ with zero boundary conditions defined on $[0,1]$. Hence the condition number of B_j can be shown as about $\frac{12}{h}$. One may use an interpolation operator to obtain the asymptotic behavior of condition numbers like $O(\frac{1}{h})$, but such technique does not yield the explicit bound for the eigenvalues (see [5]).

For piecewise linear hierarchical multilevel elements, the condition numbers are analyzed in [4, 6, 7]. One can easily see that the stiffness matrix for the unidimensional case using piecewise linear hierarchical basis becomes a diagonal matrix. This phenomenon is quite different from the case of piecewise quadratic hierarchical multilevel elements mainly because of their non-orthogonalities in the H^1 sense.

Define the bilinear form corresponding to $Lu = -u''$ with zero boundary conditions as

$$b(u,v) = \int_0^1 u'v'dx. \tag{1.1}$$

Following the ideas [6], we will give the upper bound $\frac{16(\sqrt{2}+1)}{3(\sqrt{2}-1)\,h}$ of $b(u,u)$ in terms of the Euclidean norm of the hierarchical coefficient vector of u when u is represented by the piecewise quadratic hierarchical basis where h denotes the uniform mesh size of the final level space. In order to get the uniform lower bound $\frac{8}{3}$ of $b(u,u)$, we will use a lower bound for eigenvalues of certain symmetric matrices which will be given in Appendix. There are lots of literature on the

* Received September 29, 2000.
1) The paper was supported by KOSEF 1999-1-103-002-3.
† Email address: skim@knu.ac.kr.
‡ Email address: bcshin@chonnam.ac.kr.

multilevel or two level hierarchical bases. One may refer to, for example, [1, 2, 3, 4, 5, 6, 7] for understanding of them.

The rest of paper is as follows. In section 2, we provide some of definitions and preliminaries on the piecewise quadratic hierarchical basis. In section 3, we analyze the explicit bounds of the stiffness matrix B_j using the ideas given in [6] and the results given in Appendix. We analyze a preconditioner which is the main diagonal of the stiffness matrix and give a numerical experiment in section 4. Appendix which plays an important role to investigate the result in section 3 will be provided in the last of this paper.

2. Preliminaries

For $I = [0,1]$, let us denote π a uniform partition of I, that is to say, any elements of π has same length such that the union of these intervals is I and such that the intersection of two subintervals of π either consists of a common knot of both intervals or is empty. Let $\pi_0 = I$ be a coarse initial partition, beginning with this partition we construct a nested family $\pi_0, \pi_1, \pi_2, \cdots$ of partitions of I where π_{k+1} is obtained from π_k by subdividing each interval of π_k into two subintervals having the same size. Note that the partition π_k of level k has the mesh size $h_k := (1/2)^k$.

Denote by $\|\cdot\|_{0,D}$ the usual $L^2(D)$-norm, $\|\cdot\|_{1,D}$ the usual Sobolev $H^1(D)$-norm and $|\cdot|_{1,D}$ the Sobolev $H^1(D)$-seminorm. Throughout this paper, we will use k and ℓ for levels, j for last level, p and q denote indices for nodes.

Let $\mathcal{N}_k (k = 0, 1, \cdots, j)$ be the set consisting of the nodes of the intervals of π_k, their mid-points and end-points of I and let $\mathcal{S}_k$ be the subspace of the Sobolev space $H_0^1(I)$ which consists of all continuous functions on I that is quadratic on the intervals of π_k vanishing at two boundary points 0 and 1. We call the function in $\mathcal{S}_k$ finite element functions of level k. Obviously we have $\mathcal{S}_k \subset \mathcal{S}_{k+1}$ from the fact that $\mathcal{N}_k \subset \mathcal{N}_{k+1}$, and a function $u \in \mathcal{S}_k$ is determined by its values at the nodes $x \in \mathcal{N}_k$.

Let J_k be the interpolation operator from $\mathcal{S}_j$ to $\mathcal{S}_k$ such that

$$J_k u \in \mathcal{S}_k; \quad (J_k u)(x) = u(x), \quad x \in \mathcal{N}_k.$$

For $k \le j$, a function $u \in \mathcal{S}_k$ can be reproduced by the interpolation operator J_j, so that any function $u \in \mathcal{S}_j$ has the representation

$$u = J_0 u + \sum_{k=1}^{j} (J_k u - J_{k-1} u), \tag{2.1}$$

which is a decomposition of u into fast oscillating functions corresponding to the different refinement levels. Note that $J_0 u$ is a function of the finite element space corresponding to the initial partition and $J_k u - J_{k-1} u \in \mathcal{S}_k$ vanishes at all nodal points of level $k-1$.

Let $\mathcal{V}_k (k = 1, 2, \cdots, j)$ be the subspace of $\mathcal{S}_k$ consisting of all finite element functions vanishing at the nodes of level $k-1$. Let us denote the nodes in $\mathcal{N}_k \setminus \mathcal{N}_{k-1}$ as $\{x_p^k \,|\, p = 1, 2, \cdots, d_k\}$ and let $\mathcal{V}_0 := \mathcal{S}_0$ with $\mathcal{N}_0 := \{x_1^0\}$. Then we can easily check that the dimension of $\mathcal{V}_k$ is $d_k := 2^k$, $\mathcal{V}_k$ is the range of $J_k - J_{k-1}$, and (2.1) means that $\mathcal{S}_j$ is the direct sum of $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_{j-1}$ and $\mathcal{V}_j$. The hierarchical basis of $\mathcal{S}_k$, $k \ge 1$, consists of the old hierarchical basis functions of level $k-1$ and the functions forming a nodal basis of $\mathcal{V}_k$. For a function $u \in \mathcal{S}_j$ with hierarchical basis of this space, we can represent u as

$$u = \sum_{k=0}^{j} u^k \in \mathcal{S}_j \quad \text{with} \quad u^k = \sum_{p=1}^{d_k} u_p^k \phi_p^k \in \mathcal{V}_k \tag{2.2}$$

where the quadratic hierarchical basis $\{\phi_p^k : k = 0, 1, \cdots, j,\ p := p(k) = 1, 2, \cdots, d_k\ \}$ of $\mathcal{S}_j$ is

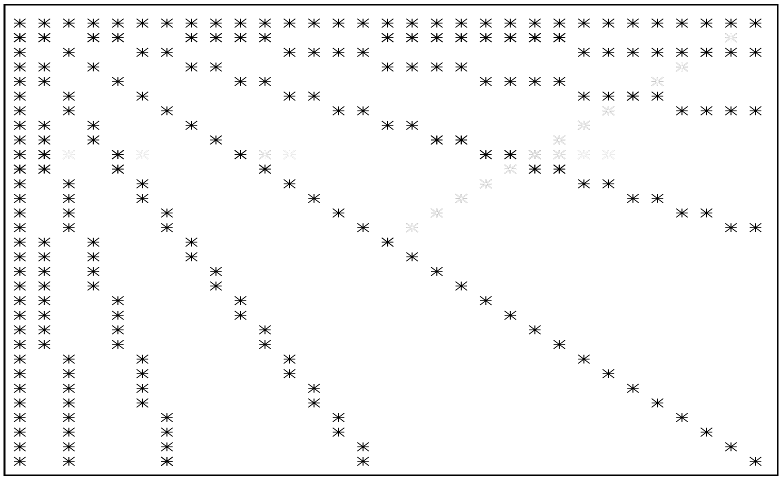

Figure 1: The stiffness matrix B_j when $j = 4$

given by

$$\phi_p^k(x) := \begin{cases} -\frac{4}{h_k^2}\,(x-(p-1)h_k)\,(x-ph_k)\,, & \text{if } (p-1)h_k \le x \le ph_k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that ϕ_p^k is the nodal basis function corresponding to the nodal point x_p^k of $\mathcal{V}_k$.

With the multilevel splitting (2.1) of the finite element space $\mathcal{S}_j$, we associate a mesh dependent seminorm defined by

$$|u|^2 = \sum_{k=1}^{j}\sum_{p=1}^{d_k} |(J_k u - J_{k-1}u)(x_p^k)|^2, \quad u \in \mathcal{S}_j.$$

This seminorm has a very simple representation when the function u is represented with respect to the hierarchical basis of $\mathcal{S}_j$. It is the Euclidean length of the vector of its coefficients with the exception of that corresponding to the initial level.

The above observations together with some results with respect to the space $\mathcal{V}_k$ are summarized in the following proposition.

Proposition 2.1. *We have the followings:*

(a) For $v \in \mathcal{V}_\ell$ with $\ell > k$, $J_k v \equiv 0$.

(b) For $1 \le k \le j$, $\{\phi_p^k\}_{p=1}^{d_k}$ is the orthogonal basis of the subspace $\mathcal{V}_k$ of Sobolev space $H_0^1(I)$.

(c) For $0 \le k \le \ell \le j$, we have

$$b(\phi_p^k,\phi_p^k) = |\phi_p^k|_{1,I}^2 = \frac{16}{3h_k} = \left(\frac{16}{3}\right)2^k, \quad \forall p = 1,2,\cdots,2^k,$$

$$b(\phi_p^k,\phi_q^\ell) = \begin{cases} \left(\dfrac{1}{2\sqrt{2}}\right)^{\ell-k} |\phi_p^k|_{1,I}\,|\phi_q^\ell|_{1,I} = \left(\dfrac{16}{3}\right)\,2^{2k-\ell} \\ \qquad\qquad\qquad \text{if } 1+2^{\ell-k}(p-1) \le q \le 2^{\ell-k}p, \\ 0 \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

(d) For $u \in \mathcal{S}_j$, u has its hierarchical basis representation such that

$$u = \sum_{k=0}^{j}\sum_{p=1}^{d_k} u_p^k \phi_p^k \in \mathcal{S}_j$$

and then the seminorm $|u|$ is the Euclidean length of the vector of its coefficient except of the initial level:

$$|u|^2 = \sum_{k=1}^{j} \sum_{p=1}^{d_k} |u_p^k|^2.$$

The topological structure of the stiffness matrix B_j associated with hierarchical piecewise quadratic basis is shown in FIGURE 1 for the finest level $j = 4$, which comes from (c) of Proposition 2.1.

3. Bounds of $b(u,u)$

In this section, using the ideas in [6] and a lower bound of the eigenvalues for a symmetric matrix which will be given in Appendix, we will give the explicit lower and upper bounds of $b(u,u)$ in terms of the Euclidean norm $|u_1^0|^2 + |u|^2$ of the hierarchical coefficient vector of u. Indeed the following theorem follows from Theorem 3.2 and 3.3 which will be proved in the rest of this section.

Theorem 3.1 *For every*

$$u = \sum_{k=0}^{j} \sum_{p=1}^{2^k} u_p^k \phi_p^k \in \mathcal{S}_j,$$

we have

$$c\ (|u_1^0|^2 + |u|^2) \le b(u,u) \le C\ h_j^{-1}\ (|u_1^0|^2 + |u|^2)$$

where $c = \frac{8}{3}$, $C = \frac{16(\sqrt{2}+1)}{3(\sqrt{2}-1)}$ and $h_j = 1/2^j$ is the mesh size of the $\mathcal{S}_j$. That is, for any nonzero vector $U = (u_1^0, u_1^1, \cdots, u_{d_1}^1, \cdots, u_1^j, \cdots, u_{d_j}^j)^t$, we have

$$c\, U^t U \le U^t B_j U \le C h_j^{-1}\ U^t U,$$

so that the condition number of B_j is about $\frac{11.6}{h_j}$.

3.1 Upper bound

In this subsection, we discuss an explicit upper bound for $b(u,u)$.

Lemma 3.1. *For all $u \in \mathcal{S}_j$, we have*

$$\frac{16}{3h_1}\ |u|^2 \le \sum_{k=1}^{j} |J_k u - J_{k-1} u|_{1,I}^2 \le \frac{16}{3h_j}\ |u|^2.$$

Proof. Let $u^k := J_k u - J_{k-1} u$ for each $k \le j$ and let $\mathcal{I}_p^{k-1}$ be the p-th interval of π_{k-1} which is subdivided into two intervals $\mathcal{I}_{2p-1}^k, \mathcal{I}_{2p}^k$ of π_k. Then $u^k \in \mathcal{V}_k$. Hence u^k is piecewise quadratic on these two intervals and vanishes at the nodes of $\mathcal{I} \cap \mathcal{N}_{k-1}$. Then $u^k|_{\mathcal{I}_p^{k-1}}$ is written as

$$u^k|_{\mathcal{I}_p^{k-1}} = u^k(x_{2p-1}^k)\ \phi_{2p-1}^k + u^k(x_{2p}^k)\ \phi_{2p}^k.$$

By (c) of Proposition 2.1, we have

$$|u^k|_{1,\mathcal{I}_p^{k-1}}^2 = \frac{16}{3h_k}(|u^k(x_{2p-1}^k)|^2 + |u^k(x_{2p}^k)|^2)$$

so that

$$|J_k u - J_{k-1} u|_{1,I}^2 = \sum_{p=1}^{d_{k-1}} |u^k|_{1,\mathcal{I}_p^{k-1}}^2 = \frac{16}{3h_k} \sum_{p=1}^{d_k} |u^k(x_p^k)|^2.$$

By summing over k from 1 to j, we have the conclusion.

Lemma 3.2. *For all $u \in \mathcal{V}_k$ and all $v \in \mathcal{V}_\ell$, we have*

$$b(u,v) \le \left(\frac{1}{\sqrt{2}}\right)^{|\ell-k|} |u|_{1,I}|v|_{1,I}.$$

Proof. Without loss of generality, we assume that $\ell > k$. Since there are $2^{\ell-k}$-subintervals of level ℓ in $\mathcal{I}_p^k \in \pi_k$, we can write $v \in \mathcal{V}_\ell$ as the form

$$v = \sum_{q=1+2^{\ell-k}(p-1)}^{2^{\ell-k}p} v(x_q^\ell)\phi_q^\ell \quad \text{on } \mathcal{I}_p^k.$$

On the other hand, since the interval $\mathcal{I}_p^k$ of π_k contains only one nodal point x_p^k in $\mathcal{I}_p^k$ for $\mathcal{V}^k$, $u = u(x_p^k)\ \phi_p^k$ on $\mathcal{I}_p^k$. Then, using (c) of Proposition 2.1, we have

$$\begin{aligned}\int_{\mathcal{I}_p^k} u'v'\,dx &= \sum_{q=1+2^{\ell-k}(p-1)}^{2^{\ell-k}p} u(x_p^k)v(x_q^\ell)\int_{\mathcal{I}_p^k}(\phi_p^k)'\ (\phi_q^\ell)'\,dx \\ &\le \left(\frac{1}{2\sqrt{2}}\right)^{\ell-k}|u|_{1,\mathcal{I}_p^k}\sum_{q=1+2^{\ell-k}(p-1)}^{2^{\ell-k}p}|v(x_q^\ell)\phi_q^\ell|_{1,\mathcal{I}_p^k}.\end{aligned}$$

Using Cauchy-Schwarz inequality and the orthogonality of ϕ_q^ℓ yields

$$\int_{\mathcal{I}_p^k} u'v'\,dx \le \left(\frac{1}{\sqrt{2}}\right)^{l-k}|u|_{1,\mathcal{I}_p^k}|v|_{1,\mathcal{I}_p^k}.$$

By summing these inequalities over $p = 1, 2, \cdots, d_k$ and then applying the Cauchy-Schwarz inequality to the right hand side of their sum, we have the conclusion.

Lemma 3.3 *For all $u \in \mathcal{S}_j$ we have*

$$b(u,u) \le \frac{\sqrt{2}+1}{\sqrt{2}-1}\left(|J_0u|_{1,I}^2 + \frac{16}{3h_j}|u|^2\right).$$

Proof. Let $u^0 := J_0u$ and $u^k := J_ku - J_{k-1}u$ $(k = 1, 2, \cdots, j)$ so that $u = \sum_{k=0}^j u^k$. By Lemma 3.2, we have that

$$b(u,u) = |u|_{1,I}^2 \le \sum_{k,\ell=0}^{j}\left(\frac{1}{\sqrt{2}}\right)^{|k-\ell|}|u^k|_{1,I}\ |u^\ell|_{1,I} = \tilde{U}^tA\tilde{U},$$

where the matrix $A = (a_{k\ell})$ and the vector $\tilde{U} = (\tilde{u}^k)$ are defined as

$$a_{k\ell} = \left(\frac{1}{\sqrt{2}}\right)^{|k-\ell|} \quad \text{and} \quad \tilde{u}^k := |u^k|_{1,I}.$$

Then, since A is symmetric and the spectral radius $\rho(A)$ of A is bounded by the largest row sum of A, we have

$$b(u,u) \le \rho(A)\tilde{U}^t\tilde{U} \le \frac{\sqrt{2}+1}{\sqrt{2}-1}\sum_{k=0}^{j}|u^k|_{1,I}^2. \tag{3.1}$$

By Lemma 3.1, we are led to

$$b(u,u) \le \frac{\sqrt{2}+1}{\sqrt{2}-1}\left(|J_0u|_{1,I}^2 + \frac{16}{3h_j}|u|^2\right)$$

which completes the proof.

Let $u = \sum_{k=0}^{j} \sum_{p=1}^{d_k} u_p^k \phi_p^k \in \mathcal{S}_j$ be the representation with respect to the hierarchical basis of $\mathcal{S}_j$. Then, by Proposition 1 (d), we have

$$|u|^2 = \sum_{k=1}^{j} \sum_{i=1}^{d_k} |u_i^k|^2 = |u_1^1|^2 + \cdots + |u_{d_1}^1|^2 + \cdots + |u_1^j|^2 + \cdots + |u_{d_j}^j|^2,$$

and

$$|J_0 u|_{1,I}^2 = |u_1^0 \ \phi_1^0|_{1,I}^2 = \frac{16}{3}|u_1^0|^2.$$

Hence combining these facts with Lemma 3.3 yields the upper bound of $b(u,u)$, which is stated as a theorem.

Theorem 3.2. *For every*

$$u = \sum_{k=0}^{j} \sum_{p=1}^{2^k} u_p^k \phi_p^k \in \mathcal{S}_j,$$

we have

$$b(u,u) \le C \ h_j^{-1}(|u_1^0|^2 + |u|^2) = C \ h_j^{-1} \sum_{k=0}^{j} \sum_{p=1}^{2^k} |u_p^k|^2$$

where $C := \frac{16(\sqrt{2}+1)}{3(\sqrt{4}2-1)}$ *and* $h_j = 1/2^j$ *is the mesh size of the* $\mathcal{S}_j$*.*

3.2 Lower bound

Here we reformulate the result given in Theorem A.2 in Appendix which plays an important role to get an explicit lower bound of $b(u,u)$.

Lemma 3.4. *For every sequence* $\{a_k\}_{k=0}^{j} \subset \mathbb{R}$ *with an integer* $j \ge 1$*, we also have*

$$\sum_{k=0}^{j} 4^k |a_k|^2 + 2\sum_{k=0}^{j} \sum_{\ell=k+1}^{j} 4^k a_k \ a_\ell \ge \frac{1}{2} \sum_{k=0}^{j} 2^k |a_k|^2. \tag{3.2}$$

Proof. Let $m = j+1$. By taking $x = 4$ and $\alpha = 2$ in Theorem A.2 (2), we have

$$\mathbf{x}_m^t M_m \mathbf{x}_m \ge \left(\frac{1}{2}\right) \mathbf{x}_m^t \hat{D}_m \mathbf{x}_m, \quad \mathbf{x}_m = (a_0, a_1, \cdots, a_j)^t, \tag{3.3}$$

where the symmetric matrix M_m is given by

$$M_m(p,q) = 4^{\ell-1} \quad \text{with } \ell = \min\{p,q\}, \ 1 \le p, \ q \le m$$

and the diagonal matrix $\hat{D}_m$ is given by $\hat{D}_m(p,p) = 2^{p-1}$. Now, expanding the inequality (3.3) in terms of the sequence $\{a_k\}_{k=0}^{j}$ yields the conclusion (3.2).

Let us turn to a lower bound of $b(u,u)$. First, recall that if $u \in \mathcal{S}_j$, then by (2.2)

$$b(u,u) = \sum_{k=0}^{j} b(u^k, u^k) + 2\sum_{k=0}^{j} \sum_{\ell=k+1}^{j} b(u^k, u^\ell). \tag{3.4}$$

We will reorder the sums in (3.4) to a particular form to be easily handled. Using (2.2), the orthogonality and Proposition 2.1, the first term of the right hand side in (3.4) can be written as

$$\begin{aligned} \sum_{k=0}^{j} b(u^k, u^k) &= \sum_{k=0}^{j} \left(\sum_{p,q=1}^{2^k} u_p^k u_q^k b(\phi_p^k, \phi_q^k) \right) = \sum_{k=0}^{j} \sum_{p=1}^{2^k} |u_p^k|^2 b(\phi_p^k, \phi_p^k) \\ &= \frac{16}{3} \sum_{k=0}^{j} \sum_{p=1}^{2^k} 2^k |u_p^k|^2. \end{aligned} \tag{3.5}$$

For $1 \le k \le j$, $1 \le q \le 2^j$, we define

$$\hat{u}_q^k := u_{q_k}^k \quad \text{where} \quad q_k := q_k(k,q) = \left\lfloor \frac{q-1}{2^{j-k}} \right\rfloor + 1$$

where q_k is the largest integer less than or equal to $\frac{q-1}{2^{j-k}}$. Then we have

$$\hat{u}_q^k = u_{q_k}^k \quad \text{for } 1 + (q_k - 1)2^{j-k} \le q \le q_k 2^{j-k},$$

that is to say,

$$u_p^k = \hat{u}_{1+2^{j-k}(p-1)}^k = \hat{u}_{2+2^{j-k}(p-1)}^k = \cdots = \hat{u}_{2^{j-k}p}^k, \tag{3.6}$$

hence

$$u_p^k = \left(\frac{1}{2}\right)^{j-k} \sum_{q=1+2^{j-k}(p-1)}^{2^{j-k}p} \hat{u}_q^k \quad \text{and} \quad |u_p^k|^2 = \left(\frac{1}{2}\right)^{j-k} \sum_{q=1+2^{j-k}(p-1)}^{2^{j-k}p} |\hat{u}_q^k|^2. \tag{3.7}$$

By applying (3.7) to (3.5) we have

$$\sum_{k=0}^{j} b(u^k,u^k) = \frac{16}{3}\left(\frac{1}{2}\right)^j \sum_{k=0}^{j}\sum_{p=1}^{2^k} \sum_{q=1+2^{j-k}(p-1)}^{2^{j-k}p} 4^k |\hat{u}_q^k|^2 = \frac{16}{3}\left(\frac{1}{2}\right)^j \sum_{k=0}^{j}\sum_{p=1}^{2^j} 4^k |\hat{u}_p^k|^2$$

or

$$\sum_{k=0}^{j} b(u^k,u^k) = \frac{16}{3}\left(\frac{1}{2}\right)^j \sum_{p=1}^{2^j} \left(\sum_{k=0}^{j} 4^k |\hat{u}_p^k|^2\right). \tag{3.8}$$

On the other hand, $b(u^k,u^\ell)$ in the second term of the right hand side in (3.4) is in fact

$$b(u^k,u^\ell) = \sum_{p=1}^{2^k}\sum_{q=1}^{2^\ell} u_p^k u_q^\ell \; b(\phi_p^k,\phi_q^\ell). \tag{3.9}$$

Note that if we let $\pi_j = \{\mathcal{I}_p^j\}_{p=1}^{2^j}$ be the partition of last level j, then we have the followings:

i) For fixed k and p $(0 \le k \le \ell \le j,\ 1 \le p \le 2^k)$, there exist only $2^{\ell-k}$ basis functions $\{\phi_q^\ell\}_{q=1+2^{\ell-k}(p-1)}^{2^{\ell-k}p}$ of $\mathcal{V}_\ell$ whose supports are contained in the support of ϕ_p^k.

ii) Since $\mathrm{supp}(\phi_q^\ell) \subset \mathrm{supp}(\phi_p^k)$, by Proposition 2.1 (c), we have

$$b(\phi_p^k,\phi_q^\ell) = \left(\frac{16}{3}\right) 2^{2k-\ell} \quad \text{for } q = 1 + 2^{\ell-k}(p-1), \cdots, 2^{\ell-k}p.$$

Using (3.9) and the above facts, we can rewrite

$$b(u^k,u^\ell) = \left(\frac{16}{3}\right) 2^{2k-\ell} \sum_{p=1}^{2^k} \left(u_p^k \sum_{q=1+2^{\ell-k}(p-1)}^{2^{\ell-k}p} u_q^\ell \right), \tag{3.10}$$

and then, by using (3.7) and (3.6) we have

$$\begin{aligned} b(u^k,u^\ell) &= \left(\frac{16}{3}\right) 2^{2k-\ell} \sum_{p=1}^{2^k} \left(u_p^k \sum_{q=1+2^{j-k}(p-1)}^{2^{j-k}p} \left(\frac{1}{2}\right)^{j-\ell} \hat{u}_q^\ell \right) \\ &= \frac{16}{3}\left(\frac{1}{2}\right)^j \sum_{p=1}^{2^k} \sum_{q=1+2^{j-k}(p-1)}^{2^{j-k}p} \left(4^k \hat{u}_q^k \hat{u}_q^\ell\right) \\ &= \frac{16}{3}\left(\frac{1}{2}\right)^j \sum_{q=1}^{2^j} 4^k \hat{u}_q^k \hat{u}_q^\ell. \end{aligned} \tag{3.11}$$

Hence, by substituting (3.8) and (3.11) into (3.4), we have

$$b(u,u) = \frac{16}{3}\left(\frac{1}{2}\right)^j \sum_{q=1}^{2^j}\left(\sum_{k=0}^{j} 4^k |\hat{u}_q^k|^2 + 2\sum_{k=0}^{j}\sum_{\ell=k+1}^{j} 4^k \hat{u}_q^k \hat{u}_q^\ell\right). \tag{3.12}$$

Theorem 3.3. *For every*

$$u = \sum_{k=0}^{j}\sum_{p=1}^{2^k} u_p^k \phi_p^k \in \mathcal{S}_j,$$

we have

$$b(u,u) \geq \frac{8}{3}(|u_1^0|^2 + |u|^2) = \frac{8}{3}\sum_{k=0}^{j}\sum_{p=1}^{2^k} |u_p^k|^2.$$

Proof. ¿From (3.2) and (3.12), we have

$$b(u,u) \geq \frac{8}{3}\left(\frac{1}{2}\right)^j \sum_{q=1}^{2^j}\sum_{k=0}^{j} 2^k\ |\hat{u}_q^k|^2 = \frac{8}{3}\left(\frac{1}{2}\right)^j \sum_{k=0}^{j}\left(2^k\ \sum_{q=1}^{2^j} |\hat{u}_q^k|^2\right).$$

Using (3.7) we have

$$\sum_{q=1}^{2^j} 2^k\ |\hat{u}_q^k|^2 = 2^k \sum_{p=1}^{2^k} 2^{j-k}\ |u_p^k|^2.$$

Combining these arguments yields that

$$b(u,u) \geq \frac{8}{3}\sum_{k=0}^{j}\sum_{p=1}^{2^k} |u_p^k|^2 = \frac{8}{3}(|u_1^0|^2 + |u|^2).$$

4. Preconditioning

Let us consider the linear system corresponding to the problem $Lu = f$ in $(0,1)$ with homogeneous Dirichlet boundary conditions based on the finite element space $\mathcal{S}_j$ using piecewise quadratic hierarchical basis such that

$$B_j\, U = F \tag{4.1}$$

where

$$B_j(p,q) = b(\phi_p, \phi_q), \quad F(p) = (f, \phi_p)$$

and

$$U = (u_1^0, u_1^1, \cdots, u_{d_1}^1, \cdots, u_1^j, \cdots, u_{d_j}^j)^t$$

with the basis $\{\phi_p\}_{p=1}^{2^{j+1}-1}$ reordered as

$$\phi_1^0,\ \phi_1^1,\ \phi_2^1,\ \phi_1^2, \cdots, \phi_{2^2}^2, \cdots, \phi_1^j, \cdots, \phi_{2^j}^j.$$

In this section we analyze a preconditioner P_j which is the diagonal matrix of B_j. In deed, P_j is the Jacobi preconditioner. The following theorem shows that the iteration numbers to solve the linear system (4.1) by Jacobi method do not depend on the finest level number.

Theorem 4.1. *Let P_j be the diagonal matrix such that*

$$P_j(p,p) = B_j(p,p) \quad \textit{for } p = 1, 2, \cdots, N$$

where $N := 2^{j+1} - 1$ *is the dimension of* $\mathcal{S}_j$. *Then we have*

$$\left(\frac{1}{4}\right) U^t P_j U \le U^t B_j U \le \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) U^t P_j U \quad \text{for } \forall j \ge 1. \tag{4.2}$$

That is, if λ *is an eigenvalue of* $P_j^{-1} B_j$, *then*

$$\frac{1}{4} \le \lambda \le \frac{\sqrt{2}+1}{\sqrt{2}-1}.$$

Proof. Using (b) of Proposition 2.1 and the fact that the main diagonal elements consist of $b(\phi_p^k, \phi_p^k)$ for $k = 0, 1, \cdots, j$ and $p = 1, 2, \cdots, d_k$, we get

$$U^t P_j U = \sum_{k=0}^{j} \sum_{p=1}^{d_k} b(u_p^k \phi_p^k, u_p^k \phi_p^k) = \sum_{k=0}^{j} b(u^k, u^k) \quad \text{with } u^k = \sum_{p=1}^{d_k} u_p^k \phi_p^k. \tag{4.3}$$

¿From (3.12) and (3.8) we have

$$\begin{aligned} U^t B_j U - \left(\frac{1}{4}\right) U^t P_j U &= b(u,u) - \left(\frac{1}{4}\right) \sum_{k=0}^{j} b(u^k, u^k) \\ &= \frac{16}{3} \left(\frac{1}{2}\right)^j \sum_{q=1}^{2^j} \left(\frac{3}{4} \sum_{k=0}^{j} 4^k |\hat{u}_q^k|^2 + 2 \sum_{k=0}^{j} \sum_{\ell=k+1}^{j} 4^k \hat{u}_q^k \hat{u}_q^\ell \right) \ge 0, \end{aligned}$$

where the last inequality comes from Theorem A.2 (1) with $x = 4$ and $\alpha = 2$. This completes the first inequality of (4.2). Now, from (3.1) in Lemma 3.3 we have

$$U^t B_j U = b(u,u) = |u|_{1,I}^2 \le \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \sum_{k=0}^{j} |u^k|_{1,I}^2 = \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) U^t P_j U.$$

This implies the second inequality of (4.2). Thus the proof is completed.

For numerical evidence, we compute the eigenvalues and condition numbers of B_j and the preconditioned system $P_j^{-1} B_j$. The following table reveals that numerical results are coincide with our theory developed.

Table 1. Eigenvalues and condition numbers of B_j and $P_j^{-1} B_j$

j	N	B_j			$P_j^{-1}B_j$		
level	Dim.	λ_{min}	λ_{max}	Cond.	λ_{min}	λ_{max}	Cond.
2	3	3.3812	12.6188	3.7321	0.5000	1.5000	3.1861
3	7	3.2747	26.0587	7.9577	0.4069	1.8431	4.9529
4	15	3.2728	52.3969	16.0098	0.3750	2.0856	6.2432
5	31	3.2728	104.8760	32.0448	0.3602	2.2619	7.2131
6	63	3.2728	209.7743	64.0964	0.3522	2.3934	7.9642
7	127	3.2728	419.5544	128.1945	0.3473	2.4937	8.5593
8	255	3.2728	839.1102	256.3894	0.3441	2.5716	9.0395
9	511	3.2728	1678.2209	512.7789	0.3419	2.6333	9.4325
10	1023	3.2728	3356.4418	1025.5579	0.3403	2.6828	9.7583
11	2047	3.2728	6712.8836	2051.1158	0.3391	2.7231	10.0311

A A Lower bound of eigenvalues for certain symmetric matrices

Consider positive symmetric matrices $M_m := M_m[x]$ of order $m \ge 1$ defined by

$$M_m(i,j) = x^{\ell-1} \ (x > 1) \quad \text{with } \ell = \min\{i,j\} \text{ for } 1 \le i,j \le m,$$

that is,

$$M_m = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & x & x & x & \dots & x \\ 1 & x & x^2 & x^2 & \dots & x^2 \\ 1 & x & x^2 & x^3 & \cdots & x^3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x & x^2 & x^3 & \dots & x^{m-1} \end{pmatrix}.$$

The matrices M_m play an important role to prove the lower bound of $b(u,u)$ in section 3 and it has somewhat interesting algebraic structure itself which is defined hierarchically. In this appendix, we give a positive lower bound for the eigenvalues of M_m, which is independent of the order m of M_m when $x > 1$. The uniform positive lower bound yields such matrices to be positive definite. In the rest of section we will prove that any eigenvalue λ_m of M_m satisfies

$$\lambda_m \geq \max_{1<\alpha<x} \left\{ \left(\frac{\alpha - 1}{\alpha} \right) \left(\frac{x - \alpha}{x} \right) \right\} \quad \text{for } 1 < x \leq 2$$

and

$$\lambda_m \geq \frac{x-2}{x-1} \quad \text{for } x > 2.$$

Let us begin with a simple lemma, which is evidently clear.

Lemma A.1. *Let us define a symmetric matrix $Q_{m,\ell} := Q_{m,\ell}[\alpha]$ of order m as*

$$Q_{m,\ell}(i,j) = \begin{cases} 1 & \text{if } i = j = \ell, \\ \alpha & \text{if } (i = \ell,\ i < j) \text{ or } (j = \ell,\ i > j), \\ \alpha^2 & \text{if } (\ell < i \leq j) \text{ or } (\ell < j < i), \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leq \ell \leq m-1$. Then $Q_{m,\ell}$ is nonnegative definite for all positive integers m and ℓ satisfying $1 \leq \ell \leq m-1$, that is, for any real number α and any vector $\mathbf{x}_m \in \mathbb{R}^m$

$$\mathbf{x}_m^t\, Q_{m,\ell}\, \mathbf{x}_m = (a_\ell + \alpha a_{\ell+1} + \alpha a_{\ell+2} + \cdots + \alpha a_m)^2 \geq 0.$$

Lemma A.2. *Define the sequence of matrices $S_n := S_n[x,\alpha]$ as*

$$S_n := M_n - (\frac{s_1}{\alpha})Q_{n,1} - (\frac{s_2}{\alpha})Q_{n,2} - \cdots - (\frac{s_{n-1}}{\alpha})Q_{n,n-1} \quad \text{for } n > 1$$

where $S_1 := M_1 = [1]$ and s_k denote the k^{th} diagonal element of S_k, i.e., $s_k = S_k(k,k)$.

Then, the matrices S_n are diagonal matrices and their diagonal elements are given by

$$S_n(i,i) := \begin{cases} \dfrac{(\alpha-1)\, s_i}{\alpha} & \text{if } i \neq n \\ s_n & \text{if } i = n \end{cases} \tag{4.4}$$

where $s_i = x^{i-1} - x^{i-2} - (\alpha-1)s_{i-1}$ for $i = 2, 3, \ldots, n$ with $s_1 := 1$.

Proof. We will prove this lemma using the mathematical induction. First, for the case of $k = 2$ we can easily show from $s_1 = 1$ that S_2 is diagonal matrix and

$$S_2(1,1) = \left(\frac{\alpha-1}{\alpha} \right) s_1,$$

$$S_2(2,2) = M_2(2,2) - \frac{s_1}{\alpha} Q_{2,1}(2,2) = x - x^0 - (\alpha-1)s_1.$$

Now, in order to show (4.4), we suppose that (4.4) holds for $k = n+1$. Set

$$\tilde{S}_{n+1} := M_{n+1} - (\frac{s_1}{\alpha})Q_{n+1,1} - (\frac{s_2}{\alpha})Q_{n+1,2} - \cdots - (\frac{s_{n-1}}{\alpha})Q_{n+1,n-1}.$$

Using the fact that, for any integer $m > 1$

$$M_m(k,\ell) = M_m(k,k) \quad \text{for } \ell \geq k,$$
$$Q_{m,k}(k,\ell) = Q_{m,k}(k,k) \quad \text{for } \ell \geq k,$$
$$Q_{m,k}(i,j) = Q_{m,k}(k+1,k+1) \quad \text{for } i,j \geq k+1,$$

we have

$$\tilde{S}_{n+1} = \begin{pmatrix} \left(\frac{\alpha-1}{\alpha}\right) \operatorname{diag}(s_i) & \vdots & \mathbf{O} \\ \cdots & \cdots & \cdots \\ \mathbf{O} & \vdots & \begin{matrix} s_n & s_n \\ s_n & \tilde{s}_{n+1} \end{matrix} \end{pmatrix}$$

where $\tilde{s}_{n+1} = x^n - x^{n-1} + s_n$. Since

$$S_{n+1} := \tilde{S}_{n+1} - (\frac{s_n}{\alpha})Q_{n+1,n},$$

$$S_{n+1} = \begin{pmatrix} \left(\frac{\alpha-1}{\alpha}\right) \operatorname{diag}(s_i) & \vdots & \mathbf{O} \\ \cdots & \cdots & \cdots \\ \mathbf{O} & \vdots & \begin{matrix} \left(\frac{\alpha-1}{\alpha}\right) s_n & 0 \\ 0 & s_{n+1} \end{matrix} \end{pmatrix}$$

where s_{n+1} is given by

$$s_{n+1} = \tilde{s}_{n+1} - \alpha s_n = x^n - x^{n-1} + s_n - \alpha s_n.$$

Hence, by the mathematical induction, (4.4) holds for any integer $n > 1$.

Theorem A.1. *Assume that the hypotheses of Lemma 2 hold. For a given real number $x > 1$, if we choose $\alpha > 1$ satisfying $x - \alpha > 0$, then for any positive integer m, $M_m := M_m[x]$ are positive definite matrices such that*

$$\mathbf{x}_m^t M_m \mathbf{x}_m \geq \left(\frac{\alpha-1}{\alpha}\right) \mathbf{x}_m^t \hat{S}_m \mathbf{x}_m > 0 \quad \text{for any nonzero vector } \mathbf{x}_m \in \mathbb{R}^m$$

where $\hat{S}_m := \hat{S}_m[x,\alpha]$ is the diagonal matrix such that $\hat{S}_m(k,k) = s_k$ is given by

$$s_k := x^{k-1} - x^{k-2} - (\alpha-1)s_{k-1} \quad \text{for } k = 1,2,\cdots,m, \text{ with } s_1 := 1.$$

Proof. Since $s_1 = 1$ and $x - \alpha > 0$, we have $0 < (x-\alpha) \leq s_2 \leq x$. If we suppose that $(x-\alpha)x^{n-2} \leq s_n \leq x^{n-1}$, then we have clearly $s_{n+1} = x^n - x^{n-1} - (\alpha-1)s_n \leq x^n$ and

$$s_{n+1} \geq x^n - x^{n-1} - (\alpha-1)x^{n-1} \geq (x-\alpha)x^{n-1} > 0.$$

Therefore, by mathematical induction, $s_k > 0$ for any positive integer k. Now, using the facts that $\left(\frac{\alpha-1}{\alpha}\right) s_k > 0$ and each $Q_{m,k}$ is nonnegative definite matrix, we obtain

$$\mathbf{x}_m^t M_m \mathbf{x}_m \geq \mathbf{x}_m^t S_m \mathbf{x}_m \geq \left(\frac{\alpha-1}{\alpha}\right) \mathbf{x}_m^t \hat{S}_m \mathbf{x}_m.$$

This completes the proof of theorem.

Theorem A.2. *We have the followings.*

1) For a given real number $x > 1$, if we choose $\alpha > 1$ satisfying $x - \alpha > 0$, then for any positive integer m and for any vector $\mathbf{x}_m \in \mathbb{R}^m$ we have

$$\mathbf{x}_m^t M_m \mathbf{x}_m \geq \left(\frac{\alpha-1}{\alpha}\right)\left(\frac{x-\alpha}{x}\right) \mathbf{x}_m^t D_m \mathbf{x}_m$$

where $D_m := D_m[x]$ is the main diagonal matrix of M_m, i.e., $D_m(k,k) = x^{k-1}$. If λ is any eigenvalue of M_m, then we have

$$\lambda \geq \max_{1<\alpha<x} \left\{ \left(\frac{\alpha-1}{\alpha}\right)\left(\frac{x-\alpha}{x}\right) \right\}.$$

2) For $x > 2$, if we choose $\alpha > 1$ satisfying $x - \alpha \geq 1$, then for any positive integer m and for any vector $\mathbf{x}_m \in \mathbb{R}^m$ we have

$$\mathbf{x}_m^t M_m \mathbf{x}_m \geq \left(\frac{\alpha - 1}{\alpha}\right) \mathbf{x}_m^t \hat{D}_m \mathbf{x}_m$$

where $\hat{D}_m := \hat{D}_m[x, \alpha]$ is the diagonal matrix such that $\hat{D}_m(k,k) = (x-\alpha)^{k-1}$. Hence, if λ is any eigenvalue of M_m, then by choosing $\alpha = x - 1$ we have

$$\lambda \geq \left(\frac{x-2}{x-1}\right).$$

Proof. Using the argument in the proof of Theorem 1, for the case of $x > 1$, the following inequality

$$s_k \geq (x-\alpha)x^{k-2} = \left(\frac{x-\alpha}{\alpha}\right) x^{k-1}$$

yields the conclusion (1) and, for the case of $x > 2$, using the facts that $x > x - \alpha$ and

$$s_k \geq (x-\alpha)x^{k-2} > (x-\alpha)^{k-1},$$

we get the conclusion (2). The results connected with eigenvalues are easily checked by observing that the diagonal elements of both D_m and $\hat{D}_m$ are all greater than or equal to 1. Thus, the proof is completed.

References

[1] R. Bank, Hierarchical bases and the finite element method, *Acta Numerica*, **5**(1996), 1–43.

[2] R. Bank and T. Dupont, Analysis of a two-level scheme for solving finite element equations, Tech. Report CNA-159, Center for Numerical Analysis, University of Texas at Austin, TX, 1980.

[3] R. Bank, T. Dupont and H. Yserantant, The hierarchical basis multigrid method, *Numer. Math.*, **52**(1988), 427–458.

[4] E. Ong, Hierarchical basis preconditioners for second order elliptic problems in three dimensions, PhD thesis, University of Washington, 1989.

[5] P. Oswald, Multilevel finite element approximation, Teubner Skripten Zur Numerik. Hrsg.: H. G. Bock, W. Hackbusch and R. Rannacher, Teubner, Stuttgart, 1994.

[6] H. Yserentant On the multi-level splitting of finite element spaces, *Numer. Math.*, **49**(1986), 379–412.

[7] O. C. Zienciewicz, D. W. Kelly, J. Gago and I. Babuška, Hierarchical finite element approaches, error estimates and adaptive refinement. In: The mathematics of finite elements and applications IV. (J. R. Whiteman ed.) London: Academic Press 1982.