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LEAST-SQUARES SOLUTIONS OF $X^TAX=B$ OVER POSITIVE SEMIDEFINITE MATRIXES A *1)

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Abstract

This paper is mainly concerned with solving the following two problems:

Problem I. Given $X \in R^{n\times m}$, $B \in R^{m\times m}$. Find $A \in P_n$ such that

$$\|X^TAX-B\|_F=\min,$$

where $$P_n=\{A\in R^{n\times n}|\quad x^TAx\geq 0,\quad \forall x\in R^n\}.$$

Problem II. Given $\tilde{A}\in R^{n\times n}$. Find $\hat{A}\in S_E$ such that

$$\|\tilde{A}-\hat{A}\|_F=\min_{A\in S_E}\|\tilde{A}-A\|_F,$$

where $\|\cdot\|_F$ is Frobenius norm, and S_E denotes the solution set of Problem I.

The general solution of Problem I has been given. It is proved that there exists a unique solution for Problem II. The expression of this solution for corresponding Problem II for some special case will be derived.

Key words: positive semidefinite matrix, Least-square problem, Frobenins norm

1. Introduction

[2] pointed out that $X^TAX=B$ comes from an inverse problem vibration theory. [2] has studied least-squares solutions where the unknown **A** is symmetric positive semidefinite, given the expression of general solution. It is more difficult to study least-squares solutions for the case that the unknown **A** is positive semidefinite (may be unsymmetric). In this paper we will discuss this problem. We will give the expression of general solution. Then we will discuss so called optimal approximation problem associated with $X^TAX=B$. That is: to find the optimal approximate of a given matrix $\tilde{A}$ by $A\in S_E$, where S_E is the solution set of the least-square problem of $X^TAX=B$. The existence and uniqueness of the solution for the problem is proved, the expression of the solution is derived for some conditions.

We denote the real $n\times m$ matrix space by $R^{n\times m}$, and $R^n=R^{n\times 1}$, the set of all matrices in $R^{n\times m}$ with rank r by $R_r^{n\times m}$, the set of all $n\times n$ orthogonal matrices by $OR^{n\times n}$, the set of all $n\times n$ symmetric matrices by $SR^{n\times n}$, the set of all $n\times n$ anti-symmetric matrices by $ASR^{n\times n}$, the column space, the null space and the Moore–penrose generalized inverse of a matrix A by $R(A), N(A), A^+$ respectively, the identity matrix of order n by I_n, the Frobenius norm of A by $\|A\|_F$. We define inner product in space $R^{n\times m}$, $(A,B)=\mathrm{tr}B^TA=\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}b_{ij}, \quad \forall A,B\in R^{n\times m}$. Then $R^{n\times m}$ is a Hilbert inner product space. The norm of a matrix defined by the inner product is Frobenius norm.

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Definition 1. $A \in R^{n\times n}$ is called positive semidefinite if $x^TAx \geq 0$ for every non-vanishing vector x in R^n and denoted by $A \geq 0$.

Let

$$P_n = \{A \in R^{n\times n} | x^TAx \geq 0, \quad \forall x \in R^n\},$$
$$SR_{\geq}^{n\times n} = \{A \in R^{n\times n} | \quad A = A^T, \quad x^TAx \geq 0, \quad \forall x \in R^n\}.$$

Now we consider the following problems:

Problem I. Given $B \in R^{m\times m}, X \in R_r^{n\times m}$. Find $A \in P_n$ such that

$$f(A) = \|X^TAX - B\|_F = \min.$$

Problem II. Given $\tilde{A} \in R^{n\times n}$. Find $\hat{A} \in S_E$ such that

$$\|\tilde{A} - \hat{A}\|_F = \min_{A\in S_E} \|\tilde{A} - A\|_F,$$

where S_E is the solution set of Problem I.

In [2] the symmetric positive semidefinite and positive definite real solutions of $\|X^TAX - B\|_F = \min$ have been considered. And Problem II has not been studied (or the optimal approximate solution has not been studied).

At first, in this paper, we will discuss the optimal approximate problem on P_n. Then we will give the general solution of Problem I. At last, we will prove that there exists a unique solution for Problem II and derive the expression of this solution for some special case.

2. THE OPTIMAL APPROXIMATE PROBLEM ON P_n

Problem MA. Given nonempty closed convex cone $S \subseteq R^{n\times n}$, $F \in R^{n\times n}$, $D = \text{diag}\,(d_1, \cdots, d_n)$, $d_i > 0$, $i = 1, \cdots, n$. Find $\hat{E} \in S$ such that

$$\|D(\hat{E} - F)D\| \leq \|D(E - F)D\|, \quad \forall E \in S.$$

To solve Problem MA we introduce a conclusion.

Lemma 2.1[4]**.** *Suppose V is a real Hilbert space, $(\cdot,\cdot)$ denotes inner product , $\|u\|_V = \sqrt{(u,u)}$ represents norm in V, $K \subset V$ is a nonempty closed convex cone. $K^\perp$ represents the set of all elements which are orthogonal to K in V. It is obvious that $K^\perp$, $K^{\perp\perp} \triangleq (K^\perp)^\perp$ are closed linear subspace in V. $K^{\perp\perp}$ is the minimum subspace that concludes K. K^* is the dual cone of K in $K^{\perp\perp}$. Then, for every $u \in V$, there is an unique $u_0 \in K^\perp, u_1 \in K, u_2 \in K^*$ such that*

$$(u_1, u_2) = 0, \qquad u = u_0 + u_1 - u_2$$

and

$$\|u - u_1\|_V \leq \|u - v\|_V, \quad \forall v \in K$$

In $R^{n\times n}$ we define a new inner product and norm:

$$(A, B)_D = (DAD, DBD) = tr(DB^TD^2AD), \quad \|A\|_D = \sqrt{(A,A)_D} = \sqrt{(DAD, DAD)}$$

where $D = \text{diag}(d_1, \cdots, d_n)$, $d_i > 0$, $i = 1, \cdots, n$. This new Euclidean space is noted by $R_D^{n\times n}$. Therefore Problem MA is equivalent to

$$\|\hat{E} - F\|_D \leq \|E - F\|_D, \quad \forall E \in S \subseteq R_D^{n\times n}.$$

We point out P_n as a closed convex cone with vertex at zero point. In fact, it is evident that P_n is closed. And for any $\alpha \geq 0, \beta \geq 0$, there is $\alpha P_n + \beta P_n \subseteq P_n$. According to the definition of convex cone P_n is a closed convex cone.

By Lemma 2.1 Problem MA has a unique optimal approximate solution.

Lemma 2.2[4]**.** *For every matrix F of order n there are an anti-symmetric matrix F_0, a symmetric nonnegative definite matrix F_+ and a symmetric nonpositive definite matrix F_- such that*

$$F = F_0 + F_+ + F_-,$$

$$\text{and} \qquad (F_+,F_-)=tr(F_+^TF_-)=0, \quad F_+F_-=0.$$

We denote the symmetric nonegative definite matrix and the symmetric nonpositive definite matrix above decomposition of F by $[F]_+$ and $[F]_-$ respectively. The computing method of $[F]_+$ is the same as [4]

Theorem 2.1. *When $S=P_n$ in problem MA. For every given $F\in R^{n\times n}$ there exists a unique $\hat{E}\in P_n$ such that*

$$\|F-\hat{E}\|_D=\min_{E\in P_n}\|F-E\|_D, \tag{2.1}$$

and

$$\hat{E}=D^{-1}[\frac{DFD+DF^TD}{2}]_+D^{-1}+D^{-1}\frac{DFD-DF^TD}{2}D^{-1}. \tag{2.2}$$

Proof. At first, by above discussed conclusion we know that there exists a unique $\hat{E}\in P_n$ such that (2.1) holds.

Next, we give the expression of $\hat{E}$. From Lemma 2.2 we know $D^{-1}[\frac{DFD+DF^TD}{2}]_+D^{-1}$, $D^{-1}[\frac{DFD+DF^TD}{2}]_-D^{-1}$, $D^{-1}\frac{DFD-F^TD}{2}D^{-1}$ are orthogonal in $R_D^{n\times n}$. Hence there is a matrix C which is orthogonal to $D^{-1}[\frac{DFD+DF^TD}{2}]_+D^{-1}$, $D^{-1}[\frac{DFD+DF^TD}{2}]_-D^{-1}$, $D^{-1}\frac{DFD-DF^TD}{2}D^{-1}$ in $R_D^{n\times n}$ and for any $E\in R^{n\times n}$,

$$\begin{aligned} E &= \lambda_1D^{-1}[\frac{DFD+DF^TD}{2}]_+D^{-1}+\lambda_2D^{-1}[\frac{DFD+DF^TD}{2}]_-D^{-1} \\ &+\lambda_3D^{-1}\frac{DFD-DF^TD}{2}D^{-1}+C. \end{aligned}$$

When $E\in P_n$, taking the inner product with $D^{-1}[\frac{DFD+DF^TD}{2}]_-D^{-1}$ on above equation, from
$(E,D^{-1}[\frac{DFD+DF^TD}{2}]_-D^{-1})_D\le 0$ we have $\lambda_2\le 0$. Therefore

$$\begin{aligned} &\|E-F\|_D^2 \\ &=\|E-D^{-1}[\frac{DFD+DF^TD}{2}]_-D^{-1} \\ &-D^{-1}[\frac{DFD+DF^TD}{2}]_+D^{-1}-D^{-1}\frac{DFD-D^TD}{2}D^{-1}\|_D^2 \\ &=\|(1-\lambda_1)D^{-1}[\frac{DFD+DF^TD}{2}]_+D^{-1}\|_D^2+\|(1-\lambda_2)D^{-1}[\frac{DFD+DF^TD}{2}]_-D^{-1}\|_D^2 \\ &+\|(1-\lambda_3)D^{-1}\frac{DFD-DF^TD}{2}D^{-1}\|_D^2+\|C\|_D^2\ge\|(1-\lambda_2)D^{-1}[\frac{DFD+DF^TD}{2}]_-D^{-1}\|_D^2 \\ &\ge\|D^{-1}[\frac{DFD+DF^TD}{2}]_-D^{-1}\|_D^2. \end{aligned}$$

i.e $\|E-F\|_D\ge\|D^{-1}[\frac{DFD+DF^TD}{2}]_-D^{-1}\|_D$, $\forall E\in P_n$.
On the other hand, let

$$E=D^{-1}[\frac{DFD+DF^TD}{2}]_+D^{-1}+D^{-1}\frac{DFD-DF^TD}{2}D^{-1}.$$

Then

$$\|E-F\|_D=\|D^{-1}[\frac{DFD+DF^TD}{2}]_-D^{-1}\|_D.$$

Hence

$$\hat{E}=D^{-1}[\frac{DFD+DF^TD}{2}]_+D^{-1}+D^{-1}\frac{DFD-DF^TD}{2}D^{-1}.$$

Corollary 1[4]. When $D=I_n$. Then $\hat{E}$ in Problem MA is

$$\hat{E}=[\frac{F+F^T}{2}]_++\frac{F-F^T}{2}$$

3. Solution of Problem I

Lemma 3.1[5]. *Suppose $\bar{H} \in R^{n\times n}$ has following block matrix*

$$\bar{H} = \begin{pmatrix} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{21} & \bar{H}_{22} \end{pmatrix}.$$

If $\bar{H} = \bar{H}^T$, then $\bar{H} \geq 0$ if and only if

$$\begin{cases} \bar{H}_{11} = \bar{H}_{11}^T, \quad \text{and} \quad \bar{H}_{11} \geq 0, & (3.1) \\ R(\bar{H}_{12}) \subseteq R(\bar{H}_{11}\bar{H}_{11}^+), \quad \text{or} \quad \bar{H}_{11}\bar{H}_{11}^+\bar{H}_{12} = \bar{H}_{12} & (3.2) \\ \bar{H}_{22} = \bar{H}_{22}^T, \quad \text{and} \quad \bar{H}_{22} - \bar{H}_{12}^T\bar{H}_{11}^+\bar{H}_{12} \geq 0. & (3.3) \end{cases}$$

Suppose $U \in OR^{n\times n}$, $A \in R^{n\times n}$. Let

$$\bar{A} = U^TAU = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}, \quad \bar{A}_{11} \in R^{r\times r}. \tag{3.4}$$

We know $\bar{A} \geq 0$ is equivalent to $\frac{\bar{A}+\bar{A}^T}{2} \geq 0$, using of Lemma 3.1 we have $\bar{A}_{11} \geq 0$ from $\bar{A} \geq 0$.

Now we discuss Problem I.

Theorem 3.1. *Given $B \in R^{n\times m}, X \in R_r^{n\times m}$ and X has following factorization*

$$X = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^T = U_1\Sigma V_1^T. \tag{3.5}$$

where $U = (U_1, U_2) \in OR^{n\times n}$, $U_1 \in R^{n\times r}$, $V = (V_1, V_2) \in OR^{m\times m}$, $V_1 \in R^{m\times r}$, $\Sigma = diag(\sigma_1, \cdots, \sigma_r)$, $\sigma_1 > \sigma_2 > \cdots > \sigma_r > 0$. Then the general solution of Problem I can be represented as

$$A = U \begin{pmatrix} A_{11}^0 & -Z^T + \Sigma^{-1}[V_1^TBV_1 + V_1^TB^TV_1]_+\Sigma^{-1}[\Sigma^{-1}(V_1^TBV_1 + V_1^TB^TV_1)\Sigma^{-1}]_+^+Y \\ Z & \frac{Y^T(\Sigma^{-1}[V_1^TBV_1 + V_1^TB^TV_1]_+\Sigma^{-1})^+Y}{2} + P \end{pmatrix} U^T,$$
$$\forall P \in P_{n-r}, \qquad Z \in R^{(n-r)\times r}, Y \in R^{r\times(n-r)} \tag{3.6}$$

where

$$A_{11}^0 = \Sigma^{-1}[\frac{V_1^TBV_1 + V_1^TB^TV_1}{2}]_+\Sigma^{-1} + \Sigma^{-1}\frac{V_1^TBV_1 - V_1^TB^TV_1}{2}\Sigma^{-1}. \tag{3.7}$$

Proof. Let $\bar{A}$ has the partition in (3.4). Let

$$V^TBV = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad B_{11} \in R^{r\times r}. \tag{3.8}$$

Using orthogonal invariance of norm and attention to (3.5), (3.8) and (3.4) we have

$$\begin{aligned} f^2(A) = \|X^TAX - B\|^2 &= \left\| \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} U^TAU \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} - V^TBV \right\|^2 \\ &= \|\Sigma\bar{A}_{11}\Sigma - B_{11}\|^2 + \|B_{21}\|^2 + \|B_{12}\|^2 + \|B_{22}\|^2. \end{aligned} \tag{3.9}$$

It is obvious that $\bar{A} \geq 0$ by $A \geq 0$. Therefore $\|X^TAX - B\| = \min_{\forall A\in P_n}$ is equivalent to

$$\|\Sigma\bar{A}_{11}\Sigma - B_{11}\|_F = \|\bar{A}_{11} - \Sigma^{-1}B_{11}\Sigma^{-1}\|_\Sigma = \min, \quad \bar{A}_{11} \in P_r. \tag{3.10}$$

From Theorem 2.1 we obtain the solution A_{11}^0 of (3.10) as

$$A_{11}^0 = \Sigma^{-1}[\frac{B_{11} + B_{11}^T}{2}]_+\Sigma^{-1} + \Sigma^{-1}\frac{B_{11} - B_{11}^T}{2}\Sigma^{-1}. \tag{3.11}$$

We known $\bar{A} \geq 0$ is equivalent to $\dfrac{\bar{A}+\bar{A}^T}{2} \geq 0$, therefore from Lemma 3.1

$$\bar{A}_{12} = -\bar{A}_{21}^T + L \tag{3.12}$$

$$R(L) \subset R((A_{11}^0 + (A_{11}^0)^T)(A_{11}^0 + (A_{11}^0)^T)^+)$$

or

$$L = (A_{11}^0 + (A_{11}^0)^T)(A_{11}^0 + (A_{11}^0)^T)^+ Y, \quad \forall Y \in R^{r\times(n-r)} \tag{3.13}$$

$\bar{A}_{22}$ satisfies

$$\begin{aligned} & \bar{A}_{22} + \bar{A}_{22}^T - (\bar{A}_{12} + \bar{A}_{21}^T)^T (A_{11}^0 + (A_{11}^0)^T)^+ (\bar{A}_{12} + \bar{A}_{21}^T) \\ = \; & \bar{A}_{22} + \bar{A}_{22}^T - Y^T (A_{11}^0 + (A_{11}^0)^T)^+ Y \in SR_{\geq}^{(n-r)\times(n-r)} \end{aligned}$$

Write

$$\bar{A}_{21} = Z, \qquad Z \in R^{(n-r)\times r}. \tag{3.14}$$

$$\bar{A}_{22} + \bar{A}_{22}^T - Y^T (A_{11}^0 + (A_{11}^0)^T)^+ Y \triangleq Q, \quad Q = Q^T, \quad Q \in SR_{\geq}^{(n-r)\times(n-r)}$$

i.e

$$Q = \bar{A}_{22} + \bar{A}_{22}^T - Y^T (A_{11}^0 + (A_{11}^0)^T)^+ Y$$

Let

$$P = \frac{Q}{2} + \frac{\bar{A}_{22} - \bar{A}_{22}^T}{2}.$$

Then $P \in P_{n-r}$ and

$$\bar{A}_{22} = \frac{Y^T (A_{11}^0 + (A_{11}^0)^T)^+ Y}{2} + P. \tag{3.15}$$

Because $[\dfrac{B_{11}+B_{11}^T}{2}]_+ = \dfrac{[B_{11}+B_{11}^T]_+}{2}$ from (3.11), (3.13), (3.15) we have

$$L = \Sigma^{-1}[B_{11}+B_{11}^T]_+ \Sigma^{-1} (\Sigma^{-1}[B_{11}+B_{11}^T]_+ \Sigma^{-1})^+ Y. \tag{3.16}$$

$$\bar{A}_{22} = \frac{Y^T (\Sigma^{-1}[B_{11}+B_{11}^T]_+ \Sigma^{-1})^+ Y}{2} + P.$$

Attention to (3.8) we have

$$\bar{A}_{22} = \frac{Y^T (\Sigma^{-1}[V_1^T B V_1 + V_1^T B^T V_1]_+ \Sigma^{-1})^+ Y}{2} + P. \tag{3.17}$$

Substituting (3.11), (3.12), (3.14), (3.16), (3.17) into (3.4) We can obtain the general solution of Problem I as (3.6)

Theorem 3.2. *Suppose $B \in R^{n\times m}$, $X \in R_r^{n\times m}$ and X has the factorization in the form (3.5), then $f(A) = 0$ in Problem I has a solution if and only if*

$$B \geq 0, \quad BX^+X = B, \quad X^+XB = B \tag{3.18}$$

in which case the general nonnegative definite solution are

$$A = U \begin{pmatrix} \Sigma^{-1}V_1^T B V_1 \Sigma^{-1} & -Z^T + \Sigma^{-1}V_1^T(B+B^T)V_1\Sigma^{-1}(\Sigma^{-1}V_1^T(B+B^T)V_1\Sigma^{-1})^+ Y \\ Z & \dfrac{Y^T(\Sigma^{-1}V_1^T(B+B^T)V_1\Sigma^{-1})^+ Y}{2} + P \end{pmatrix} U^T,$$
$$\forall P \in P_{n-r}, \qquad Z \in R^{(n-r)\times r}, Y \in R^{r\times(n-r)} \tag{3.19}$$

Proof. We known $\bar{A} \geq 0$ is equivalent to $\dfrac{\bar{A}+\bar{A}^T}{2} \geq 0$, therefore from Lemma 3.1 and (3.9) $f(A) = 0$ has a solution $A \in P_n$ if and only if

$$\bar{A}_{11} = \Sigma^{-1} B_{11} \Sigma^{-1} \geq 0, \quad B_{21} = 0, \quad B_{12} = 0, \quad B_{22} = 0. \tag{3.20}$$

in the form (3.20) the second, the third and the forth equation is equivalent to $BV_2 = 0$, $V_2^T B = 0$ i.e $BX^+X = B$, $X^+XB = B$. From the first equation of (3.20) we can demonstrated that $B \geq 0$. On the other hand, we can demonstrated that $\tilde{A}_{11} \geq 0$ from $B \geq 0$. Therefore $f(A) = 0$ has a solution in Problem I if and only if (3.18) holds.

When B, X satisfy the (3.18). It follows that $[V_1^T BV_1 + V_1^T B^T V_1]_+ = V_1^T BV_1 + V_1^T B^T V_1$. Therefore in (3.6) $A_{11}^0 = \Sigma^{-1} V_1^T BV_1 \Sigma^{-1}$ Therefore in the form (3.6) becomes (3.19).

4. The Expression of the Solution for Problem II

In general solution (3.6) of Problem I we fix a Y at randon. Then we get a subset of the solution set S_E for Problem I, write $S_{E,Y}$.

Lemma 4.1. *Let*

$$A_Y = U \begin{pmatrix} A_{11}^0 & -Z^T + \Sigma^{-1}[V_1^T BV_1 + V_1^T B^T V_1]_+ \Sigma^{-1}(\Sigma^{-1}[V_1^T BV_1 + V_1^T B^T V_1]_+ \Sigma^{-1})^+ Y \\ Z & \dfrac{Y^T(\Sigma^{-1}[V_1^T BV_1 + V_1^T B^T V_1]_+ \Sigma^{-1})^+ Y}{2} \end{pmatrix} U^T,$$

$$\forall Y \in R^{r\times(n-r)}, \quad \forall Z \in R^{(n-r)\times r}.$$

Then $S_{E,Y}$ is a closed convex cone with vertex A_Y.

Proof. In the form of (3.6) taking P as zero matrix we can know clearly $A_Y \in S_{E,Y}$.Take any two matrices of $S_{E,Y}$

$$A_1 = A_Y + U_2 P_1 U_2^T, \quad A_2 = A_Y + U_2 P_2 U_2^T, \quad P_1, P_2 \in P_{n-r},$$

where U_2 is the same as Theorem 3.1. Let

$$C' \triangleq A_Y + \alpha(A_1 - A_Y) + \beta(A_2 - A_Y), \qquad \forall \alpha, \beta \geq 0.$$

Then

$$C' = A_Y + U_2(\alpha P_1 + \beta P_2)U_2^T \in P_n$$

By the definition, we know that $S_{E,Y}$ is a convex cone with vertex A_Y, clearly it is closed.

Corollary 2. Solution set

$$S_E = \bigcup_{Y\in R^{r\times(n-r)}} S_{E,Y}$$

i.e. S_E is union set of many closed convex cones. And S_E is a closed convex set.

Proof. It is clear that S_E is a closed set from Lemma 4.1. Next, we will prove that S_E is convex. $\forall A_1 \in S_E$, $\forall A_2 \in S_E$, A_1, $A_2 \in P_n$. Therefore, $\forall \alpha \geq 0$, $\forall \beta \geq 0$ and $\alpha + \beta = 1$. We have $\alpha A_1 + \beta A_2 \in P_n$ and if $A_1, A_2 \in S_E$ we have $\|X^T A_1 X - B\| = \min = c_0$, $\|X^T A_2 X - B\| = \min = c_0$. Hence $c_0 \leq \|X^T(\alpha A_1 + \beta A_2)X - B\| \leq \|\alpha(X^T A_1 X - B)\| + \|\beta(X^T A_2 X - B)\| = \alpha c_0 + \beta c_0 = c_0$. It implies that $\|X^T(\alpha A_1 + \beta A_2)X - B\| = \min$. Thus $\alpha A_1 + \beta A_2 \in S_E$. By the definition of convex set we obtain that S_E is convex. This corollary has been completed.

Lemma 4.2[3]**.** *Given $W \in R^{n\times m}, R \in R^{n\times m}$. Then*

$$\|T - R\|^2 + \|T - W\|^2 = \min, \quad \forall T \in R^{n\times m}$$

has the solution $T = \dfrac{R+W}{2}$.

Theorem 4.1. *Let $B \in R^{n\times m}, X \in R_r^{n\times m}$, $\tilde{A} \in R^{n\times n}$ and X has the factorization in the form (3.5). The notation is the same as Theorem 3.1. Let*

$$U^T \tilde{A} U = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \quad \tilde{A}_{11} \in R^{r\times r}. \tag{4.1}$$

Then Problem II has a unique optimal approximate solution. When $B = -B^T$ the optimal approximate solution of corresponding Problem II can be represented as

$$\hat{A} = U \begin{pmatrix} A_{11}^0 & \tilde{A}_{12} - \tilde{A}_{21}^T \\ \tilde{A}_{21} - \tilde{A}_{12}^T & \hat{P} \end{pmatrix} U^T \tag{4.2}$$

where

$$A_{11}^0 = \Sigma^{-1}\frac{V_1^TBV_1 - V_1^TB^TV_1}{2}\Sigma^{-1} \tag{4.3}$$

$$\hat{P} = [\frac{\tilde{A}_{22}+\tilde{A}_{22}^T}{2}]_+ + \frac{\tilde{A}_{22}-\tilde{A}_{22}^T}{2}. \tag{4.4}$$

Proof.

$$N = [V_1^TBV_1 + V_1^TB^TV_1]_+ = 0 \in SR_{\geq}^{r\times r}. \tag{4.5}$$

Using orthogonal invariance of norm attention (3.7) we have

$$\begin{aligned}&\left\|\tilde{A} - U\begin{pmatrix} A_{11}^0 & -Z^T \\ Z & P\end{pmatrix}U^T\right\|^2\\ &= \left\|U^T\tilde{A}U - \begin{pmatrix} A_{11}^0 & -Z^T \\ Z & P\end{pmatrix}\right\|^2\\ &= \|\tilde{A}_{11} - A_{11}^0\|^2 + \|\tilde{A}_{12} + Z^T\|^2 + \|\tilde{A}_{21} - Z\|^2 + \|\tilde{A}_{22} - P\|^2\end{aligned}$$

Therefore $\|\tilde{A} - A\|^2 = \min\limits_{\forall A\in S_E}$ is equivalent to

$$\|\tilde{A}_{21} - Z\|^2 + \|\tilde{A}_{12}^T + Z\|^2 = \min, \quad Z \in R^{(n-r)\times r}, \tag{4.6}$$

$$\|\tilde{A}_{22} - P\| = \min, \quad \forall Y \in R^{r\times(n-r)}. \tag{4.7}$$

Hence from Lemma 4.2, (4.5) reaches minimum if and only if

$$Z = \tilde{A}_{21} - \tilde{A}_{12}^T \tag{4.8}$$

From Corollary 1 (4.6) reaches minimum if and only if

$$P = \hat{P} \triangleq [\frac{\tilde{A}_{22}+\tilde{A}_{22}^T}{2}]_+ + \frac{\tilde{A}_{22}-\tilde{A}_{22}^T}{2} \tag{4.9}$$

Substituting (4.5), (4.6) and (4.7) into (3.6) we obtain the unique optimal approximation solution as (4.2).

5. The Algorithm Analysis of the Unique Solution for Problem II in Case $B = -B^T$

According to Theorem 4.1, now we give an algorithm of the optimal approximate solution of Problem II as the following steps:

(1) if $B + B^T = 0$, then next,

(2) calculate Singular Values Decomposition of X we get U, V

(3) calculate A_{11}^0 according to (4.3),

(4) calculate $P = \dfrac{\tilde{A}_{22}+\tilde{A}_{22}^T}{2}$.

(5) calculate eigenvalues of P, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-r}$, and corresponding unit eigenvectors u_1,u_2, $\cdots$, u_{n-r}.

(6) find the minimum positive eigenvalue and write it as λ_k, calculate $\hat{P} = \sum\limits_1^k \lambda_i u_i u_i^T + \dfrac{\tilde{A}_{22}-\tilde{A}_{22}^T}{2}$.

(7) according to (4.2), calculate $\hat{A}$.

(8) stop.

In above steps we use stable Singular Values Decomposition. In [4] stability has been analysied for step (5). Hence, this algorithm is stable.

Example.

$$X = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 2 & 6 \\ 0.5 & 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 & -4 \\ -3 & 0 & 1 \\ 4 & -1 & 0 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} -3 & 2 & -6 & 0.7 \\ -4 & 0 & 2.5 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -2 & -1 \end{pmatrix},$$

Suppose Singular Values Decomposition of X as

$$X = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^T$$

$$U = \begin{pmatrix} -0.0787 & 0.2253 & 0.6064 & 0.7585 \\ 0.2359 & 0.2470 & 0.7096 & -0.6163 \\ 0.8865 & 0.3410 & -0.2488 & 0.1896 \\ 0.3902 & -0.8786 & 0.2585 & 0.0948 \end{pmatrix}, \quad V = \begin{pmatrix} -0.0754 & -0.0882 & 0.9932 \\ 0.4820 & -0.8752 & -0.0411 \\ 0.8729 & 0.4756 & 0.1085 \end{pmatrix},$$

$$\Sigma = \mathrm{diag}(7.0807, 3.4935, 1.7056), \quad \Sigma^{-1} = \mathrm{diag}(0.1412, 0.2862, 0.5863)$$

$$A_{11}^0 = \begin{pmatrix} 0.0000 & 0.0467 & 0.1790 \\ -0.0467 & 0.0000 & 0.7503 \\ -0.1790 & -0.7503 & 0.0000 \end{pmatrix}, \quad \hat{P} = 0$$

The unique solution of corresponding Problem II is

$$\hat{A} = \begin{pmatrix} 0.0000 & -1.2942 & 0.2576 & -1.4684 \\ 1.2942 & 0.0000 & -0.4950 & 2.1360 \\ -0.2576 & 0.4950 & 0.00000 & -0.6181 \\ 1.4684 & -2.1360 & 0.6181 & 0.0000 \end{pmatrix}$$

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