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FOURIER-CHEBYSHEV COEFFICIENTS AND GAUSS-TURÁN QUADRATURE WITH CHEBYSHEV WEIGHT*1)

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Abstract

The main purpose of this paper is to derive an explicit expression for Fourier-Chebyshev coefficient $A_{kn}(f) = \frac{2}{\pi}\int_{-1}^{1} f(x)T_{kn}(x)\frac{dx}{\sqrt{1-x^2}}, k, n \in \mathcal{N}_0$, which is initiated by L.Gori and C.A.Micchelli.

Key words: Fourier-Chebyshev coefficient, Gauss-Turán quadrature

1. Introduction

Throughout this paper let $x_1, \cdots, x_n$ be zeros of the Chebyshev polynomial of first kind $T_n(x) = \cos(n \arccos x), |x| \leq 1$ and $\mathcal{N}$ the set of the natural numbers. Let the points $\xi_1, ..., \xi_n$ be arbitrary and $\mathcal{P}_k$ the space of all polynomials of degree $\leq k$,then there exist weights $\lambda_1, ..., \lambda_n$ such that the numerical quadrature of the type

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} \lambda_i f(\xi_i) \tag{1}$$

is exact for $f \in \mathcal{P}_{n-1}$. But it is exact for $f \in \mathcal{P}_{2n-1}$ if the points $\xi_1, ..., \xi_n$ are the zeros of the Legendre polynomial of degree n. Moreover, there is no quadrature using a linear combination of n values of f such that Eq.(1) holds for all polynomials of degree $2n$. This classical result is the well-known Gauss-Legendre quadrature. Because of the above theorem of Gauss it is natural to ask whether the points $\xi_1, ..., \xi_n$ can be chosen so that quadrature rules of the form

$$\int_{-1}^{1} f(x)w(x)dx = \sum_{i=1}^{n}\sum_{j=0}^{2s} \lambda_{ij} f^{(j)}(\xi_i) \tag{2}$$

will be exact for all $f \in \mathcal{P}_{2(s+1)n-1}$,where $w(x)$ is a weight function. In his interesting paper [13],Turán showed that the answer is positive. Moreover,he showed that the n zeros $\xi_1, ..., \xi_n$ of the monic polynomials of degree n minimizing the expression

$$\int_{-1}^{1} |p(x)|^{2s+2} w(x)dx \tag{3}$$

over all such polynomials gives a quadrature of maximum degree of accuracy,

$$\int_{-1}^{1} f(x)w(x)dx = \sum_{i=1}^{n} \lambda_i f(\xi_i), \qquad f \in \mathcal{P}_{2(s+1)n-1}. \tag{4}$$

As Turán pointed out in [14],particularly interesting is the case when

$$w(x) = (1-x^2)^{-\frac{1}{2}}. \tag{5}$$

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In 1930, S.Bernstein [1] showed that $2^{1-n}T_n(x)$ minimizes all integrals of the type

$$\int_{-1}^{1} \frac{|p_n(x)|^k}{\sqrt{1-x^2}} dx, \qquad k \in \mathcal{N}. \tag{6}$$

So the Turán-Chebyshev formula

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \sum_{i=1}^{n} \sum_{j=0}^{2s} \lambda_{ij} f^{(j)}(x_{in}) \tag{7}$$

with $x_i = \cos\frac{(2i-1)\pi}{2n}, i = 1, ..., n$, is exact for $f \in \mathcal{P}_{2(s+1)n-1}$. Turán [14] has raised

Problem 26. Give an explicit formula for λ_{ij} and determine its asymptotic behavior as $n \to \infty$.

In this regard, Micchelli and Rivlin [6] have proved the following

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{n}\{\sum_{i=1}^{n} f(x_i) + \sum_{j=1}^{s} \frac{1}{2j4^{jn}} \binom{2j}{j} f'[x_1^{2j}, ..., x_n^{2j}]\}, \tag{8}$$

where $f[x_1^{2j}, ..., x_n^{2j}]$ designates the divided difference of the function f with each x_i repeated $2j$ times. For related work, see [5],[7]-[11] and references cited therein. Recently, Gori and Micchelli [3] considered the class $\mathcal{W}_n$ of weight functions to consist of all nonnegative integrable functions w on $[-1, 1]$ such that

$$w\sqrt{1-x^2} = \sum_{k=0}^{\infty}{}' \ \ \rho_k T_{2kn}(x), \tag{9}$$

where the prime on the summation indicates that the term corresponding to $k = 0$ is halved.

Accordingly, for every $w \in \mathcal{W}_n$ and $f \in C[-1, 1]$ we have

$$\int_{-1}^{1} f(x)w(x)dx = \frac{\pi}{2} \sum_{k=0}^{\infty}{}' \ \ \rho_k A_{2kn}(f), \tag{10}$$

where

$$A_n(f) = \frac{2}{\pi} \int_{-1}^{1} f(x)T_n(x) \frac{dx}{\sqrt{1-x^2}}. \tag{11}$$

Thus formula (10), and consequently (7), reduces to explicit expression for $A_{2kn}(f)$. Gori and Micchelli [3] obtained

Theorem A Let $j, k, s \in \mathcal{N}_0, \forall f \in \mathcal{P}_{2(s+1)n-1}$. Then

$$A_{2kn}(f) = \sum_{j=0}^{s} H_{kj} f'[x_1^{2j}, ..., x_n^{2j}], \tag{12}$$

where H_{kj} is implicitly defined by the following formal power series for $j, k \geq 1$, $|z| < 4^{n-1}$,

$$\sum_{j=1}^{\infty} H_{kj} j z^j = n^{-1} 4^{(n-1)k} z^{-k} (1 - \sqrt{1 - 4^{-n+1}z})^{2k} (1 - 4^{-n+1}z)^{-\frac{1}{2}}, \tag{13}$$

for $k = 0, j \geq 1$,

$$\sum_{j=1}^{\infty} H_{0j} j z^j \quad = \quad n^{-1}((1 - 4^{-n+1}z)^{-\frac{1}{2}} - 1), \qquad |z| < 4^{n-1}, \tag{14}$$

$$H_{00} \quad = \quad \frac{2}{n}, \tag{15}$$

$$k \geq 1, H_{k0} \quad = \quad 0. \tag{16}$$

Theorem B Let $j, k, s \in \mathcal{N}_0, \forall f \in \mathcal{P}_{(2s+3)n-1}$,

$$A_{(2k+1)n}(f) = \frac{2}{n} \sum_{j=0}^{s} \hat{H}_{kj} f'[x_1^{2j+1}, ...x_n^{2j+1}], \tag{17}$$

where $\hat{H}_{kj}$ is defined by

$$\sum_{j=0}^{\infty} \hat{H}_{kj}(2j+1)z^j = \frac{2^n}{n} 4^{(n-1)k} z^{-k-1}(1-\sqrt{1-4^{-n+1}z})^{2k+1}(1-4^{-n+1}z)^{-\frac{1}{2}},$$

if $|z| < 4^{n-1}$.

Two special cases, i.e., $A_0(f)$ and $A_n(f)$ were considered by Bojanov [2]. As we see, it is not very convenient to use Theorems A and B because the coefficients H_{kj} for $A_{2kn}(f)$ and $\hat{H}_{kj}$ for $A_{(2k+1)n}(f)$ are implicitly defined. The purpose of this paper is to find explicit expression for all $A_{kn}(f), k \in \mathcal{N}_0$. Following the way and main idea used in [2], we give a simple and unified approach to this question.

2. Main Result

Now we state our main results.

Theorem *Let $j, k, s \in \mathcal{N}_0$, then $\forall f \in \mathcal{P}_{(2s+k+2)n-1}$,*

$$\begin{aligned} A_{kn}(f) &= \frac{2}{\pi}\int_{-1}^{1} f(x)T_{kn}(x)\frac{dx}{\sqrt{1-x^2}} \\ &= \frac{2}{n}\Big\{\sum_{i=1}^{n}\frac{1}{2^{kn}} f[x_1^k, ..., x_{i-1}^k, x_i^{k+1}, x_{i+1}^k, ..., x_n^k] \\ &+ \sum_{j=1}^{s}\frac{1}{(2j+k)2^{(2j+k)n}}\binom{2j+k}{j} f'[x_1^{2j+k}, ..., x_n^{2j+k}]\Big\}, \end{aligned} \tag{18}$$

Corllary If $k > 0, \forall f \in \mathcal{P}_{(2s+k+2)n-1}$, we have

$$A_{kn}(f) = \frac{2}{n}\sum_{j=0}^{s}\frac{1}{(2j+k)2^{(2j+k)n}}\binom{2j+k}{j} f'[x_1^{2j+k}, ..., x_n^{2j+k}], \tag{19}$$

and $\forall f \in \mathcal{P}_{2(s+1)n-1}$,

$$A_0(f) = \frac{2}{n}\Big\{\sum_{i=1}^{n} f(x_i) + \sum_{j=1}^{s}\frac{1}{2j4^{jn}}\binom{2j}{j} f'[x_1^{2j}, ..., x_n^{2j}]\Big\}. \tag{20}$$

Remark. Note that Corollary 2.2 can be easily derived from (18).

In order to state our next result we need some more notation:

$$\begin{aligned} \omega_n(x) &= \prod_{i=1}^{n}(x-x_i) = 2^{1-n}T_n(x), \\ l_i(x) &= \frac{\omega_n(x)}{\omega_n'(x_i)(x-x_i)}, \quad i = 1, 2, ..., n. \end{aligned} \tag{21}$$

According to [12], let $j \in \mathcal{N}$,

$$b_{lij} = \frac{1}{l!}(l_i(x)^{-j})^{(l)}_{x=x_i}, \quad l = 0, 1, ...; i = 1, 2, ..., n. \tag{22}$$

Obviously,

$$\omega_n'(x_i) = 2^{1-n}(-1)^{i-1}n(1-x_i^2)^{-\frac{1}{2}}, \quad i = 1, 2, ..., n.$$

If we expand the second term in the right-hand side of $A_0(f)$ in (20) by proposition 96 of chapter 4(P.235) in [15], we easily obtain

$$\begin{aligned}
&\int_{-1}^{1} f(x)\frac{dx}{\sqrt{1-x^2}}\\
&=\frac{\pi}{n}\sum_{i=1}^{n} f(x_i)+\frac{\pi}{n}\sum_{j=1}^{s}\frac{1}{2j4^{jn}}\binom{2j}{j}f'[x_1^{2j},...,x_n^{2j}]\\
&=\frac{\pi}{n}\sum_{i=1}^{n}\Big\{f(x_i)+\sum_{j=1}^{s}\frac{(1-x_i^2)^j}{4^j(j!)^2n^{2j}}(l_i(x)^{-2j}f'(x))_{x=x_i}^{(2j-1)}\Big\}\\
&=\frac{\pi}{n}\sum_{i=1}^{n}\Big\{f(x_i)+\sum_{j=1}^{s}\sum_{l=0}^{2j-1}\frac{(2j-1)!(1-x_i^2)^j}{l!((2j)!!)^2n^{2j}}b_{2j-l-1,i,2j}f^{(l+1)}(x_i)\Big\}\\
&=\frac{\pi}{n}\sum_{i=1}^{n}\Big\{f(x_i)+\sum_{j=1}^{s}\sum_{l=1}^{2j}\frac{(2j-1)!(1-x_i^2)^j}{(l-1)!((2j)!!)^2n^{2j}}b_{2j-l,i,2j}f^{(l)}(x_i)\Big\}\\
&=\frac{\pi}{n}\sum_{i=1}^{n}\Big\{f(x_i)+\sum_{l=1}^{2s}(\frac{1}{(l-1)!}\sum_{j=[\frac{l+1}{2}]}^{s}\frac{(2j-1)!(1-x_i^2)^j}{((2j)!!)^2n^{2j}}b_{2j-l,i,2j})f^{(l)}(x_i)\Big\},
\end{aligned}$$

which is a main result in [9]. Similarly, it is not hard to derive Theorems 3 and 4 from Theorem 5 in [8].

3. Proof of our main result

To prove our main result, we need the following auxiliary lemmas.

Lemma 3.1[2]. *Let $f(x)$ be sufficiently differentiable on $[a,b]$, $m \in \mathcal{N}$, then*

$$m\sum_{i=1}^{n} f[x_1^m,...,x_{i-1}^m,x_i^{m+1},x_{i+1}^m,...,x_n^m]=f'[x_1^m,...,x_n^m]. \tag{23}$$

Lemma 3.2. *Let $k > j \in \mathcal{N}_0$, then*

$$\int_{-1}^{1} T_n^j(x)T_{kn}(x)\frac{dx}{\sqrt{1-x^2}}=0.$$

Proof. $T_n^j(x)\in\mathcal{P}_{jn}, j<k$, and orthogonality prove the result.

Lemma 3.3[2]. *If $j \in \mathcal{N}_0$, then*

$$\int_{-1}^{1} T_n^{2j+1}(x)\Omega_{n-1}(x)\frac{dx}{\sqrt{1-x^2}}=0, \quad \forall\Omega_{n-1}\in\mathcal{P}_{n-1}.$$

Lemma 3.4. *Let $j,k \in \mathcal{N}_0$ and $j+k$ be odd, then*

$$\int_{-1}^{1} T_n^j(x)T_{kn}(x)\Omega_{n-1}(x)\frac{dx}{\sqrt{1-x^2}}=0, \qquad \forall\Omega_{n-1}\in\mathcal{P}_{n-1}. \tag{24}$$

Proof. Observing $T_{kn}(x)=T_k(T_n(x))$ and recalling the expansion of $T_k(x)$, we see that $T_{kn}(x)$, a polynomial of degree k in $T_n(x)$, has only power terms of $T_n(x)$ of degrees with the same oddity as k. Noticing that $k+j$ is odd, we conclude that the expansion of $T_n^j(x)T_{kn}(x)$ has only odd power terms of $T_n(x)$. Hence (24) follows from Lemma 3.3.

Lemma 3.5. *If $j,k \in \mathcal{N}_0$, then*

$$\int_{-1}^{1} T_n^{2j+k}(x)T_{kn}(x)\frac{dx}{\sqrt{1-x^2}}=\frac{\pi}{2^{2j+k}}\binom{2j+k}{j}. \tag{25}$$

Proof. By making the change of variable $x = \cos\theta$, we get

$$\int_{-1}^{1} T_n^{2j+k}(x)T_{kn}(x)\frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi} \cos^{2j+k} n\theta \cos kn\theta d\theta$$

$$= \frac{1}{n}\int_0^{n\pi} \cos^{2j+k}\theta\cos k\theta d\theta = \frac{1}{n}\sum_{i=0}^{n-1}\int_{i\pi}^{(i+1)\pi} \cos^{2j+k}\theta\cos k\theta d\theta$$

$$= \int_0^{\pi} \cos^{2j+k}\theta\cos k\theta d\theta = \frac{\pi}{2^{2j+k}}\binom{2j+k}{j}.$$

The last equality can be found in [4](see Formula 3.631.17 on P.374).

Lemma 3.6. *If $j, k \in \mathcal{N}_0, v \in \mathcal{N}$ and $v \le 2n-1$, then*

$$\int_{-1}^{1} T_n^{2j+k}(x)T_{kn}(x)T_v(x)\frac{dx}{\sqrt{1-x^2}} = 0. \tag{26}$$

Proof. It is not very hard to check that

$$T_n^{2j+k}(x)T_{kn}(x) = \sum_{i=0}^{j+k}\alpha_i T_n^{2i}(x) = \sum_{i=0}^{j+k}\beta_i T_{2in}(x) \tag{27}$$

for $\alpha_i' s$ and $\beta_i' s$ being constant. Now Eq.(26) follows from Eq.(27) and orthogonality.

Lemma 3.7. *If $j, k \in \mathcal{N}_0$, then*

$$\int_{-1}^{1} T_n^{2j+k}(x)T_{kn}(x)l_i(x)\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2^{2j+k}n}\binom{2j+k}{j}, \qquad i = 1, ..., n. \tag{28}$$

Proof. First note that $l_i(x)$ can be rewritten as

$$l_i(x) = \frac{1}{n} + \sum_{v=1}^{n-1}\gamma_v T_v(x), \tag{29}$$

where $\gamma_v' s$ are constant. According to Eq.(29), Eq. (25) and Eq.(26) , we obtain

$$\int_{-1}^{1} T_n^{2j+k}(x)T_{kn}(x)l_i(x)\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{1}{n}\int_{-1}^{1} T_n^{2j+k}(x)T_{kn}(x)\frac{dx}{\sqrt{1-x^2}} + \sum_{v=1}^{n-1}\gamma_v\int_{-1}^{1} T_n^{2j+k}(x)T_{kn}(x)T_v(x)\frac{dx}{\sqrt{1-x^2}}$$
$$= \frac{\pi}{2^{2j+k}n}\binom{2j+k}{j}.$$

Lemma 3.8[2]**.** *If $f(x)$ is sufficiently differentiable on $[a, b]$ and $m \in \mathcal{N}$, then*

$$\begin{aligned} f(x) \quad = \quad & \sum_{i=1}^{n}\sum_{v=0}^{m} f[x_1^v, ..., x_{i-1}^v, x_i^{v+1}, x_{i+1}^v, ..., x_n^v]\omega_n^v(x)l_i(x) \\ & + f[x_1^{m+1}, ..., x_n^{m+1}, x]\omega_n^{m+1}(x). \end{aligned} \tag{30}$$

Proof of the Theorem

Proof. Multiplying both sides of (30) by $T_{kn}(x)$ and then integrating from -1 to 1 with respect to weight $\frac{dx}{\sqrt{1-x^2}}$,we obtain from Lemma 3.2 and also the fact $\omega_n(x) = 2^{1-n}T_n(x)$ that

$$\frac{\pi}{2}A_{kn}(f) = \int_{-1}^{1} f(x)T_{kn}(x)\frac{dx}{\sqrt{1-x^2}}$$
$$= \sum_{i=1}^{n}\sum_{v=k}^{m} 2^{v(1-n)} f[x_1^v, ..., x_{i-1}^v, x_i^{v+1}, x_{i+1}^v, ..., x_n^v]\int_{-1}^{1} T_n^v(x)T_{kn}(x)l_i(x)\frac{dx}{\sqrt{1-x^2}} + R_{m+1}(f)$$
$$\equiv I_1 + R_{m+1}(f),$$

where

$$R_{m+1}(f)=2^{(m+1)(1-n)}\int_{-1}^{1}f[x_1^{m+1},...,x_n^{m+1},x]T_n^{m+1}(x)T_{kn}(x)\frac{dx}{\sqrt{1-x^2}}.$$

According to Lemma 3.4, all but the terms such as $v=2j+k, k\in\mathcal{N}_0$ in I_1 are equal to zero. So without loss of generality, we suppose that $m=2s+k, s\in\mathcal{N}_0$. It follows from Lemma 3.7 that

$$\begin{aligned}I_1 &= \sum_{i=1}^{n}\sum_{j=0}^{s}2^{(2j+k)(1-n)}f[x_1^{2j+k},...,x_{i-1}^{2j+k},x_i^{2j+k+1},x_{i+1}^{2j+k},...,x_n^{2j+k}]\\ &\quad\times\int_{-1}^{1}T_n^{2j+k}(x)T_{kn}(x)l_i(x)\frac{dx}{\sqrt{1-x^2}}\\ &=\frac{\pi}{n}\sum_{j=0}^{s}\sum_{i=1}^{n}\frac{1}{2^{(2j+k)n}}\binom{2j+k}{j}f[x_1^{2j+k},...,x_{i-1}^{2j+k},x_i^{2j+k+1},x_{i+1}^{2j+k},...,x_n^{2j+k}].\end{aligned}$$

Dividing $\sum_{j=0}^{s}$ into $j=0$ and $\sum_{j=1}^{s}$, simplified by Lemma 3.1,we obtain the desired formula as claimed.

Next, we consider $R_{m+1}(f)=R_{2s+k+1}(f)$. It is easy to check that $f[x_1^{2s+k+1},...,x_n^{2s+k+1},x]\in\mathcal{P}_{n-1}$, since $f\in\mathcal{P}_{(2s+k+2)n-1}$. And therefore $R_{2s+k+1}(f)=0$ follows from Lemma 3.4.

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