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# On the United Theory of the Family of Euler-Halley Type Methods with Cubical Convergence in Banach Spaces[*1)]

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**Abstract**

The convergence problem of the family of Euler-Halley methods is considered under the Lipschitz condition with the $L$-average, and a united convergence theory with its applications is presented.

*Key words*: Operator equation, The family of Enler-Halley, Iterations, Cubical convergence

## 1. Introduction

Let $E$ and $F$ be real or complex Banach space with norm $\|.\|$, and let $f : D \subset E \longrightarrow F$ be a nonlinear twice differentiable operator. The family of Euler-Halley iterations with the parameter $\lambda \in [0, 2]$ for solving the operator equation $f(x) = 0$ is defined as follows:

$$x_{n+1} = T_{f,\lambda}(x_n) = x_n + u_f(x_n) + v_{f,\lambda}(x_n), \quad n = 0, 1, \cdots, \tag{1.1}$$

where

$$\begin{aligned} u_f(x) &= -f'(x)^{-1} f(x), \\ v_{f,\lambda}(x) &= -\tfrac{1}{2} f'(x)^{-1} f''(x) u_f(x) Q_{f,\lambda}(x) u_f(x), \\ Q_{f,\lambda}(x) &= \left\{ I + \tfrac{\lambda}{2} f'(x)^{-1} f''(x) u_f(x) \right\}^{-1}. \end{aligned}$$

This family includes, as particular cases, the well known Euler method ($\lambda = 0$),[1, 4, 12], the Halley method ($\lambda = 1$), [3, 5, 10, 12, 18] and the convex acceleration of Newton's method or supper-Halley method ($\lambda = 2$),[6, 7, 11], so that recent interests are focused in this direction, see for example [2, 8, 9]. In particular, using a quadratic majorizing function, Argyros et al analyze the convergence of the method (1.1). However it is incorrect as shown by Han [9]. In [8], Gutierrez and Hernandez established the convergence with a cubic polynomial as the majorizing function under the classical Lipschitz condition of $f''$ while Han [9] established the convergence under the weak condition, so-called, $\gamma$-condition of $f''$, which was first presented by Wang [13, 14] when he investigated the convergence of the family of Halley methods. The purpose of the present paper is to give a united convergence theory for the family of Euler-Halley iterations such that all the known results are included as its special cases. Also some new results are obtained as the corollaries. It should be noted that this work is in sprit of Wang's idea in [15, 16].

## 2. Preliminaries and Lemmas

Let $D \subset E$ be a convex subset, open or closed. For $x_0 \in E, r > 0$, let $B(x_0, r)$ denote the open ball with the radius $r$ and the center $x_0$ while the corresponding closed ball is denoted

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by $\overline{B(x_0,r)}$. Through the paper, we always assume that $f'(x_0)^{-1}$ exists. In order to study the convergence we require some definitions and lemmas, some of which are directly taken from [15, 16].

**Definition 2.1**[15]**.** A function $f$ from $D$ to $F$ is called to satisfy the center Lipschitz condition in the ball $B(x_0,r)$ with the $L$ average if

$$\|f(x)-f(x_0)\| \le \int_0^{\rho(x)} L(u)\mathrm{d}u, \quad \forall x \in B(x_0,r), \tag{2.1}$$

where $\rho(x)=\|x-x_0\|$ and $L$ is a positive integrable function on the interval $[0,R]$ for some sufficient large number $R>0$, for example, with $\int_0^R (R-u)L(u)\mathrm{d}u = R$.

Take $r_0>0$ such that

$$\int_0^{r_0} L(u)\mathrm{d}u = 1 \tag{2.2}$$

and set

$$b = \int_0^{r_0} uL(u)\mathrm{d}u. \tag{2.3}$$

For $\beta \in (0,b]$, define

$$h(t) = \beta - t + \int_0^t (t-u)L(u)\mathrm{d}u, \quad \forall t \in [0,R]. \tag{2.4}$$

**Lemma 2.1**[15]**.** The function $h$ is decreasing monotonically in $[0,r_0]$, while it is increasing monotonically in $[r_0,R]$. Moreover, if $\beta \le b$,

$$h(\beta)>0, \quad h(r_0)=\beta-b\le 0, \quad h(R)=\beta>0.$$

Consequently, $h$ has a unique zero in each interval, respectively, which are denoted by $r_1$ and $r_2$. They satisfy

$$\beta < r_1 < \frac{r_0}{b}\beta < r_0 < r_2 < R \tag{2.5}$$

when $\beta<b$ and $r_1=r_2$ when $\beta=b$.

Furthermore, we assume that $L$ is a positive nondecreasing differentiable function in $[0,R]$. Then we have the following lemma.

**Lemma 2.2.** Let $h$ be defined as (2.4) and $\beta \le b$. Then, for each $t \in [0,r_1]$,

**(i)** $H_h(t) = h'(t)^{-2}h''(t)h(t) < 1$;

**(ii)** $T_{h,\lambda}(t) \in [0,r_1]$;

**(iii)** $t \le T_{h,\lambda}(t)$.

*Proof.* **(i)** It suffices to show that

$$g(t) = h'(t)^2 - h''(t)h(t) > 0.$$

Since

$$g'(t) = h'(t)h''(t) - h'''(t)h(t) = h'(t)L(t) - L'(t)h(t) \le 0,$$

so that $g(t) \ge g(r_1) = h'(r_1)^2 > 0$ and proves (i).

**(ii)** Observe that

$$T'_{h,\lambda}(t) = \frac{H_h(t)^2}{2(1-\frac{\lambda}{2}H_h(t))^2}[3(1-\frac{\lambda}{2}) + \frac{\lambda}{2}(\lambda-1)H_h(t) - H_{h'}(t)].$$

Since $H_{h'}(t)$ is negative and $0<H_h(t)<1$ for each $t\in[0,r_1]$, it follows that $T'_{h,\lambda}(t)>0$ for all $t\in[0,r_1]$ and each $\lambda\in[1,2]$. Hence $T_{h,\lambda}(t)$ is monotonically increasing on $[0,r_1]$ for each $\lambda\in[1,2]$. Consequently, $T_{h,\lambda}(t) \le T_{h,\lambda}(r_1) = r_1$ for each $\lambda\in[1,2]$. On the other hand, for any $\lambda\in[0,1]$, we have

$$T_{h,\lambda}(t) \le T_{h,1}(t) \le r_1.$$

Thus (ii) holds.

**(iii)** This results from that $u_h(t)\ge 0$ and $v_{h,\lambda}(t)\ge 0$. ■

The following two lemmas are due to [16]. Note that, under the assumption of the following Lemma 2.3, we may assume that $f'(x_0)^{-1}f'$ satisfies the center Lipschitz condition in the closed ball $\overline{B(x_0,r)}$ with the $L$ average.

**Lemma 2.3.** Suppose that $f'(x_0)^{-1}f'$ satisfies the center Lipschitz condition in the ball $B(x_0,r)$ with the $L$ average and $r > r_0$. Then for each $x \in B(x_0,r_0)$, $f'(x)^{-1}$ exists and

$$\left\|f'(x)^{-1}f'(x_0)\right\| \leq \frac{1}{1-\int_0^{\rho(x)} L(u)\mathrm{d}u}. \tag{2.6}$$

**Lemma 2.4.** Let $\beta = \|f'(x_0)^{-1}f(x_0)\| \leq b$. Assume that $r_1 \leq r < r_2$ if $\beta < b$, or $r = r_1$ if $\beta = b$. Then, under the hypotheses of the conditions (2.1), the equation $f(x) = 0$ has a unique solution

$$x^* \in \overline{B(x_0 - f'(x_0)^{-1}f(x_0), r_1 - \beta)} \subset \overline{B(x_0,r_1)} \tag{2.7}$$

in the closed ball $\overline{B(x_0,r)}$.

We still need a lemma below.

**Lemma 2.5.** Suppose that

$$\|f'(x_0)^{-1}f''(x_0)\| = L(0) \tag{2.8}$$

and

$$\begin{array}{rl} \left\|f'(x_0)^{-1}(f''(x') - f''(x))\right\| & \leq L(\rho(\overline{xx'})) - L(\rho(x)), \\ \forall x \in \overline{B(x_0,r)}, & \forall x' \in \overline{B(x, r-\rho(x))}, \end{array} \tag{2.9}$$

where $\rho(x) = \|x - x_0\|, \rho(\overline{xx'}) = \rho(x) + \|x' - x\|$. Then, for each $x \in B(x_0,r_0)$,

**(i)** $\|f'(x_0)^{-1}f''(x)\| \leq h''(\|x - x_0\|)$;

**(ii)** $f'(x)^{-1}$ exists and

$$\|f'(x)^{-1}f'(x_0)\| \leq -\frac{1}{h'(\|x - x_0\|)}.$$

*Proof.* It follows from (2.8) and (2.9) that

$$\|f'(x_0)^{-1}f''(x)\| \leq L(0) + \|f'(x_0)^{-1}(f''(x) - f''(x_0))\| \leq L(\|x - x_0\|) = h''(\|x - x_0\|),$$

using the fact that $h''(t) = L(t)$ and (i) follows.

To prove (ii), by Lemma 2.2, it suffices to show that $f'(x_0)^{-1}f'$ satisfies the center Lipschitz condition in the closed ball $B(x_0,r_0)$ with the $L$ average since $h'(\|x - x_0\|) = -1 + \int_0^{\|x-x_0\|} L(u)\mathrm{d}u$. By Taylor formula,

$$f'(x) = f'(x_0) + f''(x_0)(x - x_0) + \int_0^1 (f''(x_0 + t(x - x_0)) - f''(x_0))\,\mathrm{d}t(x - x_0).$$

This, togather with (2.8) and (2.9), implies that

$$\begin{array}{rcl} \|f'(x_0)^{-1}f'(x) - I\| & \leq & L(0)\|x - x_0\| + \displaystyle\int_0^1 \|f''(x_0 + t(x - x_0)) - f''(x_0)\|\,\mathrm{d}t\|x - x_0\| \\ & \leq & \displaystyle\int_0^{\rho(x)} L(u)\mathrm{d}u. \end{array}$$

Hence (ii)holds and the proof is complete. ■

## 3. The United Convergence Theorem

Let $\{x_n\}$ be defined as (1.1) and let $\{t_n\}$ be the corresponding sequence of the majorizing function $h$, that is,

$$t_{n+1} = T_{h,\lambda}(t_n), \quad n = 0, 1, \cdots, \tag{3.1}$$

where $t_0 = 0$. Then it follows from Lemma 2.2 that $\{t_n\}$ is well defined, increasing monotonically and tending to $r_1$.

**Lemma 3.1**[8]. For any $n = 0, 1, 2, \cdots$,

$$\begin{aligned} f(x_{n+1}) &= \frac{1}{2}f''(x_n)\left\{(2-\lambda)u_f(x_n) + v_{f,\lambda}(x_n)\right\} v_{f,\lambda}(x_n) \\ &+ \int_0^1 \left\{f''(x_n + \tau(x_{n+1} - x_n)) - f''(x_n)\right\}(1-\tau)\mathrm{d}\tau(x_{n+1} - x_n)^2. \end{aligned} \tag{3.2}$$

Now we are ready to give the main theorem of this section.

**Theorem 3.1.** Suppose that $r > r_1$ and (2.8),(2.9) hold on $\overline{B(x_0, r)}$. If

$$\beta = \|f'(x_0)^{-1}f(x_0)\| \le b = \int_0^{r_0} uL(u)\mathrm{d}u,$$

then the iteration

$$x_{n+1} = T_{f,\lambda}(x_n), \quad n = 0, 1, \cdots,$$

is well defined for all $\lambda \in [0, 2]$ and that $\{x_n\}$ converges to a solution $x^*$ of the equation $f(x) = 0$ satisfying

$$x^* \in \overline{B(x_0 - f'(x_0)^{-1}f(x_0), r_1 - \beta)} \subset \overline{B(x_0, r_1)}. \tag{3.8}$$

Moreover, for each $r$ satisfying $r_1 \le r < r_2$ if $\beta < b$ and $r = r_1$ if $\beta = b$, the equation $f(x) = 0$ has a unique solution in the closed ball $\overline{B(x_0, r)}$.

*Proof.* Using the mathematical induction, we can prove that following four statements hold for any $n = 0, 1, \cdots$,:

**(a)** $f'(x_n)^{-1}$ exists and $\|f'(x_n)^{-1}f'(x_0)\| \le h'(t_n)^{-1}h'(t_0)$;

**(b)** $\|u_f(x_n)\| \le u_h(t_n)$;

**(c)** $Q_{f,\lambda}(x_n)$ exists and $\|Q_{f,\lambda}(x_n)\| \le Q_{h,\lambda}(t_n)$;

**(d)** $\|v_{f,\lambda}(x_n)\| \le v_{h,\lambda}(t_n)$.

In fact, (a)-(b) are clearly true for $n = 0$. Consequently, $\|x_1 - x_0\| \le t_1$. Now assume **(a)-(b)** are also true for $n$. Then

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n \tag{3.9}$$

so that $\|x_{n+1} - x_0\| \le t_{n+1} < r_1$. Thus Lemma 2.5 and Lemma 3.1 imply that (a) and (b) hold for $n + 1$. This again implies that (c) and (d) hold for $n + 1$. Hence $x_{n+1}$ is well defined for all $n = 0, 1, \cdots$ and $\lambda \in [0, 2]$. Furthermore, we also have that (3.9) holds for any $n = 0, 1, 2, \cdots$ so that $\{x_n\}$ converges to a solution $x^*$ of the equation $f(x) = 0$ satisfying $x^* \in \overline{B(x_0, r)}$ while (3.8) and the uniqueness of the solution $x^*$ result from Lemma 2.4. The proof is complete. ■

## 4. Corollaries of the Convergence Theorem

In this section we will take $L$ to be some particular functions and then obtain a series of concrete results, some of which have been given by some authors, see for example [8, 9], while the others seem new up to the present.

### 4.1 Kantorovith's Type Theorem

Given fixed positive constants $\gamma$ and $K$, take

$$L(u) = \gamma + Ku. \tag{4.1}$$

Then $r_0$ is the solution of the equation

$$\int_0^{r_0} (\gamma + Ku)\mathrm{d}u = \gamma r_0 + \frac{1}{2}Kr_0^2 = 1,$$

i.e.,

$$r_0 = \frac{2}{\gamma + \sqrt{\gamma^2 + 2K}}. \tag{4.2}$$

Therefore

$$b = \int_0^{r_0} u(\gamma + Ku)\mathrm{d}u = \frac{2(\gamma + 2\sqrt{\gamma^2 + 2K})}{3(\gamma + \sqrt{\gamma^2 + 2K})^2}. \tag{4.3}$$

In this case the majorizing function is

$$h(t) = \beta - t + \frac{1}{2}\gamma t^2 + \frac{1}{6}Kt^3, \tag{4.4}$$

and $r_1 \leq r_2$ are its two positive solutions when $\beta \leq b$. Thus from Theorem 3.1 we immediately obtain the Kantorovith's type theorem.

**Theorem 4.1.** Suppose that $r > r_1$, $\|f'(x_0)^{-1}f''(x_0)\| = \gamma$ and

$$\begin{aligned} \left\|f'(x_0)^{-1}(f''(x') - f''(x))\right\| &\leq K\|x' - x\|, \\ \forall x \in \overline{B(x_0, r)}, &\quad \forall x' \in \overline{B(x, r - \|x - x_0\|)}. \end{aligned} \tag{4.5}$$

If

$$\beta = \|f'(x_0)^{-1}f(x_0)\| \leq \frac{2(\gamma + 2\sqrt{\gamma^2 + 2K})}{3(\gamma + \sqrt{\gamma^2 + 2K})^2}, \tag{4.6}$$

then the iteration

$$x_{n+1} = T_{f,\lambda}(x_n), \quad n = 0, 1, \cdots,$$

is well defined for all $\lambda \in [0, 2]$ and $\{x_n\}$ converges to a solution $x^*$ of the equation $f(x) = 0$ satisfying

$$x^* \in \overline{B(x_0 - f'(x_0)^{-1}f(x_0), r_1 - \beta)} \subset \overline{B(x_0, r_1)}.$$

Moreover, for each $r$ satisfying $r_1 \leq r < r_2$ if $\beta < b$ and $r = r_1$ if $\beta = b$, the equation $f(x) = 0$ has a unique solution in the closed ball $\overline{B(x_0, r)}$.

### 4.2 Smale's Type Theorem

For fixed $\gamma > 0$, let

$$L(u) = \frac{2\gamma}{(1 - \gamma u)^3}. \tag{4.7}$$

Then by [15] we have

$$r_0 = (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma}, \quad b = (3 - 2\sqrt{2})\frac{1}{\gamma}, \tag{4.8}$$

and the majorizing function is

$$h(t) = \beta - t + \frac{\gamma t^2}{(1 - \gamma t)}. \tag{4.9}$$

So its two positive solutions are

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = \frac{1 + \alpha \mp \sqrt{(1 + \alpha)^2 - 8\alpha}}{4\gamma}, \tag{4.10}$$

where $\alpha = \beta\gamma$. From Theorem 3.1 we again have the Smale's type theorem.

**Theorem 4.2.** Suppose that $r > r_1$, $\|f'(x_0)^{-1}f''(x_0)\| = 2\gamma$ and

$$\begin{aligned} \left\|f'(x_0)^{-1}(f''(x') - f''(x))\right\| &\leq \frac{2\gamma}{(1 - \gamma\|x - x_0\| - \gamma\|x' - x\|)^3} - \frac{2\gamma}{(1 - \gamma\|x - x_0\|)^3}, \\ \forall x \in \overline{B(x_0, r)}, &\quad \forall x' \in \overline{B(x, r - \|x - x_0\|)}. \end{aligned} \tag{4.11}$$

Let $\beta = \|f'(x_0)^{-1}f(x_0)\|$ and $\alpha = \beta\gamma \leq 3 - 2\sqrt{2}$. Then the iteration

$$x_{n+1} = T_{f,\lambda}(x_n), \quad n = 0, 1, \cdots,$$

is well defined for all $\lambda \in [0, 2]$ and $\{x_n\}$ converges to a solution $x^*$ of the equation $f(x) = 0$ satisfying

$$x^* \in \overline{B(x_0 - f'(x_0)^{-1}f(x_0), r_1 - \beta)} \subset \overline{B(x_0, r_1)}.$$

Moreover, for each $r$ satisfying $r_1 \leq r < r_2$ if $\alpha < 3 - 2\sqrt{2}$ and $r = r_1$ if $\alpha = 3 - 2\sqrt{2}$, the equation $f(x) = 0$ has a unique solution in the closed ball $\overline{B(x_0, r)}$.

### 4.3 Other Examples

Let $c$ be a positive constant. Then, for different functions $L$ given in the following table, we can get a series of concrete results. The corresponding $r_0$ and $b$, which have been computed in [17], are illustrated in the following table.

| $L(u)$ | $\gamma r_0$ | $\gamma b$ |
|---|---|---|
| $c\gamma e^{\gamma u}$ | $\log\dfrac{c+1}{c}$ | $(c+1)\log\dfrac{c+1}{c}-1$ |
| $\dfrac{c(m+1)\gamma}{(1-\gamma u)^{m+2}}$ | $1-\left(\dfrac{c}{1+c}\right)^{\frac{1}{m+1}}$ | $1+c\dfrac{m+1}{m}\left(1-\left(\dfrac{c+1}{c}\right)^{\frac{m}{m+1}}\right)$ |
| $\dfrac{c\gamma}{(1-\gamma u)^2}$ | $\dfrac{1}{c+1}$ | $1-c\log\dfrac{c+1}{c}$ |
| $\dfrac{c\gamma}{1-\gamma u}$ | $1-e^{-\frac{1}{c}}$ | $1-c+ce^{-\frac{1}{c}}$ |

where $m > -1, m \neq 0$. Thus using Theorem 3.1 we obtain the corresponding convergence results.

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