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# MATHEMATICAL ANALYSIS FOR QUADRILATERAL ROTATED $\mathcal{Q}_1$ ELEMENT III: THE EFFECT OF NUMERICAL INTEGRATION*[1)]

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**Abstract**

This is the third part of the paper for the rotated $\mathcal{Q}_1$ nonconforming element on quadrilateral meshes for general second order elliptic problems. Some optimal numerical formulas are presented and analyzed. The novelty is that it includes a formula with only two sampling points which excludes even a $\mathcal{Q}_1$ unisolvent set. It is the optimal numerical integration formula over a quadrilateral mesh with least sampling points up to now.

*Key words*: Quadrilateral rotated $\mathcal{Q}_1$ element, Numerical quadrature.

## 1. Numerical Integration Formulas

Throughout this paper, we adopt the notations appeared in [4]. Moreover, for any bounded domain $D$ or its subdomain $D_1$, we denote $⨍_D f\,dx$ or $⨍_{D_1} f\,dx$ by the integral mean for any function $f \in L^1(D)$ or $f \in L^1(D_1)$.

We define the quadrature formulas on the reference square $\hat{K} = [-1,1] \times [-1,1]$ as follows:

$$\int_{\hat{K}} \hat{\phi}(\xi,\eta)d\xi d\eta \approx \sum_{i=1}^{I} \hat{\omega}_i \hat{\phi}(\hat{Q}_i), \qquad \hat{\phi} \in C(\hat{K}),$$

where the weight $\hat{\omega}_i > 0$, the quadrature point $\hat{Q}_i = (\xi_i,\eta_i) \in \hat{K},\ i = 1,\cdots,I$. Let $\hat{Q} = Span\{1,\xi,\eta,\xi^2-\eta^2\}$, we assume that the quadrature is exact on $\hat{Q}$, hence it is also exact on $\mathcal{P}_1(\hat{K})$. The following four schemes will be considered:

$$\begin{array}{lll}
\text{Scheme1}: I=4, & \hat{\omega}_i = 1, & \{\hat{Q}_i\}_{i=1}^4 = (-1,-1),(1,-1),(1,1),(-1,1),\\
\text{Scheme2}: I=4, & \hat{\omega}_i = 1, & \{\hat{Q}_i\}_{i=1}^4 = (-1,0),(0,-1),(1,0),(0,1),\\
\text{Scheme3}: I=3, & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (-1,-1),(1,0),(0,1),\\
 & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (1,-1),(-1,0),(0,1),\\
 & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (1,1),(-1,0),(0,-1),\\
 & \hat{\omega}_i = 4/3, & \{\hat{Q}_i\}_{i=1}^4 = (-1,1),(1,0),(0,-1).\\
\text{Scheme4}: I=2, & \hat{\omega}_i = 2, & \{\hat{Q}_i\}_{i=1}^2 = (-1,-1),(1,1),\ \text{or}\ (1,-1),(-1,1).
\end{array}$$

In Figure 1, we only draw one case of Scheme 3 and Scheme 4, the other cases can be obtained symmetrically.

**Remark 1.1.** Unlike the standard quadrature formula, the above four formulas are not required to be exact either for the quadratic or for the bilinear polynomial space, but for the finite element space itself only. In particular, the scheme 4 does not contain even a $\mathcal{Q}_1(\hat{K})$-unisolvent set.

**Remark 1.2.** In fact, there are some other possibilities for obtaining a quadrature formula. For example, in the scheme 1, if we denote the weights $\hat{\omega}_i$ in the counterclockwise manner, then the following choices are also possible:

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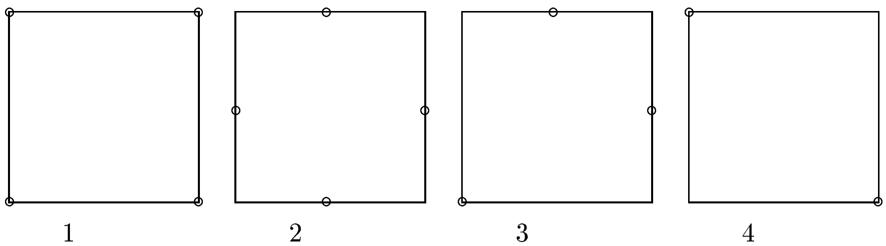


Figure 1

1. $\hat{\omega}_1 + \hat{\omega}_2 + \hat{\omega}_3 + \hat{\omega}_4 = 4$,

2. $\hat{\omega}_1 = \hat{\omega}_3$ and $\hat{\omega}_2 = \hat{\omega}_4$.

## 2. Analysis of Quadrature Formulas

The quadrature on an element $K$ is given by

$$\int_K \phi dx \approx \sum_{i=1}^{I} \Omega_{i,K}\phi(Q_{i,K}) \equiv \mathcal{Q}_K(\phi),$$

where $\phi(x) = \hat{\phi}(\hat{x})$, $\omega_{i,K} = \hat{\omega}_i J_K(\hat{Q}_i)$, $Q_{i,K} = F_K(\hat{Q}_i)$. The quadrature error functional is denoted by

$$\begin{aligned} E_K(\phi) &= \int_K \phi(x)dx - \sum_{i=1}^{I} \omega_{i,K}\phi(Q_{i,K}), \\ \hat{E}_{\hat{K}}(\hat{\phi}) &= \int_{\hat{K}} \hat{\phi} - \sum_{i=1}^{I} \hat{\omega}_i \hat{\phi}(\hat{Q}_i), \end{aligned}$$

where $E_K(\phi) = \hat{E}_{\hat{K}}(\hat{\phi} J_K)$. Now we apply the quadrature formula $\mathcal{Q}_K$ to the finite element equation (2.11) in [4]. Define

$$a_h(u,v) \equiv \sum_{K\in\mathcal{T}_h} \mathcal{Q}_K[a_{11}\partial_x u \partial_x v + a_{12}(\partial_x u\partial_y v + \partial_y u \partial_x v) + a_{22}\partial_y u \partial_y v + auv],$$

and $(f,v)_h \equiv \sum_{K\in\mathcal{T}_h} \mathcal{Q}_K(fv)$, we solve the following equation:

$$a_h(u_h, v) = (f,v)_h \quad \forall v \in V_{0,h}. \tag{2.1}$$

From now on, we always assume that the *Bi-Section Condition* [7] holds.

**Theorem 2.1.** *Suppose $a_{ij}, a \in W^{1,\infty}(\Omega), f \in W^{1,q}(\Omega), q > 2$, and $u \in H_0^1(\Omega), u_h \in V_{0,h}$ are the solution of (1.1) in [4] and (2.1), respectively, then*

$$|u - u_h|_h \le Ch\left(\sum_{i,j=1}^{2}(\|a_{ij}\|_{1,\infty} + \|a\|_{1,\infty})\|u\|_2 + \|f\|_{1,q}\right).$$

This theorem is a direct consequence of following lemmas.

**Lemma 2.1.** *The modified bilinear form $a_h(\cdot,\cdot)$ with the quadrature $\mathcal{Q}_K$ is $V_h$-ellipticity, that is*

$$a_h(v_h, v_h) \ge C\|v_h\|_h^2 \quad \forall v_h \in V_{0,h}. \tag{2.2}$$

*Proof.* For the quadrature scheme 1, 2 and 3, (2.2) is obvious (e.g. [2]). We consider the scheme 4. It is a special scheme which does not contain even a $\mathcal{Q}_1(\hat{K})$-unisolvent set, so the existing result in [2] is not available. Now we give a proof of Lemma 2.1 for the case $v_h \in V^p_{0,h}$. The ellipticity of the given differential operator $\mathcal{L}$ implies that

$$\sum_{i=1}^{2} \omega_{i,K} \sum_{i,j=1}^{2} a_{ij}\partial_i v_h \partial_j v_h(Q_{i,K}) \geq C \sum_{i=1}^{2} \omega_{i,K} \sum_{i=1}^{2} |\partial_i v_h(Q_{i,K})|^2. \tag{2.3}$$

Following the same line of [2, Theorem 4.2], we have

$$a_h(v_h, v_h) \geq C \sum_{K\in\mathcal{T}_h} \inf_{(\xi,\eta)\in\hat{K}} J_{F_K}(\xi,\eta) \inf_{(\xi,\eta)\in\hat{K}} \frac{1}{\|DF_K(\xi,\eta)\|^2} \sum_{i=1}^{2} \hat{\omega}_i \sum_{i=1}^{2} |\partial_i \hat{v}_h(\hat{Q}_i)|^2.$$

It is easy to see that

$$\begin{aligned} \hat{v}_h \quad &= (\tfrac{1}{4}(\xi^2-\eta^2) - \tfrac{1}{2}\xi + \tfrac{1}{4})\hat{v}_h(1) + (\tfrac{1}{4}(\eta^2-\xi^2) - \tfrac{1}{2}\eta + \tfrac{1}{4})\hat{v}_h(2) \\ &+ (\tfrac{1}{4}(\xi^2-\eta^2) + \tfrac{1}{2}\xi + \tfrac{1}{4})\hat{v}_h(3) + (\tfrac{1}{4}(\eta^2-\xi^2) + \tfrac{1}{2}\eta + \tfrac{1}{4})\hat{v}_h(4). \end{aligned}$$

Note that

$$\frac{\partial \hat{v}_h}{\partial \hat{\xi}} = \frac{1}{2}A\xi - \frac{1}{2}B, \quad \frac{\partial \hat{v}_h}{\partial \hat{\eta}} = -\frac{1}{2}A\eta - \frac{1}{2}C,$$

where $A = \hat{v}_h(1) - \hat{v}_h(2) + \hat{v}_h(3) - \hat{v}_h(4)$, $B = \hat{v}_h(1) - \hat{v}_h(3)$, $C = \hat{v}_h(2) - \hat{v}_h(4)$. Therefore, $\sum_{i=1}^2 \hat{\omega}_i \sum_{j=1}^2 |\partial_j \hat{v}_h(\hat{Q}_i)|^2 = 2A^2 + B^2 + C^2$.

On the other hand, it is easy to verify that $|\hat{v}_h|^2_{1,K} = \frac{2}{3}A^2 + B^2 + C^2$. Hence

$$\begin{aligned} a_h(v_h, v_h) \quad &\geq \quad C \sum_{K\in\mathcal{T}_h} \inf_{(\xi,\eta)\in\hat{K}} J_{F_K}(\xi,\eta) \inf_{(\xi,\eta)\in\hat{K}} \frac{1}{\|DF_K(\xi,\eta)\|^2} |\hat{v}_h|^2_{1,K} \\ &\geq \quad C \sum_{K\in\mathcal{T}_h} |v_h|^2_{1,K} \geq C\|v_h\|_h^2. \end{aligned}$$

In the last step we have used *Poincarè Inequality* [5]. Thus Lemma 2.1 is proved for the case $v_h \in V^p_{0,h}$. As for the case $v_h \in V^a_{0,h}$, the proof is similar.

Lemma 2.1 implies that equation 2.1 admits a unique solution.

**Lemma 2.2.** *Suppose $u \in H^1_0(\Omega), u_h \in V_{0,h}$ are the solution of (1.1) in [4], (2.1), respectively. Then*

$$\begin{aligned} \|u - u_h\|_h \quad &\leq \quad C \inf_{v\in V_{0,h}} (\|u - v\|_h + \sup_{w\in V_{0,h}} \frac{|a_h(v,w) - \bar{a}_h(v,w)|}{\|w\|_h} \\ &+ \sup_{w\in V_{0,h}} \frac{|(f, w_h) - (f, w_h)_h|}{\|w\|_h} + \sup_{w\in V_{0,h}} \frac{|\bar{a}_h(u,w) - (f,w)|}{\|w\|_h}). \end{aligned}$$

The main point in the derivation of Lemma 2.2 is the $V_h$-ellipticity of the bilinear form $a_h$, which has been proved in Lemma 2.1. The other steps are the same as in [2].

**Lemma 2.3.**

$$\sup_{w\in V_{0,h}} \frac{|\bar{a}_h(u,w) - (f,w)|}{\|w\|_h} \leq Ch\|a\|_{1,\infty}\|u\|_2. \tag{2.4}$$

*Proof.* We denote the consistency error functional by

$$d_h(\phi, w) = \bar{a}_h(\phi, w) - (\mathcal{L}\phi, w).$$

Integrating by parts gives

$$\begin{aligned} d_h(u, w) \quad &= \quad \sum_{K\in\mathcal{T}_h} \int_{\partial K} \left( a_{11}\frac{\partial u}{\partial x} w n_1 + a_{12}\frac{\partial u}{\partial x} w n_2 + a_{12}\frac{\partial u}{\partial y} w n_1 + a_{22}\frac{\partial u}{\partial y} w n_2 \right) ds \\ &= \quad I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By virtue of the $V_h$ approximation, we have

$$\begin{aligned} I_1 &= \sum_{K\in\mathcal{T}_h}\int_{\partial K} a_{11}\frac{\partial u}{\partial x} w n_1 ds \\ &= \sum_{K\in\mathcal{T}_h}\sum_{\mathcal{F}\subset\partial K}\int_{\mathcal{F}}(a_{11}-\frac{1}{|K|}\int_K a_{11}dx)\frac{\partial u}{\partial x}(w-T_K(w))n_1 ds \\ &\quad+\sum_{K\in\mathcal{T}_h}\sum_{\mathcal{F}\subset\partial K}\frac{1}{|K|}\int_K a_{11}dx\int_{\mathcal{F}}\frac{\partial u}{\partial x}(w-T_K(w))n_1 ds \\ &= I_5+I_6. \end{aligned}$$

Using [4, Lemma 2.2], $I_5$ can be estimated as follows:

$$|I_5| \le Ch\|a\|_{1,\infty}\|u\|_2\|w\|_h. \tag{2.5}$$

To estimate $I_6$, we first consider the $V_h^a$ -approximation:

$$\begin{aligned} |I_6| &= |\sum_{K\in\mathcal{T}_h}\sum_{\mathcal{F}\subset\partial K}\frac{1}{|K|}\int_K a_{11}dx\int_{\mathcal{F}}[\frac{\partial u}{\partial x}-T_{\mathcal{F}}^a(\frac{\partial u}{\partial x})][w-T_{\mathcal{F}}^a(w)]n_1 ds| \\ &\le Ch\|a\|_{0,\infty}\|u\|_2\|w\|_h. \end{aligned}$$

As to the $V_h^p$-approximation, following the same technique as in the proof of Theorem 4.1 in [4], it follows that

$$|I_6| \le Ch\|a\|_{1,\infty}\|u\|_2\|w\|_h.$$

Therefore, the above three inequalities give

$$|I_1| \le Ch\|a\|_{1,\infty}\|u\|_2\|w\|_h.$$

Similar estimates hold for $I_2, I_3$ and $I_4$. Thus,

$$|d_h(u,w)| \le Ch\sum_{i,j=1}^{2}\|a_{ij}\|_{1,\infty}\|u\|_2\|w\|_h.$$

We still need to estimate the following two terms in Lemma 2.2, i.e. $a_h(v,w)-\bar{a}_h(v,w)$ and $(f,w)_h-(f,w)$.

**Lemma 2.4.** *Let $a_{ij}, a\in W^{1,\infty}(\Omega)$. Then for any $K\in\mathcal{T}_h$ and $v,w\in B_K=V_{0,h}|_K$, we have*

$$\begin{aligned} |E_K(a_{11}\frac{\partial v}{\partial x}\frac{\partial w}{\partial x})|+|E_K(a_{12}\frac{\partial v}{\partial x}\frac{\partial w}{\partial y})| &+|E_K(a_{12}\frac{\partial v}{\partial y}\frac{\partial w}{\partial x})|+|E_K(a_{22}\frac{\partial v}{\partial y}\frac{\partial w}{\partial y})|+|E_K(avw)| \\ &\le Ch_K(\sum_{i,j=1}^{2}\|a_{ij}\|_{1,\infty,K}+\|a\|_{1,\infty,K})\|v\|_{2,K}\|w\|_{1,K}. \end{aligned}$$

*Proof.* In fact, we only need to estimate the following two terms:

$$|E_K(a\partial p\partial p')| \le Ch_K\|a\|_{1,\infty,K}\|p\|_{2,K}\|p'\|_{1,K} \quad \forall p,p'\in B_K. \tag{2.6}$$

$$|E_K(app')| \le Ch_K\|a\|_{1,\infty,K}\|p\|_{1,K}\|p'\|_{0,K} \quad \forall p,p'\in B_K. \tag{2.7}$$

Noting that $E_K(a\partial p\partial p')=\hat{E}_{\hat{K}}(\hat{b}\hat{v}\hat{w})$ with

$$\begin{cases} \hat{b}=J_K^{-1}\hat{a}\prod_{s\neq r}\partial_s F_{js}\prod_{l\neq k}\partial_l F_{jl}\in W^{2,\infty}(\hat{K}), \\ \hat{v}=\partial_r\hat{p}\in\mathcal{P}_1(\hat{K}), \\ \hat{w}=\partial_k\hat{p}'\in\mathcal{P}_1(\hat{K}), \end{cases}$$

we have following estimates for $\hat{b}$ ([2], pp. 262)

$$|\hat{b}|_{i,\infty} \le Ch^i\|a\|_{i,\infty}, i=0,1,2.$$

Let $\hat{\phi} = \hat{b}\hat{v}$, then

$$|\hat{E}_{\hat{K}}(\hat{\phi}\hat{w})| \le C\|\hat{\phi}\hat{w}\|_{0,\infty,\hat{K}} \le C\|\hat{\phi}\|_{1,\infty,\hat{K}}\|\hat{w}\|_{0,\hat{K}}.$$

Note $\hat{E}_{\hat{K}}(\hat{\phi}\hat{w}) = 0 \ \ \forall\hat{\phi} \in \mathcal{P}_0(\hat{K})$, then an application of Bramble-Hilbert Lemma yields

$$|\hat{E}_{\hat{K}}(\hat{\phi}\hat{w})| \le C|\hat{\phi}|_{1,\infty,\hat{K}}\|\hat{w}\|_{0,\hat{K}},$$

$$\begin{aligned}|\hat{E}_{\hat{K}}(\hat{b}\hat{v}\hat{w})| &\le C(|\hat{b}|_{0,\infty,\hat{K}}|\hat{v}|_{1,\hat{K}} + |\hat{b}|_{1,\infty,\hat{K}}|\hat{v}|_{0,\hat{K}})|\hat{w}|_{0,\hat{K}} \\ &\le C(|\hat{b}|_{0,\infty,\hat{K}}|\hat{p}|_{2,\hat{K}} + |\hat{b}|_{1,\infty,\hat{K}}|\hat{p}|_{1,\hat{K}})|\hat{p'}|_{1,\hat{K}} \\ &\le Ch_K\|a\|_{1,\infty,K}|p|_{2,K}|p'|_{1,K},\end{aligned}$$

which implies (2.6). Following the same line, we can prove (2.7).

The following Lemma concerns the quadrature error bound of the right side term of (2.1).

**Lemma 2.5.** *Suppose $f \in W^{1,q}(\Omega)$, $q > 2$, then for each $K \in \mathcal{T}_h$ and $v \in B_K$, we have*

$$|E_K(fv)| \le Ch_K mes(K)^{\frac{1}{2}-\frac{1}{q}}\|f\|_{1,q,K}\|v\|_{1,K}. \tag{2.8}$$

*Proof.* $E_K(fv) = \hat{E}_{\hat{K}}(J_K\hat{f}\hat{v})$. Let $\hat{g} = J_K\hat{f}$ and $\hat{\psi} = \hat{g}\hat{v}$, then

$$E_K(fv) = \hat{E}_{\hat{K}}(\hat{g}\hat{v}) = \hat{E}_{\hat{K}}(\hat{\psi}) \le C\|\hat{\psi}\|_{0,\infty} \le C\|\hat{\psi}\|_{1,q}.$$

Note that $E_K(\hat{p}) = 0 \ \ \forall\hat{p} \in \mathcal{P}_0(\hat{K})$, then

$$E_K(fv) \le C|\hat{\psi}|_{1,q} \le C(|\hat{g}|_{1,q,\hat{K}}\|\hat{v}\|_{0,\hat{K}} + \|\hat{g}\|_{0,q,\hat{K}}|\hat{v}|_{1,\hat{K}}). \tag{2.9}$$

Using the scaling trick, we can conclude (2.8) from (2.9).

Summing up Lemma 2.1-2.5, we come to Theorem 2.1.

To derive the $\mathcal{L}^2$ error bound, we use the Aubin-Nitsche trick [6].

**Theorem 2.2.** *Let $u \in \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega)$ and $u_h$ be the solution of (1.1) in [4] and (2.1), respectively. Assume $a_{ij}, a \in W^{2,\infty}(\Omega)$, and $f \in W^{2,q}(\Omega)$, then*

$$\|u - u_h\|_0 \le Ch^2. \tag{2.10}$$

Before giving the proof, we need several lemmas.

**Lemma 2.6**[5]**.** *Let $u \in \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega)$ and $u_h$ be the solution of (1.1) in [4] and (2.1), respectively, then*

$$\begin{aligned}\|u-u_h\|_0 \le &\sup_{0\ne g\in L^2(\Omega)} \frac{1}{\|g\|_{0,\Omega}} \inf_{\psi_h\in V_{0,h}}[\ |\bar{a}_h(u-u_h,\phi_g-\psi_h)| + |d_h(u,\phi_g-\psi_h)| \\ &+|d_h(\phi_g,u-u_h)| + |\bar{a}_h(u_h,\psi_h) - a_h(u_h,\psi_h)| + |(f,\psi_h)-(f,\psi_h)_h|\ ],\end{aligned} \tag{2.11}$$

*where $\phi_g$ is the solution of the following auxiliary problem:*

$$\mathcal{L}\phi_g = g \quad \text{in} \quad \Omega, \quad \phi_g = 0 \ \text{ on } \ \partial\Omega \tag{2.12}$$

Now we estimate the terms in the right side of (2.11).

**Lemma 2.7.**

$$|d_h(u,\phi_g-\psi_h)| + |d_h(\phi_g,u-u_h)| \le Ch^2\|u\|_2\|g\|_0. \tag{2.13}$$

*Proof.* Let $W_{0,h} \in C(\bar{\Omega}) \cap \mathcal{H}_0^1(\Omega)$ be the bilinear element space over the mesh $\mathcal{T}_h$, and $\mathcal{Q}_h : C(\bar{\Omega}) \to W_{0,h}$ be the associated bilinear interpolation operator. Choosing $\psi_h = \pi_h(\mathcal{Q}_h\phi_g)$, we have

$$\begin{aligned}d_h(u,\psi_h-\phi_g) &= d_h(u,\psi_h-\mathcal{Q}_h\phi_g+\mathcal{Q}_h\phi_g-\phi_g) \\ &= d_h(u,\psi_h-\mathcal{Q}_h\phi_g).\end{aligned} \tag{2.14}$$

Let $\chi_h = \pi_h(\mathcal{Q}_h\phi_g) - \mathcal{Q}_h\phi_g$, then $\int_{\mathcal{F}} \chi_h ds = 0$ due to Lemma 2.3. We have

$$\begin{aligned} d_h(u, \phi_g - \psi_h) &= \sum_{K\in\mathcal{T}_h} \int_{\partial K} a_{11}\frac{\partial u}{\partial x}\chi_h n_1 ds + a_{12}\frac{\partial u}{\partial x}\chi_h n_2 ds \\ &+ a_{12}\frac{\partial u}{\partial y}\chi_h n_1\, ds + a_{22}\frac{\partial u}{\partial y}\chi_h n_2 ds \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

We estimate $J_1$ to $J_4$ as follows.

$$\begin{aligned} |J_1| &= \left|\sum_{K\in\mathcal{T}_h} \int_{\partial K} a_{11}\frac{\partial u}{\partial x}\chi_h n_1\, ds\right| \\ &\le \left|\sum_{K\in\mathcal{T}_h, \mathcal{F}\subset\partial K} \int_{\mathcal{F}} (a_{11} - \int\!\!\!\!\!\!-_K a_{11} dx)\frac{\partial u}{\partial x}(\chi_h - \int\!\!\!\!\!\!-_{\mathcal{F}}\chi_h) n_1\, ds\right| \\ &+ \left|\sum_{K\in\mathcal{T}_h, \mathcal{F}\subset\partial K} \int\!\!\!\!\!\!-_K a_{11} dx \int_{\mathcal{F}} [\frac{\partial u}{\partial x} - \int\!\!\!\!\!\!-_{\mathcal{F}}\frac{\partial u}{\partial x}][\chi_h - \int\!\!\!\!\!\!-_{\mathcal{F}}\chi_h] n_1\, ds\right| \\ &\le Ch\|a_{11}\|_{1,\infty,\Omega}\|u\|_2\|\chi_h\|_1 \le Ch^2\|a_{11}\|_{1,\infty,\Omega}\|u\|_2\|\phi_g\|_2 \\ &\le Ch^2\|a_{11}\|_{1,\infty,\Omega}\|u\|_2\|g\|_0. \end{aligned}$$

Similar result holds for $J_2, J_3$ and $J_4$. Thus

$$|d_h(u, \phi_g - \psi_h)| \le Ch^2\|a_{11}\|_{1,\infty,\Omega}\|u\|_2\|g\|_0.$$

Next,

$$\begin{aligned} d_h(\phi_g, u - u_h) &= d_h(\phi_g, u - \mathcal{Q}_h u + \mathcal{Q}_h u - u_h) \\ &= d_h(\phi_g, \mathcal{Q}_h u - \pi_h(\mathcal{Q}_h u)) + d_h(\phi_g, \pi_h(\mathcal{Q}_h u) - u_h) \\ &= J_5 + J_6. \end{aligned}$$

$$|J_5| \le ch^2\|\phi_g\|_2\|\mathcal{Q}_h u\|_{2,h} \le Ch^2\|g\|_0\|u\|_2.$$

$$\begin{aligned} |J_6| &\le Ch\|\phi_g\|_2\ |\pi_h(\mathcal{Q}_h u) - u_h\|_{1,h} \\ &\le Ch\|\phi_g\|_2(\|\pi_h(\mathcal{Q}_h u) - \mathcal{Q}_h u\|_{1,h} + \|\mathcal{Q}_h u - u\|_{1,h} + \|u - u_h\|_{1,h}) \\ &\le Ch^2\|g\|_0\|u\|_2. \end{aligned}$$

Therefore,

$$|d_h(\phi_g, u - u_h)| \le Ch^2\|g\|_0\|u\|_2, \tag{2.15}$$

and (2.13) follows from (2.14) and (2.15).

The following Lemma concerns the consistency error due to the numerical integration.

**Lemma 2.8.** *Suppose $a_{ij}, a \in W^{2,\infty}(\Omega)$, then for any $K \in \mathcal{T}_h$ and $v, w \in B_K$, the quadrature scheme 1-4 admit the error bound as follows:*

$$\begin{aligned} &|E_K(a_{11}\frac{\partial v}{\partial x}\frac{\partial w}{\partial x})| + |E_K(a_{12}\frac{\partial v}{\partial x}\frac{\partial w}{\partial y})| + |E_K(a_{12}\frac{\partial v}{\partial y}\frac{\partial w}{\partial x})| + |E_K(a_{22}\frac{\partial v}{\partial y}\frac{\partial w}{\partial y})| + |E_K(avw)| \\ &\qquad \le Ch_K^2(\sum_{i,j=1}^{2}\|a_{ij}\|_{2,\infty,K} + \|a\|_{2,\infty,K})\|v\|_{2,K}\|w\|_{2,K}. \end{aligned}$$

*Proof.* As before, we only need to prove the following two facts:

1) $|E_K(a\partial p\partial p')| \le Ch_K^2\|a\|_{2,\infty,K}\|p\|_{2,K}\|p'\|_{2,K} \quad \forall p, p' \in B_K,$
2) $|E_K(app')| \le Ch_K^2\|a\|_{2,\infty,K}\|p\|_{1,K}\|p'\|_{1,K} \quad \forall p, p' \in B_K.$

We prove the first one, the second is almost the same. Note that $E_K(a\partial p\partial p') = \hat{E}_{\hat{K}}(\hat{b}\hat{v}\hat{w})$, where $\hat{b}, \hat{v}, \hat{w}$ are defined as in Lemma 2.4. Let $\hat{\Pi}_0$ be the $L^2$-projection from $\mathcal{L}^2(\hat{K})$ onto the piecewise constant space, i.e. $\hat{\Pi}_0\hat{v} = 1/|\hat{K}|\int_{\hat{K}}\hat{v}$. Based upon this projection, we can decompose $\hat{E}_{\hat{K}}(\hat{b}\hat{v}\hat{w})$ as follows:

$$\begin{aligned}\hat{E}_{\hat{K}}(\hat{b}\hat{v}\hat{w}) &= \hat{E}_{\hat{K}}(\hat{b}\hat{v}(\hat{w}-\hat{\Pi}_0\hat{w})) + \hat{E}_{\hat{K}}(\hat{b}(\hat{v}-\hat{\Pi}_0\hat{v})\hat{\Pi}_0\hat{w}) + \hat{E}_{\hat{K}}(\hat{b}\hat{\Pi}_0\hat{v}\hat{\Pi}_0\hat{w})\\ &= A_1 + A_2 + A_3.\end{aligned}$$

$A_1$ can be estimated as in Lemma 5.4:

$$\begin{aligned}|A_1| &\le C\|\hat{b}\hat{v}\|_{1,\infty,\hat{K}}\|\hat{w}-\hat{\Pi}_0\hat{w}\|_{0,\hat{K}} \le C\|\hat{b}\hat{v}\|_{1,\infty,\hat{K}}|\hat{w}|_{1,\hat{K}}\\ &\le C|\hat{b}\hat{v}|_{1,\infty,\hat{K}}|\hat{w}|_{1,\hat{K}}\\ &\le C(|\hat{b}|_{1,\infty,\hat{K}}\|\hat{v}\|_{0,\hat{K}} + \|\hat{b}\|_{0,\infty,\hat{K}}|\hat{v}|_{1,\hat{K}})|\hat{w}|_{1,\hat{K}}.\end{aligned}$$

Exchange the place of $\hat{v}, \hat{w}$, we get

$$|A_2| \le C|\hat{b}|_{1,\infty,\hat{K}}\|\hat{\Pi}_0\hat{w}\|_{0,\hat{K}}|\hat{v}|_{1,\hat{K}} \le C|\hat{b}|_{1,\infty,\hat{K}}\|\hat{w}\|_{0,\hat{K}}|\hat{v}|_{1,\hat{K}}.$$

An application of Bramble-Hilbert Lemma [1] yields

$$\begin{aligned}|A_3| &\le C\|\hat{b}\|_{2,\infty,\hat{K}}\|\hat{\Pi}_0\hat{v}\|_{0,\hat{K}}\|\hat{\Pi}_0\hat{w}\|_{0,\hat{K}} \le C\|\hat{b}\|_{2,\infty,\hat{K}}\|\hat{v}\|_{0,\hat{K}}\|\hat{w}\|_{0,\hat{K}}\\ &\le C|\hat{b}|_{2,\infty,\hat{K}}\|\hat{v}\|_{0,\hat{K}}\|\hat{w}\|_{0,\hat{K}}.\end{aligned}$$

In view of the above three inequalities, it follows

$$\begin{aligned}|\hat{E}_{\hat{K}}(\hat{b}\hat{v}\hat{w})| &\le (\|\hat{b}\|_{0,\infty,\hat{K}}|\hat{v}|_{1,\hat{K}}|\hat{w}|_{1,\hat{K}} + |\hat{b}|_{1,\infty,\hat{K}}\|\hat{v}\|_{0,\hat{K}}|\hat{w}|_{1,\hat{K}}\\ &\quad + |\hat{b}|_{1,\infty,\hat{K}}\|\hat{w}\|_{0,\hat{K}}|\hat{v}|_{1,\hat{K}} + |\hat{b}|_{2,\infty,\hat{K}}\|\hat{v}\|_{0,\hat{K}}\|\hat{w}\|_{0,\hat{K}}.)\\ &\le Ch_K^2\sum_{i,j=1}^{2}\|a_{ij}\|_{2,\infty,K}\|p\|_{2,K}\|p'\|_{2,K}.\end{aligned}$$

We complete the proof.

Based upon Lemma 2.8, we can estimate the term $\bar{a}_h(u_h,\psi_h) - a_h(u_h,\psi_h)$ as follows.

**Lemma 2.9.**

$$|\bar{a}_h(u_h,\psi_h) - a_h(u_h,\psi_h)| \le Ch^2\|u\|_2\|g\|_0. \tag{2.16}$$

*Proof.* Note that $\pi_h u$ is an interpolation of $u$, then

$$\begin{aligned}\bar{a}_h(u_h,\psi_h) - a_h(u_h,\psi_h) &= (\bar{a}_h(u_h-\pi_h u,\psi_h) - a_h(u_h-\pi_h u,\psi_h))\\ &\quad + (\bar{a}_h(\pi_h u,\psi_h) - a_h(\pi_h u,\psi_h))\\ &= B_1 + B_2.\end{aligned}$$

By virtue of Lemma 2.4, we estimate $B_1$ as follows:

$$\begin{aligned}|B_1| \le Ch\|\psi_h\|_2\|u_h-\pi_h u\|_h &\le Ch\|g\|_0(\|u_h-u\|_h + \|u-\pi_h u\|_h)\\ &\le Ch^2\|g\|_0\|u\|_2,\end{aligned}$$

where we have used Theorem 2.1 and Lemma 2.1. Using Lemma 2.8, we have

$$|B_2| \le Ch^2\|\psi_h\|_2\|\pi_h u\|_h \le Ch^2\|g\|_0\|u\|_2.$$

A combination of the above two estimates yields (2.16).

Now we turn to the last term on the right hand side of (2.11).

**Lemma 2.10.** *Suppose $f \in W^{2,q}(\Omega), q > 2$, then for each $K \in \mathcal{T}_h$ and $v \in B_K$, we have*

$$|E_K(fv)| \le Ch_K^2 mes(K)^{\frac{1}{2}-\frac{1}{q}}\|f\|_{2,q,K}\|v\|_{2,K}.$$

*Proof.* $E_K(fv) = \hat{E}_{\hat{K}}(J_K\hat{f}\hat{v})$. Let $\hat{g} = J_K\hat{f}$. If we denote $\hat{\Pi}_1$ be the $\mathcal{L}^2$ projection operator from $B_{\hat{K}}$ onto $P_1(\hat{K})$, then

$$E_K(fv) = \hat{E}_{\hat{K}}(\hat{g}\hat{v}) = \hat{E}_{\hat{K}}(\hat{g}\hat{\Pi}_1\hat{v} + \hat{g}(\hat{v} - \hat{\Pi}_1\hat{v})).$$

Note that in the present situation, $E_{\hat{K}}(\hat{p}) = 0\ \forall \hat{p} \in \mathcal{P}_1(\hat{K})$, thus

$$\begin{aligned} |\hat{E}_{\hat{K}}(\hat{g}\hat{\Pi}_1\hat{v})| &\leq C\|\hat{g}\hat{\Pi}_1\hat{v}\|_{0,\infty,\hat{K}} \leq C\|\hat{g}\hat{\Pi}_1\hat{v}\|_{2,q,\hat{K}} \\ &\leq C|\hat{g}\hat{\Pi}_1\hat{v}|_{2,q,\hat{K}} \\ &\leq C(|\hat{g}|_{2,q,\hat{K}}\|\hat{v}\|_{0,\hat{K}} + |\hat{g}|_{1,q,\hat{K}}|\hat{v}|_{1,\hat{K}}). \end{aligned} \tag{2.17}$$

For any $q > 2$, we can select $p > 2$ and the following embedding relation holds:

$$W^{2,q}(\hat{K}) \hookrightarrow W^{1,p}(\hat{K}) \hookrightarrow \mathcal{C}(\hat{K}), \frac{1}{p} = \frac{1}{q} - \frac{1}{2}.$$

An implementation of the estimate for the $\mathcal{L}^2$ projection gives

$$\begin{aligned} |\hat{g}(\hat{v} - \hat{\Pi}_1\hat{v})| &\leq C\|\hat{g}\|_{0,\infty,\hat{K}}\|\hat{v} - \hat{\Pi}_1\hat{v}\|_{0,\hat{K}} \\ &\leq C\|\hat{g}\|_{1,p,\hat{K}}|\hat{v}|_{1,\hat{K}} \leq C|\hat{g}|_{1,p,\hat{K}}|\hat{v}|_{1,\hat{K}} \\ &\leq C(|\hat{g}|_{1,q,\hat{K}} + |\hat{g}|_{2,q,\hat{K}})|\hat{v}|_{1,\hat{K}}. \end{aligned}$$

The above two inequalities imply

$$|\hat{E}_{\hat{K}}(\hat{g}\hat{v})| \leq C[|\hat{g}|_{2,q,\hat{K}}\|\hat{v}\|_{0,\hat{K}} + (|\hat{g}|_{1,q,\hat{K}} + |\hat{g}|_{2,q,\hat{K}})|\hat{v}|_{1,\hat{K}}]. \tag{2.18}$$

Using the scaling trick, the desired result follows from (2.18).

Combining Theorem 2.1 and Lemmas 2.6-2.10 yields immediately Theorem 2.2.

It means that the optimal error bounds in both energy norm and $\mathrm{L}^2$ norm are obtained with the numerical integration schemes introduced in the beginning of the paper. Notice that the third and the last scheme employ less sampling points than the corresponding 4-node bilinear element, which may considerable reduce the computing cost, such phenomenon was also observed for quadrilateral Wilson nonconforming element in our early work [3].

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