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MULTIGRID FOR THE MORTAR FINITE ELEMENT FOR PARABOLIC PROBLEM *1)

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Abstract

In this paper, a mortar finite element method for parabolic problem is presented. Multigrid method is used for solving the resulting discrete system. It is shown that the multigrid method is optimal, i.e, the convergence rate is independent of the mesh size L and the time step parameter τ.

Key words: Multigrid, Mortar element, Parabolic problem.

1. Introduction

The mortar finite element is a new type of domain decomposition method, which can handle the situations where subdomain meshes may be separately constructed and nonmatching along the interface. We refer the reader for the general presentation of the mortar element method to [3]. In [1], some domain decomposition preconditioners were constructed for the discrete system of the mortar element method. Recently, a variable V-cycle multigrid preconditioner and a W-cycle multigrid for the mortar element method were presented in [7],[4].

The objective of this paper is to study the mortar finite element for parabolic problem. First, we extend the results in [3] to parabolic problem. An optimal energy error is obtained. Meanwhile, we consider a multigrid method for solving the discrete system resulting from the mortar finite element method. It is shown that the multigrid method is optimal, i.e., the convergence rate is independent of the mesh size L and the time step parameter τ.

2. Parabolic Problem

Consider the following parabolic problem: to find $u(x,t)$ such that

$$\begin{cases} \frac{\partial u}{\partial t}+\mathcal{L}u &= f \quad in \quad \Omega\times[0,T], \\ u(x,t) &= 0 \quad in \quad \partial\Omega\times[0,T], \\ u(x,0) &= u_0(x), \end{cases} \tag{2.1}$$

where $\Omega\subset R^2$ is a bounded domain, $f\in L^2(\Omega)$. $\mathcal{L}$ is an elliptic operator

$$\mathcal{L}u=-\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}). \tag{2.2}$$

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Here $a_{ij}(x)$ satisfies

$$c\xi^t\xi \le \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \le C\xi^t\xi \quad \forall x \in \Omega, \xi \in R^d, \tag{2.3}$$

where c,C are positive constants.

The variational form of (2.1) is to find $u \in H_0^1(\Omega)$, $u(x,0) = u_0(x)$ such that

$$(\frac{\partial u}{\partial t}, v) + B(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \quad t \in [0,T], \tag{2.4}$$

where the bilinear form B is

$$B(u,v) = \int_\Omega \sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx \quad \forall u,v \in H^1(\Omega)$$

and

$$(f,v) = \int_\Omega fvdx.$$

We refer the notations of Sobolev space to [6] for details. It is easily seen that the bilinear form $B(u,v)$ is

(1). bounded, i.e. $|B(u,v)| \le C|u|_1|v|_1 \quad \forall u,v \in H_0^1(\Omega)$.

(2). elliptic, i.e. $|B(u,u)| \ge C|u|_1^2 \quad \forall u \in H_0^1(\Omega)$.

We use the backward Euler scheme and Crank-Nicolson scheme for the time discretization [10]. Both schemes are absolutely stable [8]. Let $\triangle t_n$ be the n^{th} time step and M_1 the number of steps, then $\sum_{n=1}^{M_1} \triangle t_n = T$. We lead to the following problem: for a given function $g_{n-1} \in L^2(\Omega)$, find $w \in H_0^1(\Omega)$ such that

$$A_\tau(w,v) = \tau^{-1}(w,v) + B(w,v) = (g_{n-1},v) \ \ \forall v \in H_0^1(\Omega), \tag{2.5}$$

where τ is the time step parameter. For the backward Euler scheme, we have

$$\begin{aligned} &w = u^n - u^{n-1}, \\ &\tau = \triangle t_n, \\ &(g_{n-1},v) = (f,v) - B(u^{n-1},v), \end{aligned}$$

and for the Crank-Nicolson scheme, we have

$$\begin{aligned} &w = u^n - u^{n-1}, \\ &\tau = \triangle t_n/2, \\ &(g_{n-1},v) = 2((f,v) - B(u^{n-1},v)). \end{aligned}$$

It is known [6] that if Ω is a convex polygon, then for any $g \in L^2(\Omega)$, there exists a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of

$$B(u,v) = (g,v), \quad \forall v \in H_0^1(\Omega) \tag{2.6}$$

with

$$\|u\|_2 \le C\|g\|_0. \tag{2.7}$$

Here and throughout this paper, c and C (with or without subscript) denote generic positive constants, independent of the time step parameter τ, the mesh parameters L and h_L which will be stated below.

Based on the regularity assumption (2.7), we have

Lemma 2.1. *For any $g \in L^2(\Omega)$, the equation*

$$A_\tau(u,v) = (g,v) \quad \forall v \in H_0^1(\Omega) \tag{2.8}$$

has a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ *which satisfies*

$$\|u\|_2 \le C\|g\|_0. \tag{2.9}$$

Proof. Please refer the proof to [11].

3. The Mortar Finite Element Method

Now we partition Ω into nonoverlapping polygonal subdomains such that

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i \quad and \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j.$$

They are arranged so that the intersection of $\Omega_k \cap \Omega_j$, for $k \neq j$ is either an empty set, an edge or a vertex, i.e., the partition is geometrically conforming. The interface

$$\Gamma = \bigcup_{i=1}^{N} \partial\Omega_i \backslash \partial\Omega$$

is broken into a set of disjoint open straight segments $\gamma_m (1 \le m \le M)$ (that are the edges of subdomains) called mortars, i.e.

$$\Gamma = \bigcup_{m=1}^{M} \bar{\gamma}_m, \quad \gamma_m \cap \gamma_n = \emptyset, \quad if \quad m \neq n.$$

We denote the common open edge to Ω_i and Ω_j by γ_m. By $\gamma_{m(i)}$ we denote an edge of Ω_i is a mortar and by $\delta_{m(j)}$ an edge of Ω_j that geometrically occupies the same place called nonmortar. There is no rule of selecting as a mortar.

Let Γ_1^i be a coarset triangulation of Ω_i with the mesh size h_1. The triangulation generally do not align at the subdomain interface. Denote the global mesh $\cup_i \Gamma_1^i$ by Γ_1. We refine the triangulation Γ_1 to produce Γ_2 by jointing the mid-points of the edges of the triangles in Γ_1. Obviously, the mesh size h_2 in Γ_2 is $h_2 = h_1/2$. Repeating this process, we get the l-time refined triangulation Γ_l with mesh size $h_l = h_1 2^{-l}$ $(l = 1, ..., L)$.

Define

$$X = \{v | v|_{\Omega_i} \in H^1(\Omega_i), \ \ \forall i = 1, ..., N, \ v = 0 \ on \ \partial\Omega\}. \tag{3.1}$$

On each level l, we define the linear continuous finite element space over the triangulation Γ_l^i denoted by $V_{l,i}$, whose functions are equal to zero on $\partial\Omega$. Let

$$\tilde{V}_l = \prod_{i=1}^{N} V_{l,i} = \{v_l | v_l|_{\Omega_i} = v_{l,i} \in V_{l,i}\}, \tag{3.2}$$

for all $l = 1, ..., L$, with the norm and semi-norm as follows:

$$\|v\|_{1,l} = (\sum_{i=1}^{N} \|v\|^2_{H^1(\Omega_i)})^{1/2}, \quad |v|_{1,l} = (\sum_{i=1}^{N} |v|^2_{H^1(\Omega_i)})^{1/2}, \quad \forall v \in \tilde{V}_l. \tag{3.3}$$

It is easy to see that

$$\tilde{V}_1 \subseteq \cdots \subseteq \tilde{V}_L.$$

For any interface $\gamma_m = \gamma_{m(i)} = \delta_{m(j)}$, $1 \le m \le M$, there are two different and independent 1D triangulations with mesh size $h_{l,i}$ and $h_{l,j}$. Let $M_l(\gamma_{m(i)})$ and $M_l(\delta_{m(j)})$ be piecewise continuous linear function spaces corresponding to the triangulations Γ_l^i and Γ_l^j restricted to γ_m respectively. Additionly, we need an auxiliary test space $S_l(\delta_{m(j)})$ as a subspace of the nonmortar space $M_l(\delta_{m(j)})$ such that its functions are constants on elements that intersect the ends of $\delta_{m(j)}$. Then we define the following mortar finite element space:

$$V_l = \{v_l \in \tilde{V}_l | \forall \delta_{m(j)} \subset \Gamma,\ \forall \varphi \in S_l(\delta_{m(j)}) \quad \int_{\delta_{m(j)}} (v_{l,j} - v_{l,i}) \varphi ds = 0\}. \tag{3.4}$$

The mortar element approximation of (2.8) is to find $u_l \in V_l$ such that

$$\tilde{A}_\tau(u_l, v) \hat{=} \tau^{-1}(u_l, v) + \tilde{B}(u_l, v) = (g, v) \quad \forall v \in V_l, \tag{3.5}$$

where $\tilde{B}(u,v)$ is the bilinear form on $X \times X$

$$\tilde{B}(u,v) = \sum_{k=1}^{N} \int_{\Omega_k} \sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx \quad \forall u, v \in X \tag{3.6}$$

Define the $\tilde{A}_\tau$-norm by $\|\cdot\|^2_{\tilde{A}_\tau} = \tilde{A}_\tau(\cdot,\cdot)$ and the τ-norm by $\|\cdot\|^2_\tau = \tau^{-1}\|\cdot\|_0^2 + |\cdot|^2_{1,l}$.

According to Proposition A.1 in [3], (2.3) and the definition of τ-norm, we have

$$\|v\|_\tau \leq C \tilde{A}_\tau(v, v) \quad \forall v \in V_l, \tag{3.7}$$

and

$$|\tilde{A}_\tau(u, v)| \leq C \|u\|_\tau \|v\|_\tau \quad \forall u, v \in V_l. \tag{3.8}$$

By (3.7), (3.8) and Lax-Milgram lemma, we know that (3.5) has a unique solution $u_l \in V_l$.

According to the proof of Theorem 5.4 in [3], the following result holds.

Lemma 3.1. *For any $v|_{\Omega_i} \in H^2(\Omega_i)$, $i = 1, \cdots, N$, $w_l \in V_l$, we have*

$$|\sum_{K \in \Gamma_l} \int_{\partial K} v w_l n_k ds| \leq C (\sum_{i=1}^{N} h_{l,i}^2 |v|^2_{H^1(\Omega_i)})^{1/2} |w_l|_{1,l}, \quad k = 1, 2, \tag{3.9}$$

where (n_1, n_2) denotes the outer unit normal along ∂K and $h_{l,i} = \max_{K \in \Gamma_l^i} h_K$, h_K is the diamater of triangle $K \in \Gamma_l^i$.

Based on Lemma 3.1, we have

Lemma 3.2. *Let u, u_l be the solutions of problems (2.8), (3.5), respectively. Assume $a_{ij}(x)|_{\Omega_i} \in W^1_\infty(\Omega_i)$ and $u|_{\Omega_i} \in H^2(\Omega_i)$, $i = 1, \cdots, N$. Then*

$$\|u - u_l\|_\tau \leq C\{\inf_{v_l \in V_l} \|u - v_l\|_\tau + (\sum_{i=1}^{N} h_{l,i}^2 |u|^2_{2,\Omega_i})^{1/2}\}. \tag{3.10}$$

Proof. Using Lemma 3.1, (3.7),(3.8) and integration by parts, for any $v_l \in V_l$, we have

$$\begin{aligned} C\|u_l - v_l\|^2_\tau &\leq \tilde{A}_\tau(u_l - v_l, u_l - v_l) \\ &= \tilde{A}_\tau(u - v_l, u_l - v_l) + \{(g, u_l - v_l) - \tilde{A}_\tau(u, u_l - v_l)\} \\ &= \tilde{A}_\tau(u - v_l, u_l - v_l) - \sum_{K \in \Gamma_l} \int_{\partial K} \sum_{i,j=1}^{2} (a_{ij} \frac{\partial u}{\partial x_j}) n_i (u_l - v_l) ds \\ &\leq C\{\|u - v_l\|_\tau \|u_l - v_l\|_\tau \\ &\quad + (\sum_{i=1}^{N} h_{l,i}^2 |u|^2_{2,\Omega_i})^{1/2} |u_l - v_l|_{1,l}\}, \end{aligned}$$

which yields

$$\|u_l - v_l\|_\tau \leq C\{\|u - v_l\|_\tau + (\sum_{i=1}^{N} h_{l,i}^2 |u|^2_{2,\Omega_i})^{1/2}\}. \tag{3.11}$$

The desired result follows from above inequality and triangle inequality.

In order to estimate the approximation error, we need the following simultaneous approximation of V_l in L^2 norm and H^1 norm.

Lemma 3.3. *For any $\gamma_m = \delta_{m(j)} = \gamma_{m(i)} \subset \Gamma$, assume $C_1 h_{l,j} \le h_{l,i} \le C_2 h_{l,j}$. Let $u \in H_0^1(\Omega)$, $u|_{\Omega_i} \in H^2(\Omega_i)$, $i = 1, \cdots, N$. Then there exists an element $v_l \in V_l$, such that*

$$\|u - v_l\|_{s,l}^2 \le C \sum_{i=1}^{N} h_{l,i}^{2(2-s)} |u|_{2,\Omega_i}^2, \quad s = 0, 1, \tag{3.12}$$

where $\|\cdot\|_{0,l} \equiv \|\cdot\|_0$.

Proof. For any $\gamma_m \subset \Gamma$, define operator $\pi_{l,m} : L^2(\gamma_m) \to W_l(\delta_{m(j)})$ by

$$\int_{\gamma_m} (\pi_{l,m} v) w ds = \int_{\gamma_m} v w ds, \quad \forall w \in S_l(\delta_{m(j)}), \tag{3.13}$$

where $W_l(\gamma_m)$ defined by

$$W_l(\delta_{m(j)}) = \{v | v \ \ is \ \ a \ \ linear \ \ continuous \ \ function \ \ on \ \ \delta_{m(j)}, \ \ and \ \ v \ \ vanishes \ \ at \ \ end-points \ \ of \ \ \delta_{m(j)}\}.$$

From [7] we know

$$\|\pi_{l,m} v\|_{L^2(\delta_{m(j)})} \le C \|v\|_{L^2(\delta_{m(j)})}. \tag{3.14}$$

Let $\{y_l^i\}$ denote the nodes of $\delta_{m(j)}$ and the operator $\Xi_{l,\delta_{m(j)}} : X \to \tilde{V}_l$ is defined by

$$(\Xi_{l,\delta_{m(j)}}(v))(y_l^i) = \begin{cases} (\pi_{l,m}(v|_{\gamma_{m(i)}} - v|_{\delta_{m(j)}}))(y_l^i), & y_l^i \in \delta_{m(j)} \\ 0 & otherwise. \end{cases} \tag{3.15}$$

It is not difficult to check that for any $v \in \tilde{V}_l$,

$$v^* = v + \sum_{m=1}^{M} \Xi_{l,\delta_{m(j)}}(v) \in V_l. \tag{3.16}$$

Let $C_l^i : H^2(\Omega_i) \to V_{l,i}$ be usual interpolation operator. Define $C_l|_{\Omega_i} = C_l^i$ and $v_l = C_l u + \sum_{m=1}^{M} \Xi_{l,\delta_{m(j)}}(C_l u)$. Obviously $v_l \in V_l$. Since $\Xi_{l,\delta_{m(j)}}(C_l u)$ equals zero at every interior vertex of the mesh in Ω_j, using the discrete norm and inverse inequality, we have

$$|\Xi_{l,\delta_{m(j)}}(C_l u)|_{H^1(\Omega_j)}^2 \le C \sum_{y_l^k} \Xi_{l,\delta_{m(j)}}(C_l u)(y_l^k)^2 \le C h_{l,j}^{-1} \|\Xi_{l,\delta_{m(j)}}(C_l u)\|_{L^2(\delta_{m(j)})}^2, \tag{3.17}$$

$$\|\Xi_{l,\delta_{m(j)}}(C_l u)\|_{L^2(\Omega_j)}^2 \le C h_{l,j}^2 \sum_{y_l^k} \Xi_{l,\delta_{m(j)}}(C_l u)(y_l^k)^2 \le C h_{l,j} \|\Xi_{l,\delta_{m(j)}}(C_l u)\|_{L^2(\delta_{m(j)})}^2. \tag{3.18}$$

The above sum is taken over the vertices of the mesh in Ω_j that lie on $\delta_{m(j)}$. By (3.14), triangle inequality, trace Theorem, and standard interpolation estimate, we have

$$\begin{aligned} \|\Xi_{l,\delta_{m(j)}}(C_l u)\|_{L^2(\delta_{m(j)})}^2 &= \|\pi_{l,m}\{(C_l u)|_{\gamma_{m(i)}} - (C_l u)|_{\delta_{m(j)}}\}\|_{L^2(\delta_{m(j)})}^2 \\ &\le C\{\|C_l^i u - u\|_{L^2(\gamma_{m(i)})}^2 + \|u - C_l^j u\|_{L^2(\delta_{m(j)})}^2\} \\ &\le C\{h_{l,i}^{-1} \|C_l^i u - u\|_{L^2(\Omega_i)}^2 + h_{l,i} |C_l^i u - u|_{H^1(\Omega_i)}^2 \\ &\quad + h_{l,j}^{-1} \|C_l^j u - u\|_{L^2(\Omega_j)}^2 + h_{l,j} |C_l^j u - u|_{H^1(\Omega_j)}^2\} \\ &\le C\{h_{l,i}^3 |u|_{H^2(\Omega_i)}^2 + h_{l,j}^3 |u|_{H^2(\Omega_j)}^2\}. \end{aligned} \tag{3.19}$$

It follows from (3.17)-(3.19) and interpolation estimate that

$$\begin{aligned} \|u - v_l\|_{s,l}^2 &\le C\{\|u - C_l u\|_{s,l}^2 + \sum_{\delta_{m(j)} \in \Gamma} \|\Xi_{l,\delta_{m(j)}}(C_l u)\|_{s,l}^2\} \\ &\le C \sum_{i=1}^{N} h_{l,i}^{2(2-s)} |u|_{2,\Omega_i}^2, \quad s = 0, 1. \end{aligned}$$

Combining Lemma 3.2 with Lemma 3.3, we obtain the following error estimate.

Theorem 3.1. *For any* $\gamma_m = \delta_{m(j)} = \gamma_{m(i)} \subset \Gamma$, *assume* $C_1 h_{l,j} \le h_{l,i} \le C_2 h_{l,j}$. *Let* $u \in H_0^1(\Omega)$ *be the solution of problem (2.8),* $u_l \in V_l$ *be the solution of problem (3.5). Assume* $a_{ij}(x)|_{\Omega_i} \in W_\infty^1(\Omega_i)$ *and* $u|_{\Omega_i} \in H^2(\Omega_i)$, $i = 1, \cdots, N$. *Then*

$$\|u - u_l\|_\tau \le C\{\sum_{i=1}^N h_{l,i}^2 (1 + \tau^{-1} h_{l,i}^2)|u|_{2,\Omega_i}^2\}^{1/2}. \tag{3.20}$$

4. Multigrid Method

In this section, we will propose a W-cycle multigrid for solving (3.5). An optimal convergence factor is obtained, i.e. the convergence rate is independent of the mesh level l and time step parameter τ.

Define the operator $A_{l,\tau} : V_l \to V_l$ as:

$$(A_{l,\tau} v, w) = \tilde{A}_\tau(v, w), \quad \forall v, w \in V_l.$$

Then (3.5) can be written as

$$A_{l,\tau} u_l = g_l, \tag{4.1}$$

where $(g_l, v) = g(v), \quad \forall v \in V_l$.

In order to present the multigrid algorithm, we introduce the following intergrid transfer operator I_l for the nonnested space $V_l (l = 1, ..., L)$, which is first constructed in [7]:

$$I_l v = v + \sum_{m=1}^M \Xi_{l,\delta_{m(j)}}(v), \quad \forall v \in V_{l-1}, \tag{4.2}$$

here the operator $\Xi_{l,\delta_{m(j)}}$ is defined by (3.15).

Lemma 4.1. *For the operator* I_l, *we have*

$$(1).|I_l v|_{1,l} \le C|v|_{1,l},$$
$$(2).\|v - I_l v\|_0 \le C h_l |v|_{1,l}.$$

Proof. Please refer the proof to [7].

By Lemma 4.1 and the inverse inequality, it is easy to check that

$$\|I_l v\|_0 \le C\|v\|_0, \quad \forall v \in V_{l-1}. \tag{4.3}$$

Then we have

$$\|I_l v\|_\tau \le C\|v\|_\tau, \quad \forall v \in V_{l-1}. \tag{4.4}$$

We now define the multigrid interation. In this paper, we choose the framework in [2]. The kth-level iteration with the initial guess w_0 yields $MG(l, w_0, F_l)$ as an approximation to the following problem at level l: Find $w \in V_l$ such that

$$\tilde{A}_\tau(w, v) = (F_l, v), \quad \forall v \in V_l, \tag{4.5}$$

where $F_l \in V_l$.

Multigrid iteration.

(1). If $l = 1$, (4.5) is solved directly.

(2). If $l > 0$, let $w_0 \in V_l$ be an initial guess, a final approximation $MG(l, w_0, F_l)$ is defined as follows:

Smoothing step: For $1 \le i \le m$, w_i is defined by

$$(w_i - w_{i-1}, v) = \lambda_{l,\tau}^{-1}[(F_l, v) - \tilde{A}_\tau(w_{l-1}, v)] \quad \forall v \in V_l. \tag{4.6}$$

Correction Step: Set

$$w_{m+1} = w_m + I_l q_\mu, \tag{4.7}$$

where $q_\mu \in V_{l-1}$ is the approximation of $\bar{q} \in V_{l-1}$ obtained by applying μ iterations with zero as the initial guess of the $l-1$-level schems to the residual equation

$$\tilde{A}_\tau(\bar{q}, v) = (F^*, v) \quad \forall v \in V_{l-1}, \tag{4.8}$$

where for any $v \in V_{l-1}$

$$\begin{aligned}(F_l^*, v) &= (F_l, I_l v) - \tilde{A}_\tau(w_m, I_l v)\\ &= \tilde{A}_\tau(w - w_m, I_l v) \quad \forall v \in V_{l-1}.\end{aligned} \tag{4.9}$$

Finally, we have

$$MG(l, w_0, F_l) = w_{m+1}. \tag{4.10}$$

In the above multigrid algorithm, $\lambda_{l,\tau} = \lambda_l + \tau^{-1}$, where λ_l is the largest eigenvelue of B_l defined by (4.12), m is a positive integer to be determined and μ any positive constant bigger than or equal to two.

For the convergence analysis, we also need the operator $P_{l-1} : V_l \to V_l$ which is defined by

$$\tilde{A}_\tau(P_{l-1}v, w) = \tilde{A}_\tau(v, I_l w) \quad \forall v \in V_l, w \in V_{l-1}. \tag{4.11}$$

Let $\{\lambda_j\}_{j=1}^{N_l}$ and $\{\varphi_j\}_{j=1}^{N_l}$ be the eigenvalues and corresponding normalized eigenfunctions of B_l, i.e.

$$B_l \varphi_j = \lambda_j \varphi_j, \quad j = 1, ..., N_l,$$

and

$$(\varphi_i, \varphi_j) = \delta_{ij},$$

where δ_{ij} is Kronecker symbol and B_l is defined by

$$(B_l v, w) = \tilde{B}(v, w) \quad \forall v, w \in V_l. \tag{4.12}$$

For any $v \in V_l$, $v = \sum_{j=1}^{N_l} c_j \varphi_j$, define the following discrete norm over the space V_l by

$$\|v\|_{s,\tau} = [\sum_i c_i^2 (\lambda_i + \tau^{-1})^s]^{\frac{1}{2}}. \tag{4.13}$$

It is easy to check that $\|v\|_{0,\tau} = \|v\|_0$, $\|v\|_{1,\tau} = \|v\|_\tau$, and

$$\|v\|_{2,\tau}^2 = \sum_i c_i^2 (\lambda_i + \tau^{-1})^2 = \|A_{l,\tau} v\|_0^2. \tag{4.14}$$

Then for the smoothing operator $T_{l,\tau} = I - \frac{1}{\lambda_{l,\tau}} A_{l,\tau}$, we have (cf. [2], [9],[11] for details)

$$(1). \|T_{l,\tau}^m v\|_\tau \le (1 + \tau^{-1}\lambda_l^{-1})^{-m} \|v\|_\tau. \tag{4.15}$$

$$(2). \|T_{l,\tau}^m v\|_{2,\tau} \le C \frac{h_l^{-1}}{m^{\frac{1}{2}}} (1 + \tau^{-1}\lambda_l^{-1})^{-\frac{m}{2}} \|v\|_\tau, \quad m \ge 2. \tag{4.16}$$

Moreover, for the projection operator P_{l-1}, we have

Lemma 4.2. *It holds that*

$$\|v - I_l P_{l-1} v\|_\tau \le C h_l (1 + \tau^{-1} h_l^2) \|v\|_{2,\tau}, \quad \forall v \in V_l. \tag{4.17}$$

Proof. Consider the following auxiliary problem: Find $\xi \in H_0^1(\Omega)$ such that

$$\tau^{-1}(\xi, v) + B(\xi, v) = (g, v), \tag{4.18}$$

where $g \hat{=} A_{l,\tau} v$.

Obviously, v is the mortar finite element solution of ξ in the space V_l, so by (2.9), (3.20), we have

$$\begin{aligned}\|\xi - v\|_\tau &\le Ch_l(1+\tau^{-1}h_l^2)^{\frac{1}{2}}\|g\|_0 \\ &\le Ch_l(1+\tau^{-1}h_l^2)^{\frac{1}{2}}\|A_{l,\tau}v\|_0. \end{aligned} \tag{4.19}$$

Let $v_{l-1} \in V_{l-1}$ be the solution of the following variational problem

$$\tilde{A}_\tau(v_{l-1},\phi) = (g,\phi) \quad \forall \phi \in V_{l-1}. \tag{4.20}$$

By (2.9),(3.20), we conclude that

$$\|\xi - v_{l-1}\|_\tau \le Ch_l(1+\tau^{-1}h_l^2)^{\frac{1}{2}}\|A_{l,\tau}v\|_0, \tag{4.21}$$

here we use the fact $h_l = h_{l-1}/2$.

An application of the triangle inequality yields that

$$\begin{aligned}\|v - I_lP_{l-1}v\|_\tau &\le \|v-\xi\|_\tau + \|\xi - I_lP_{l-1}v\|_\tau \\ &\le \|v-\xi\|_\tau + \|\xi - v_{l-1}\|_\tau + \|(I-I_l)v_{l-1}\|_\tau \\ &\quad +\|I_l(v_{l-1}-P_{l-1}v)\|_\tau \\ &\hat{=} \sum_{j=1}^{4} J_i. \end{aligned} \tag{4.22}$$

Now we estimate J_i one by one, by (4.19),(4.21),(4.14) we know that

$$\begin{aligned}|J_i| &\le Ch_l(1+\tau^{-1}h_l^2)^{\frac{1}{2}}\|A_{l,\tau}v\|_0 \\ &\le Ch_l(1+\tau^{-1}h_l^2)^{\frac{1}{2}}\|v\|_{2,\tau}, \quad i=1,2. \end{aligned} \tag{4.23}$$

For J_3, note that

$$(I-I_l)v_{l-1} = \sum_{m=1}^{M} \Xi_{l,\delta_{m(j)}}(v_{l-1}).$$

By a scaling argument and definition of I_l (cf. [7] for details), we can derive

$$|(I-I_l)v_{l-1}|_{1,l}^2 \le C\sum_{m=1}^{M} \|v_{l-1}|_{\gamma_{m(i)}} - v_{l-1}|_{\delta_{m(j)}}\|_{1/2,\gamma_m}^2. \tag{4.24}$$

Note that $\xi \in H_0^1(\Omega)$, so that

$$\begin{aligned}|(I-I_l)v_{l-1}|_{1,l}^2 &\le C\sum_{m=1}^{M} \|(v_{l-1}-\xi)|_{\gamma_{m(i)}} - (v_{l-1}-\xi)|_{\delta_{m(j)}}\|_{1/2,\gamma_m}^2 \\ &\le C(\sum_{i=1}^{N}\|v_{l-1}-\xi\|_{1/2,\partial\Omega_i}^2 + \sum_{j=1}^{N}\|v_{l-1}-\xi\|_{1/2,\partial\Omega_j}^2) \\ &\le C\|v_{l-1}-\xi\|_{1,\Omega}^2 + C\|v_{l-1}-\xi\|_{1,\Omega}^2 \\ &\le Ch_l^2(1+\tau^{-1}h_l^2)\|A_{l,\tau}v\|_0^2. \end{aligned} \tag{4.25}$$

Then we get the estimation of J_3.

Similarly, we can get

$$\|(I-I_l)v_{l-1}\|_0^2 \le Ch_l^4(1+\tau^{-1}h_l^2)\|A_{l,\tau}v\|_0^2. \tag{4.26}$$

Combining (4.25),(4.26) and using the definition of τ-norm, we have

$$|J_3| \le Ch_l(1+\tau^{-1}h_l^2)\|A_{l,\tau}v\|_0. \tag{4.27}$$

For the last term J_4 in (4.22), by (4.11), we have

$$\begin{aligned} J_4 &\le C \sup_{\varphi \in V_{l-1}} \frac{\tilde{A}_\tau(v_{l-1} - P_{l-1}v, \varphi)}{\|\varphi\|_\tau} \\ &= C \sup_{\varphi \in V_{l-1}} \frac{(g, \varphi - I_l\varphi)}{\|\varphi\|_\tau} \\ &\le Ch_l\|g\|_0 = Ch_l\|A_{l,\tau}v\|_0 \\ &\le Ch_l(1+\tau^{-1}h_l^2)\|A_{l,\tau}v\|_0. \end{aligned} \tag{4.28}$$

Then combining (4.22), (4.23), (4.27),(4.28) yields Lemma 4.2.

Finally, we can prove the main result of this paper.

Theorem 4.1. *Let $\mu > 1$ in multigrid algorithm, then there exists a constant $1 < \gamma < 1$ and an integer m, all independent of level number l and the time step parameter τ, such that if*

$$\|\bar{q} - q_\mu\|_\tau \le \gamma^\mu \|\bar{q}\|_\tau. \tag{4.29}$$

Then

$$\|w - w_{m+1}\|_\tau \le \gamma\|w - w_0\|_\tau. \tag{4.30}$$

Proof. Let $e_i = w - w_i$, $i = 0, 1, ..., m+1$.

$$\begin{aligned} e_{m+1} &= e_m - I_l q_\mu \\ &= (e_m - I_l\bar{q}) + I_l(\bar{q} - q_\mu) \\ &= (I - I_lP_{l-1})e_m + I_l(\bar{q} - q_\mu) \end{aligned} \tag{4.31}$$

It follows from (4.17),(4.16) that

$$\begin{aligned} \|(I - I_lP_{l-1})e_m\|_\tau^2 &\le Ch_l^2(1+\tau^{-1}h_l^2)^2\|A_{l,\tau}e_m\|_0^2 \\ &\le Ch_l^2(1+\tau^{-1}h_l^2)^2\|e_m\|_{2,\tau} \\ &\le Ch_l^2(1+\tau^{-1}h_l^2)^2\frac{h_l^{-2}}{m}(1+\tau^{-1}\lambda_l^{-1})^{-m}\|e_0\|_\tau^2 \\ &\le C\frac{1}{m}\|e_0\|_\tau^2. \quad (if\ m \ge 2). \end{aligned} \tag{4.32}$$

On the other hand,

$$\begin{aligned} \|I_l(\bar{q} - q_\mu)\|_\tau &\le C\|\bar{q} - q_\mu\|_\tau \\ &\le C\gamma^\mu\|\bar{q}\|_\tau = C\gamma^\mu\|P_{l-1}e_m\|_\tau \\ &\le C\gamma^\mu\|e_m\|_\tau \le C\gamma^\mu\|e_0\|_\tau. \end{aligned} \tag{4.33}$$

Hence, using similar arguments as in [2], (4.30) follows from the triangle inequality, (4.32),(4.33), and choosing appropriate γ and m.

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