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THE GLOBAL ARTIFICIAL BOUNDARY CONDITIONS FOR NUMERICAL SIMULATIONS OF THE 3D FLOW AROUND A SUBMERGED BODY *1)

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Abstract

We consider the numerical approximations of the three-dimensional steady potential flow around a body moving in a liquid of finite constant depth at constant speed and distance below a free surface in a channel. One vertical side is introduced as the upstream artificial boundary and two vertical sides are introduced as the downstream artificial boundaries. On the artificial boundaries, a sequence of high-order global artificial boundary conditions are given. Then the original problem is reduced to a problem defined on a finite computational domain, which is equivalent to a variational problem. After solving the variational problem by the finite element method, we obtain the numerical approximation of the original problem. The numerical examples show that the artificial boundary conditions given in this paper are very effective.

Key words: Ship wave, Potential flow, Global artificial boundary condition, Finite element method.

1. Introduction

Consider the three-dimensional steady potential flow around a body moving in a liquid of finite constant depth at constant speed and distance below a free surface in a channel. Let d denote the depth of the liquid, c denote the width of the channel, U denote the speed of the body and g denote the acceleration of gravity. We scale the physical quantities by the length d and the velocity $\sqrt{gd}$. We describe the motion in Cartesian coordinates fixed with respect to the body, where the x-axis points opposite to the forward velocity and z-axis is directed vertically upward, y-axis points the remaining direction of the right-angle reference frame, y=0 corresponds to one side of the channel and y=c to another side of the channel, z=0 corresponds to the undisturbed free surface and z=-1 to the bottom. Let Ω_i denote the domain occupied by the body, then $\Omega = \{\mathbb{R} \times (0,c) \times (-1,0)\} \backslash \bar{\Omega}_i$ is the domain occupied by the liquid. The total velocity potential is split into a free stream potential plus a perturbation potential: $\Phi{=}\mu x + \phi(x,z)$, where $\mu{=}U/\sqrt{gd}$ is the Froude number. By linearizing the boundary condition at the free surface, see whitham[22], we obtain the following problem for the perturbation potential on the unbounded domain Ω:

$$\triangle \phi = 0 \qquad \text{in } \Omega, \tag{1.1}$$

together with the boundary conditions

$$(\mu^2 \phi_{xx} + \phi_z)|_{z=0} = 0 \qquad -\infty < x < +\infty, 0 < y < c \tag{1.2}$$

$$\phi_z|_{z=-1} = 0 \qquad -\infty < x < +\infty, 0 < y < c \tag{1.3}$$

$$\phi_y|_{y=0} = 0 \qquad -\infty < x < +\infty, -1 < z < 0 \tag{1.4}$$

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$$\phi_y|_{y=c} = 0 \qquad -\infty < x < +\infty, -1 < z < 0 \tag{1.5}$$

$$\frac{\partial \phi}{\partial n} = \mu cos\theta \qquad \partial\Omega_i, \tag{1.6}$$

$$\lim_{x\to-\infty} \phi = 0, \lim_{x\to+\infty} \phi \text{ is bounded} \qquad -1 < z < 0; \tag{1.7}$$

where $\partial/\partial n$ denote the outward normal derivative of Ω, in the following $\partial/\partial n$ always denote the outward normal derivative of a given domain. θ is the angle between the outwardly directed normal to the body and the x-direction.

There are many authors who studied the numerical simulations of the flow around a submerged body in two dimensional case. For examples, Petersson and Malmliden [20] studied the numerical solutions of the given 2-D problem using composite grids, furthermore Malmliden and Petersson [17] proposed a Schwarz-type iterative method. Doctors and Beck [3], Nakos and Sclavounos [18] presented the boundary integral methods. In this paper we will concentrate on the numerical simulations of the 3D flow around a submerged body by the artificial boundary method. The artificial boundary method is very popular used for overcoming the difficulty caused by the unboundedness of the physical domain. During the last two decades, there are many mathematicians and engineers who have worked on this field for various problems by different techniques, see references [4]-[15], [21], [23].

For the given problem (1.1)-(1.7), we introduce the upstream artificial boundary Γ_a, the downstream artificial boundary Γ_b and the auxiliary artificial boundary $\Gamma_{b'}$. We design the high-order artificial boundary conditions on Γ_a, Γ_b and $\Gamma_{b'}$, then the given problem (1.1)-(1.7) is reduced to a boundary value problem on bounded computational domain, which can be solved by the finite element method. Furthermore the numerical example shows the effectiveness of the method given in this paper.

2. The Global Artificial Boundary Conditions

Take three constants $a < b' < b$, such that $\Omega_i \subset (a, b')\times(0,c)\times(-1,0)$. Then we obtain the upstream artificial boundary $\Gamma_a = \{(x,y,z) : x = a, 0 \le y \le c, -1 \le z \le 0\}$, the downstream artificial boundary $\Gamma_b = \{(x,y,z) : x = b, 0 \le y \le c, -1 \le z \le 0\}$, and the auxiliary artificial boundary $\Gamma_{b'} = \{(x,y,z) : x = b', 0 \le y \le c, -1 \le z \le 0\}$. The artificial boundaries Γ_a, Γ_b divide the domain Ω into three parts:

$$\Omega_a = \{(x,y,z) : -\infty < x < a, 0 < y < c, -1 < z < 0\},$$
$$\Omega_T = \{(x,y,z) : a < x < b, 0 < y < c, -1 < z < 0\}\backslash\overline{\Omega_i},$$
$$\Omega_b = \{(x,y,z) : b < x < +\infty, 0 < y < c, -1 < z < 0\},$$

furthermore we denote

$$\Omega_{b'} = \{(x,y,z) : b' < x < +\infty, 0 < y < c, -1 < z < 0\}.$$

2.1. The Artificial Boundary Condition on the Downstream Artificial Boundary

We consider the artificial boundary condition on the downstream artificial boundary. The restriction of the solution of the problem (1.1)-(1.7) on the domain $\Omega_{b'}$ satisfies:

$$\triangle\phi = 0 \qquad \text{in } \Omega_{b'}, \tag{2.1}$$

$$(\mu^2\phi_{xx} + \phi_z)|_{z=0} = 0 \qquad b' < x < +\infty, 0 < y < c \tag{2.2}$$

$$\phi_z|_{z=-1} = 0 \qquad b' < x < +\infty, 0 < y < c \tag{2.3}$$

$$\phi_y|_{y=0} = 0 \qquad -\infty < x < +\infty, -1 < z < 0 \tag{2.4}$$

$$\phi_y|_{y=c} = 0 \qquad -\infty < x < +\infty, -1 < z < 0 \tag{2.5}$$

$$\lim_{x\to+\infty} \phi \text{ is bounded}; \tag{2.6}$$

where the domain $\Omega_{b'}$ is a semi-infinite channel. The problem (2.1)-(2.6) is an uncompletely posed problem. The general solution of problem (2.1)-(2.6) is given in [16] by the separation of variables:

$$\phi(x,y,z) = \widetilde{\alpha_0} + \sum_{m=0}^{\infty} \cos\frac{m\pi y}{c}\Big\{[\alpha_0^m \cos\mu_0^m(x-b) + \beta_0^m \sin\mu_0^m(x-b)]\cosh\lambda_0^m(1+z) + \sum_{k=1}^{\infty}\beta_k^m e^{-\mu_k^m(x-b)}\cos\lambda_k^m(1+z)\Big\} \qquad (x,y,z)\in\overline{\Omega}_{b'} \tag{2.7}$$

with the $\mu_0^m > 0$ and $\lambda_0^m > 0$ (m=0, 1, $\cdots$) are given by the relation

$$(\lambda_0^m)^2 = (\mu_0^m)^2 + (\frac{m\pi}{c})^2, \qquad \mu^2(\mu_0^m)^2 = \lambda_0^m \tanh\lambda_0^m \quad m\ge 0 \tag{2.8}$$

and $\mu_k^m > 0$ and $\lambda_k^m > 0$ (m=0, 1, $\cdots$, k=1, 2, $\cdots$) are given by the relation

$$(\mu_k^m)^2 = (\lambda_k^m)^2 + (\frac{m\pi}{c})^2, \qquad \mu^2(\mu_k^m)^2 = \lambda_k^m \tan\lambda_k^m \quad m\ge 0, k\ge 1 \tag{2.9}$$

We suppose that 0< μ < 1, which is the most interested case in physics. This implies a positive real μ_0^m and λ_0^m for each $m\ge 0$. The functions $\{\cos\frac{m\pi y}{c} \quad m = 0,1,\cdots\}$ are orthogonal on [0, c].

$$\int_0^c \cos\frac{i\pi y}{c}\cos\frac{j\pi y}{c}dy = 0, \qquad i\ne j \tag{2.10}$$

and for each $m\ge 0$, we denote $O_0^m(z) = \cosh\lambda_0^m(1+z)$, $O_k^m(z) = \cos\lambda_k^m(1+z)$ $k = 1,2,\cdots$, then functions $\{O_k^m(z) \quad k = 0,1,\cdots\}$ satisfy the following orthogonal relations on [-1, 0]

$$\int_{-1}^0 \frac{\partial O_k^m(z)}{\partial z}\frac{\partial O_j^m(z)}{\partial z}dz + (\frac{m\pi}{c})^2\int_{-1}^0 O_k^m(z)O_j^m(z)dz = 0, \qquad k\ne j, \quad m\ge 0 \tag{2.11}$$

On the artificial boundary Γ_b

$$\phi\Big|_{\Gamma_b} = \widetilde{\alpha_0} + \sum_{m=0}^{\infty}\cos\frac{m\pi y}{c}\Big\{\alpha_0^m\cosh\lambda_0^m(1+z) + \sum_{k=1}^{\infty}\beta_k^m\cos\lambda_k^m(1+z)\Big\} \tag{2.12}$$

$$\frac{\partial\phi}{\partial x}\Big|_{\Gamma_b} = \sum_{m=0}^{\infty}\cos\frac{m\pi y}{c}\Big\{\beta_0^m\mu_0^m\cosh\lambda_0^m(1+z) - \sum_{k=1}^{\infty}\beta_k^m\mu_k^m\cos\lambda_k^m(1+z)\Big\} \tag{2.13}$$

From the (2.10), (2,11) and (2.12) we obtain

$$\beta_k^m = \frac{1}{p_c^m[(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]}\int_0^c\Big\{\int_{-1}^0 \frac{\partial\phi}{\partial z}\Big|_{\Gamma_b}\frac{\partial\cos\lambda_k^m(1+z)}{\partial z}dz + (\frac{m\pi}{c})^2\int_{-1}^0 \phi\Big|_{\Gamma_b}\cos\lambda_k^m(1+z)dz\Big\}\cos\frac{m\pi y}{c}dy, \quad k = 1,2,\cdots \tag{2.14}$$

$$\alpha_0^m = \frac{1}{p_c^m[(\lambda_0^m)^2 d_0^m + (\frac{m\pi}{c})^2 c_0^m]}\int_0^c\Big\{\int_{-1}^0 \frac{\partial\phi}{\partial z}\Big|_{\Gamma_b}\frac{\partial\cosh\lambda_0^m(1+z)}{\partial z}dz + (\frac{m\pi}{c})^2\int_{-1}^0 \phi\Big|_{\Gamma_b}\cosh\lambda_0^m(1+z)dz\Big\}\cos\frac{m\pi y}{c}dy \tag{2.15}$$

for $m\ge 0$, with

$$p_c^m = \int_0^c (\cos\frac{m\pi y}{c})^2 dy = \begin{cases} c, & m = 0 \\ \frac{c}{2}, & m > 0 \end{cases}$$

$$
\begin{aligned}
c_0^m &= \int_{-1}^{0} (\cosh \lambda_0^m(1+z))^2 dz = \frac{e^{2\lambda_0^m} - e^{-2\lambda_0^m} + 4\lambda_0^m}{8\lambda_0^m} \\
c_k^m &= \int_{-1}^{0} (\cos \lambda_k^m(1+z))^2 dz = \frac{1}{2} \quad k = 1, 2, \cdots \\
d_0^m &= \int_{-1}^{0} (\sinh \lambda_0^m(1+z))^2 dz = \frac{e^{2\lambda_0^m} - e^{-2\lambda_0^m} - 4\lambda_0^m}{8\lambda_0^m} \\
d_k^m &= \int_{-1}^{0} (\sin \lambda_k^m(1+z))^2 dz = \frac{1}{2} \quad k = 1, 2, \cdots
\end{aligned}
$$

On the auxiliary boundary $\Gamma_{b'}$, we have

$$
\begin{aligned}
\phi\Big|_{\Gamma_{b'}} &= \widetilde{\alpha_0} + \sum_{m=0}^{\infty} \cos\frac{m\pi y}{c} \Big\{ [\alpha_0^m \cos(\mu_0^m \triangle) - \beta_0^m \sin(\mu_0^m \triangle)] \cosh \lambda_0^m (1+z) \\
&+ \sum_{k=1}^{\infty} \beta_k^m e^{(\mu_k^m \triangle)} \cos \lambda_k^m (1+z) \Big\}
\end{aligned} \tag{2.16}
$$

with $\triangle = b - b'$, and $\sin(\mu_0^m \triangle) \neq 0, \quad m = 0, 1, \cdots$

Furthurmore we have

$$
\begin{aligned}
&\alpha_0^m \cos(\mu_0^m \triangle) - \beta_0^m \sin(\mu_0^m \triangle) \\
= &\frac{1}{p_c^m[(\lambda_0^m)^2 d_0^m + (\frac{m\pi}{c})^2 c_0^m]} \int_0^c \Big\{ \int_{-1}^{0} \frac{\partial\phi}{\partial z}\Big|_{\Gamma_{b'}} \frac{\partial \cosh \lambda_0^m (1+z)}{\partial z} dz \\
+ &(\frac{m\pi}{c})^2 \int_{-1}^{0} \phi\Big|_{\Gamma_{b'}} \cosh \lambda_0^m (1+z) dz \Big\} \cos\frac{m\pi y}{c} dy
\end{aligned} \tag{2.17}
$$

combining (2.15) and (2.17), we get

$$
\begin{aligned}
\beta_0^m = &\frac{1}{p_c^m[(\lambda_0^m)^2 d_0^m + (\frac{m\pi}{c})^2 c_0^m] \sin(\mu_0^m \triangle)} \int_0^c \Big\{ \\
&\int_{-1}^{0} [\frac{\partial\phi}{\partial z}\Big|_{\Gamma_b} \cos(\mu_0^m \triangle) - \frac{\partial\phi}{\partial z}\Big|_{\Gamma_{b'}}] \frac{\partial \cosh \lambda_0^m (1+z)}{\partial z} dz \\
+ &(\frac{m\pi}{c})^2 \int_{-1}^{0} [\phi\Big|_{\Gamma_b} \cos(\mu_0^m \triangle) - \phi\Big|_{\Gamma_{b'}}] \cosh \lambda_0^m (1+z) dz \\
&\Big\} \cos\frac{m\pi y}{c} dy
\end{aligned} \tag{2.18}
$$

From here we can see the introduction of auxiliary artificial boundary $\Gamma_{b'}$ is very important, otherwise the value of parameter β_0^m can not be determined.

Substituting (2.14) and (2.18) into (2.13) we obtain

$$
\begin{aligned}
\frac{\partial\phi}{\partial x}\Big|_{\Gamma_b} = &\sum_{m=0}^{\infty} \cos\frac{m\pi y}{c} \Big\{ \frac{\mu_0^m \cosh \lambda_0^m (1+z)}{p_c^m[(\lambda_0^m)^2 d_0^m + (\frac{m\pi}{c})^2 c_0^m] \sin(\mu_0^m \triangle)} \int_0^c \Big\{ \\
&\int_{-1}^{0} [\frac{\partial\phi}{\partial z}\Big|_{\Gamma_b} \cos(\mu_0^m \triangle) - \frac{\partial\phi}{\partial z}\Big|_{\Gamma_{b'}}] \frac{\partial \cosh \lambda_0^m (1+z)}{\partial z} dz \\
+ &(\frac{m\pi}{c})^2 \int_{-1}^{0} [\phi\Big|_{\Gamma_b} \cos(\mu_0^m \triangle) - \phi\Big|_{\Gamma_{b'}}] \cosh \lambda_0^m (1+z) dz \\
&\Big\} \cos\frac{m\pi y}{c} dy \\
- &\sum_{k=1}^{\infty} \frac{\mu_k^m \cos \lambda_k^m (1+z)}{p_c^m[(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]} \int_0^c \Big\{ \int_{-1}^{0} \frac{\partial\phi}{\partial z}\Big|_{\Gamma_b} \frac{\partial \cos \lambda_k^m (1+z)}{\partial z} dz
\end{aligned}
$$

$$+ \quad (\frac{m\pi}{c})^2 \int_{-1}^{0} \phi\Big|_{\Gamma_b} \cos\lambda_k^m(1+z)dz\Big\} \cos\frac{m\pi y}{c}dy\Big\} \equiv D_b(\phi) \tag{2.19}$$

The condition (2.19) is the exact boundary condition satisfied by the solution ϕ of problem (1.1)-(1.7). The auxiliary artificial boundary $\Gamma_{b'}$ is involved in the condition (2.19). Let

$$\begin{aligned} D_b^{MK}(\phi) \quad = \quad & \sum_{m=0}^{M} \cos\frac{m\pi y}{c}\Big\{\frac{\mu_0^m \cosh\lambda_0^m(1+z)}{p_c^m[(\lambda_0^m)^2 d_0^m + (\frac{m\pi}{c})^2 c_0^m]\sin(\mu_0^m \triangle)} \int_0^c \Big\{ \\ & \int_{-1}^{0} [\frac{\partial\phi}{\partial z}\Big|_{\Gamma_b} \cos(\mu_0^m\triangle) - \frac{\partial\phi}{\partial z}\Big|_{\Gamma_{b'}}] \frac{\partial\cosh\lambda_0^m(1+z)}{\partial z}dz \\ + \quad & (\frac{m\pi}{c})^2 \int_{-1}^{0} [\phi\Big|_{\Gamma_b} \cos(\mu_0^m\triangle) - \phi\Big|_{\Gamma_{b'}}] \cosh\lambda_0^m(1+z)dz \\ & \Big\} \cos\frac{m\pi y}{c}dy \\ - \quad & \sum_{k=1}^{K} \frac{\mu_k^m \cos\lambda_k^m(1+z)}{p_c^m[(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]} \int_0^c \Big\{\int_{-1}^{0} \frac{\partial\phi}{\partial z}\Big|_{\Gamma_b} \frac{\partial\cos\lambda_k^m(1+z)}{\partial z}dz \\ + \quad & (\frac{m\pi}{c})^2 \int_{-1}^{0} \phi\Big|_{\Gamma_b} \cos\lambda_k^m(1+z)dz\Big\} \cos\frac{m\pi y}{c}dy\Big\} \end{aligned} \tag{2.20}$$

for $M \geq 0, K \geq 0$, then we obtain a sequence of approximate artificial boundary conditions on the downstream artificial boundary Γ_b.

$$\frac{\partial\phi}{\partial x}\Big|_{\Gamma_b} = D_b^{MK}(\phi) \tag{2.21}$$

2.2. The Artificial Boundary Conditions on the Upstream Artificial Boundary Γ_a

We consider the restriction of the solution of the problem (1.1)-(1.7) on the domain Ω_a satisfying:

$$\triangle\phi = 0 \qquad \text{in } \Omega_a, \tag{2.22}$$

$$(\mu^2\phi_{xx} + \phi_z)|_{z=0} = 0 \qquad -\infty < x < a, 0 < y < c \tag{2.23}$$

$$\phi_z|_{z=-1} = 0 \qquad -\infty < x < a, 0 < y < c \tag{2.24}$$

$$\phi_y|_{y=0} = 0 \qquad -\infty < x < +\infty, -1 < z < 0 \tag{2.25}$$

$$\phi_y|_{y=c} = 0 \qquad -\infty < x < +\infty, -1 < z < 0 \tag{2.26}$$

$$\lim_{x\to-\infty} \phi = 0; \tag{2.27}$$

The problem (2.22)-(2.27) is uncompletely posed. The general solution of (2.22)-(2.27) is given by separation of variables in [16]:

$$\phi(x,y,z) = \sum_{m=0}^{\infty} \cos\frac{m\pi y}{c}\Big\{\sum_{k=1}^{\infty} \alpha_k^m e^{\mu_k^m(x-a)} \cos\lambda_k^m(1+z)\Big\} \qquad (x,y,z) \in \overline{\Omega}_a \tag{2.28}$$

on the upstream boundary Γ_a we have

$$\phi\Big|_{\Gamma_a} = \sum_{m=0}^{\infty} \cos\frac{m\pi y}{c}\Big\{\sum_{k=1}^{\infty} \alpha_k^m \cos\lambda_k^m(1+z)\Big\} \tag{2.29}$$

$$\frac{\partial\phi}{\partial x}\Big|_{\Gamma_a} = \sum_{m=0}^{\infty} \cos\frac{m\pi y}{c}\Big\{\sum_{k=1}^{\infty} \alpha_k^m \mu_k^m \cos\lambda_k^m(1+z)\Big\} \tag{2.30}$$

From (2.29) we obtain

$$\alpha_k^m = \frac{1}{p_c^m[(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]} \int_0^c \Big\{\int_{-1}^{0} \frac{\partial\phi}{\partial z}\Big|_{\Gamma_a} \frac{\partial\cos\lambda_k^m(1+z)}{\partial z}dz$$

$$+ \quad (\frac{m\pi}{c})^2 \int_{-1}^{0} \phi\Big|_{\Gamma_a} \cos\lambda_k^m(1+z)dz\Big\} \cos\frac{m\pi y}{c}dy, \quad m\geq 0, k\geq 1 \tag{2.31}$$

$$R_a^m(\phi) \equiv \int_0^c \Big\{\int_{-1}^{0} \frac{\partial\phi}{\partial z}\Big|_{\Gamma_b} \frac{\partial\cosh\lambda_0^m(1+z)}{\partial z}dz + (\frac{m\pi}{c})^2 \int_{-1}^{0} \phi\Big|_{\Gamma_b} \cosh\lambda_0^m(1+z)dz\Big\} \cos\frac{m\pi y}{c}dy = 0, \quad m\geq 0 \tag{2.32}$$

Substituting (2.31) into (2.29) and integrating (2.29) on [-1, 0], we have

$$\int_{\Gamma_a} \phi dydz = \sum_{k=1}^{\infty}\Big\{\frac{\sin\lambda_k^0}{p_c^0(\lambda_k^0)^3 d_k^0}\int_0^c\Big\{ \int_{-1}^{0} \frac{\partial\phi}{\partial z}\Big|_{\Gamma_a} \frac{\partial\cos\lambda_k^0(1+z)}{\partial z}dz\Big\}dy\Big\} \equiv S_a(\phi) \tag{2.33}$$

Substituting (2.31) into (2.30), we obtain

$$\frac{\partial\phi}{\partial n}\Big|_{\Gamma_a} = -\frac{\partial\phi}{\partial x}\Big|_{\Gamma_a} = -\sum_{m=0}^{\infty}\cos\frac{m\pi y}{c}\Big\{\sum_{k=1}^{\infty}\frac{\mu_k^m\cos\lambda_k^m(1+z)}{p_c^m[(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]}\int_0^c\Big\{ \int_{-1}^{0} \frac{\partial\phi}{\partial z}\Big|_{\Gamma_a} \frac{\partial\cos\lambda_k^m(1+z)}{\partial z}dz + (\frac{m\pi}{c})^2 \int_{-1}^{0} \phi\Big|_{\Gamma_a} \cos\lambda_k^m(1+z)dz\Big\}\cos\frac{m\pi y}{c}dy\Big\} \equiv U_a(\phi) \tag{2.34}$$

On the upstream artificial boundary Γ_a we now have the exact boundary conditions (2.32), (2.33) and (2.34)

Let

$$S_a^K(\phi) = \sum_{k=1}^{K}\Big\{\frac{\sin\lambda_k^0}{p_c^0(\lambda_k^0)^3 d_k^0}\int_0^c\Big\{ \int_{-1}^{0} \frac{\partial\phi}{\partial z}\Big|_{\Gamma_a} \frac{\partial\cos\lambda_k^0(1+z)}{\partial z}dz\Big\}dy\Big\}$$

$$U_a^{MK}(\phi) = -\sum_{m=0}^{M}\cos\frac{m\pi y}{c}\Big\{\sum_{k=1}^{K}\frac{\mu_k^m\cos\lambda_k^m(1+z)}{p_c^m[(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]}\int_0^c\Big\{ \int_{-1}^{0} \frac{\partial\phi}{\partial z}\Big|_{\Gamma_a} \frac{\partial\cos\lambda_k^m(1+z)}{\partial z}dz + (\frac{m\pi}{c})^2 \int_{-1}^{0} \phi\Big|_{\Gamma_a} \cos\lambda_k^m(1+z)dz\Big\}\cos\frac{m\pi y}{c}dy\Big\}$$

for $M\geq 0, K\geq 1$

On the upstream artificial boundary Γ_a we obtain the following approximate artificial boundary conditions:

$$\int_{\Gamma_a} \phi dz = S_a^K(\phi), \tag{2.35}$$

$$\frac{\partial\phi}{\partial n}\Big|_{\Gamma_a} = U_a^{MK}(\phi), \tag{2.36}$$

$$R_a^m(\phi) = 0, \qquad 0\leq m\leq M \tag{2.37}$$

2.3. The Reduced Boundary Value Problem of Problem (1.1)-(1.7)

Using the artificial boundary conditions given in this section, the problem (1.1)-(1.7) is reduced to a boundary value problem on the computational domain Ω_T:

$$\triangle\phi = 0 \qquad \text{in } \Omega_T, \tag{2.38}$$

$$(\mu^2\phi_{xx} + \phi_z)|_{z=0} = 0 \qquad a < x < b,\ \ 0 < y < c, \tag{2.39}$$

$$\phi_z|_{z=-1} = 0 \qquad a < x < b,\ \ 0 < y < c, \tag{2.40}$$

$$\phi_y|_{y=0} = 0 \qquad a < x < b,\ \ -1 < z < 0, \tag{2.41}$$

$$\phi_y|_{y=c} = 0 \qquad a < x < b,\ \ -1 < z < 0, \tag{2.42}$$

$$\frac{\partial\phi}{\partial n} = \mu\cos\theta \qquad \text{on } \partial\Omega_i, \tag{2.43}$$

$$R_a^m(\phi) = 0 \qquad 0 \le m \le M, \tag{2.44}$$

$$\int_{\Gamma_a} \phi dz = S_a^K(\phi), \tag{2.45}$$

$$\frac{\partial\phi}{\partial n}\Big|_{\Gamma_a} = U_a^{MK}(\phi) \qquad -1 < z < 0, \tag{2.46}$$

$$\frac{\partial\phi}{\partial n}\Big|_{\Gamma_b} = D_b^{MK}(\phi) \qquad -1 < z < 0; \tag{2.47}$$

Let ϕ be a solution of problem (2.38)-(2.47). Then $S_a^K(\phi)$ is constant. Let

$$\widetilde{\phi} = \phi - S_a^K(\phi) \tag{2.48}$$

For $\widetilde{\phi}$ the condition (2.45) is simplified:

$$\int_{\Gamma_a} \widetilde{\phi} dz = 0$$

and $\widetilde{\phi}$ satisfies the equation (2.38) and the conditions (2.39)-(2.47) except (2.45). hence we consider the following simplified problem:

$$\triangle\phi = 0 \qquad \text{in } \Omega_T, \tag{2.49}$$

$$(\mu^2\phi_{xx} + \phi_z)|_{z=0} = 0 \qquad a < x < b,\ \ 0 < y < c, \tag{2.50}$$

$$\phi_z|_{z=-1} = 0 \qquad a < x < b,\ \ 0 < y < c, \tag{2.51}$$

$$\phi_y|_{y=0} = 0 \qquad a < x < b,\ \ -1 < z < 0, \tag{2.52}$$

$$\phi_y|_{y=c} = 0 \qquad a < x < b,\ \ -1 < z < 0, \tag{2.53}$$

$$\frac{\partial\phi}{\partial n} = \mu\cos\theta \qquad \text{on } \partial\Omega_i, \tag{2.54}$$

$$R_a^m(\phi) = 0 \qquad 0 \le m \le M, \tag{2.55}$$

$$\int_{\Gamma_a} \phi dz = 0, \tag{2.56}$$

$$\frac{\partial\phi}{\partial n}\Big|_{\Gamma_a} = U_a^{MK}(\phi) \qquad -1 < z < 0, \tag{2.57}$$

$$\frac{\partial\phi}{\partial n}\Big|_{\Gamma_b} = D_b^{MK}(\phi) \qquad -1 < z < 0; \tag{2.58}$$

Suppose $\widetilde{\phi}$ is the solution of problem (2.49)-(2.58), then $\phi = \widetilde{\phi} + S_a^K(\widetilde{\phi})$ is the solution of problem (2.38)-(2.47). In the following section the equivalent variational problem of problem (2.49)-(2.58) is given.

3. The Equivalent Variational Problem of Problem (2.49)-(2.58)

Let $H^m(\Omega_T)$ and $H^s(\Gamma_0)$ denote the usual Sobolev spaces on the domain Ω_T and the boundary $\Gamma_0 = \{(x,y,z)|a \le x \le b, 0 \le y \le c, z = 0\}$ with integer m and real number s [1], we introduce the space:

$$V = \{v | v \in H^1(\Omega_T) \ \text{ and } \ v|_{\Gamma_0} \in H^1(\Gamma_0)\}$$

and its subspace

$$U = \{v | v \in V, R_a^m(v) = 0, 0 \le m \le M \ \text{ and } \ \int_{-1}^{0} v|_{\Gamma_a} dz = 0\}$$

Then we have the following result:

Theorem 3.1. *The boundary value problem (2.49)-(2.58) is equivalent to the following variational problem:*

Find $\phi_{MK} \in U$*, such that*

$$A_T(\phi_{MK}, \psi) + A_0(\phi_{MK}, \psi) + A_a^{MK}(\phi_{MK}, \psi) + A_b^{MK}(\phi_{MK}, \psi) = F(\psi), \qquad \forall \psi \in V \tag{3.1}$$

where

$$\begin{aligned}
A_T(\phi, \psi) &= \int_{\Omega_T} \nabla\phi \cdot \nabla\psi dx dy dz, \\
A_0(\phi, \psi) &= -\mu^2 \int_{\Gamma_0} \frac{\partial\phi(x,y,0)}{\partial x} \frac{\partial\psi(x,y,0)}{\partial x} dx dy, \\
A_a^{MK}(\phi, \psi) &= \sum_{m=0}^{M} \sum_{k=1}^{K} \Big\{ \frac{1}{\mu_k^m p_c^m [(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]} \\
&\quad \Big\{ \int_0^c \Big\{ \int_{-1}^0 \frac{\partial\phi}{\partial z}\Big|_{\Gamma_a} \frac{\partial \cos\lambda_k^m(1+z)}{\partial z} dz \\
&+ (\frac{m\pi}{c})^2 \int_{-1}^0 \phi\Big|_{\Gamma_a} \cos\lambda_k^m(1+z) dz \Big\} \cos\frac{m\pi y}{c} dy \Big\} \\
&\quad \Big\{ \int_0^c \Big\{ \int_{-1}^0 \frac{\partial\psi}{\partial z}\Big|_{\Gamma_a} \frac{\partial \cos\lambda_k^m(1+z)}{\partial z} dz \\
&+ (\frac{m\pi}{c})^2 \int_{-1}^0 \psi\Big|_{\Gamma_a} \cos\lambda_k^m(1+z) dz \Big\} \cos\frac{m\pi y}{c} dy \Big\}\Big\} \\
A_b^{MK}(\phi, \psi) &= \sum_{m=0}^{M} \Big\{\Big\{ \frac{1}{\mu_0^m p_c^m [(\lambda_0^m)^2 d_0^m + (\frac{m\pi}{c})^2 c_0^m] \sin(\mu_0^m \triangle)} \\
&\quad \Big\{ \int_0^c \Big\{ \int_{-1}^0 [\frac{\partial\phi}{\partial z}\Big|_{\Gamma_b} \cos(\mu_0^m\triangle) - \frac{\partial\phi}{\partial z}\Big|_{\Gamma_{b'}}] \frac{\partial \cosh\lambda_0^m(1+z)}{\partial z} dz \\
&+ (\frac{m\pi}{c})^2 \int_{-1}^0 [\phi\Big|_{\Gamma_b} \cos(\mu_0^m\triangle) - \phi\Big|_{\Gamma_{b'}}] \cosh\lambda_0^m(1+z) dz \Big\} \cos\frac{m\pi y}{c} dy \Big\} \\
&\quad \Big\{ \int_0^c \Big\{ \int_{-1}^0 \frac{\partial\psi}{\partial z}\Big|_{\Gamma_b} \frac{\partial \cosh\lambda_0^m(1+z)}{\partial z} dz \\
&+ (\frac{m\pi}{c})^2 \int_{-1}^0 \psi\Big|_{\Gamma_b} \cosh\lambda_0^m(1+z) dz \Big\} \cos\frac{m\pi y}{c} dy \Big\}\Big\} \\
&+ \sum_{k=1}^{K} \Big\{ \frac{1}{\mu_k^m p_c^m [(\lambda_k^m)^2 d_k^m + (\frac{m\pi}{c})^2 c_k^m]} \\
&\quad \Big\{ \int_0^c \Big\{ \int_{-1}^0 \frac{\partial\phi}{\partial z}\Big|_{\Gamma_b} \frac{\partial \cos\lambda_k^m(1+z)}{\partial z} dz \\
&+ (\frac{m\pi}{c})^2 \int_{-1}^0 \phi\Big|_{\Gamma_b} \cos\lambda_k^m(1+z) dz \Big\} \cos\frac{m\pi y}{c} dy \Big\}
\end{aligned}$$

$$
\begin{aligned}
& \Big\{ \int_0^c \Big\{ \int_{-1}^0 \frac{\partial \psi}{\partial z}\Big|_{\Gamma_b} \frac{\partial \cos \lambda_k^m (1+z)}{\partial z} dz \\
& + \ (\frac{m\pi}{c})^2 \int_{-1}^0 \psi\Big|_{\Gamma_b} \cos \lambda_k^m (1+z) dz \Big\} \cos \frac{m\pi y}{c} dy \Big\}\Big\}\Big\} \\
F(\psi) \ &= \ \int_{\partial\Omega_i} \mu \cos\theta \psi ds.
\end{aligned}
$$

The proof is standard, which is omitted.

We note that the solution space U is a true subspace of the trial space V. The variational problem (3.1) is not suitable for obtaining the finite element approximation of the problem (2.49)-(2.58)

Let

$$
\psi_1 \ = \ 1, \tag{3.2}
$$

$$
\begin{aligned}
v_m \ &= \ \cos \frac{m\pi y}{c} \sin \mu_0^m (x-b') \cosh \lambda_0^m (1+z) \qquad 0 \le m \le M \\
\psi_{m+2} \ &= \ \begin{cases} 0, & \{a \le x \le b', 0 \le y \le c, -1 \le z \le 0\} \backslash \Omega_i \\ v_m, & \Omega_2 = \{b' \le x \le b, 0 \le y \le c, -1 \le z \le 0\} \end{cases} \\
& \quad\ 0 \le m \le M
\end{aligned} \tag{3.3}
$$

$$
A(\phi,\psi) \ = \ A_T(\phi,\psi) + A_0(\phi,\psi) + A_a^{MK}(\phi,\psi) + A_b^{MK}(\phi,\psi)
$$

Then for any $\phi \in U$ we have

$$
A_a^{MK}(\phi,\psi_i) \ = \ 0 \qquad 1 \le i \le M+2, \tag{3.4}
$$

$$
A_b^{M0}(\phi,\psi_i) \ = \ A_b^{MK}(\phi,\psi_i) \qquad 1 \le i \le M+2; \tag{3.5}
$$

For the variational problem(3.1), we have the following results:

Lemma 3.1. *For any $\phi \in U$ the following equalities hold*

$$
A(\phi,\psi_i) = F(\psi_i) \qquad \forall \ \ \phi \in U \qquad 1 \le i \le M+2 \tag{3.6}
$$

Proof. It is straightforward to check that the equality (3.6) holds for i=1.

For $0 \le i \le M$, on the domain Ω_2, ψ_{i+2} satisfies :

$$
\begin{aligned}
& \triangle \psi_{i+2} = 0 \qquad \text{in } \Omega_2, \\
& (\mu^2 (\psi_{i+2})_{xx} + (\psi_{i+2})_z)\Big|_{z=0} = 0 \qquad b' \le x \le b, 0 \le y \le c, \\
& (\psi_{i+2})_z\Big|_{z=-1} = 0 \qquad b' \le x \le b, 0 \le y \le c, \\
& \frac{\partial \psi_{i+2}}{\partial x}\Big|_{\Gamma_b} = D_b^{M0}(\psi_{i+2}) \qquad 0 \le y \le c, -1 \le z \le 0, \\
& \frac{\partial \psi_{i+2}}{\partial x}\Big|_{\Gamma_{b'}} = \frac{\frac{\partial \psi_{i+2}}{\partial x}\Big|_{\Gamma_b}}{\cos \mu_0^i \triangle} 0 \le y \le c, -1 \le z \le 0;
\end{aligned}
$$

For $0 \le m \le M$, denote

$$
\begin{aligned}
X_m^\phi(y) \ &= \ \frac{\mu_0^m \cos \frac{m\pi y}{c}}{p_c^m [(\lambda_0^m)^2 d_0^m + (\frac{m\pi}{c})^2 c_0^m] \sin(\mu_0^m \triangle)} \int_0^c \Big\{ \\
& \int_{-1}^0 [\frac{\partial \phi}{\partial z}\Big|_{\Gamma_b} \cos(\mu_0^m \triangle) - \frac{\partial \phi}{\partial z}\Big|_{\Gamma_{b'}}] \frac{\partial \cosh \lambda_0^m (1+z)}{\partial z} dz \\
& + \ (\frac{m\pi}{c})^2 \int_{-1}^0 [\phi\Big|_{\Gamma_b} \cos(\mu_0^m \triangle) - \phi\Big|_{\Gamma_{b'}}] \cosh \lambda_0^m (1+z) dz
\end{aligned}
$$

$$
\begin{aligned}
&\qquad \Big\} \cos\frac{m\pi y}{c}dy \\
W_m &= \frac{\mu_0^m}{p_c^m[(\lambda_0^m)^2 d_0^m + (\frac{m\pi}{c})^2 c_0^m]\sin(\mu_0^m\triangle)} \\
Z_b^m(\phi) &= \int_0^c \Big\{ \int_{-1}^0 \frac{\partial\phi}{\partial z}\Big|_{\Gamma_b} \frac{\partial \cos\lambda_0^m(1+z)}{\partial z} dz \\
&+ (\frac{m\pi}{c})^2 \int_{-1}^0 \phi\Big|_{\Gamma_b} \cos\lambda_0^m(1+z)dz \Big\} \cos\frac{m\pi y}{c}dy \\
Z_{b'}^m(\phi) &= \int_0^c \Big\{ \int_{-1}^0 \frac{\partial\phi}{\partial z}\Big|_{\Gamma_{b'}} \frac{\partial \cos\lambda_0^m(1+z)}{\partial z} dz \\
&+ (\frac{m\pi}{c})^2 \int_{-1}^0 \phi\Big|_{\Gamma_{b'}} \cos\lambda_0^m(1+z)dz \Big\} \cos\frac{m\pi y}{c}dy
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\frac{\partial\psi_{i+2}}{\partial x}\Big|_{\Gamma_b} &= D_b^{M0}(\psi_{i+2}) \\
&= \sum_{m=0}^{M} X_m^{\psi_{i+2}}(y)\cosh\lambda_0^m(1+z) \\
&= \sum_{m=0}^{M} W_m[Z_b^m(\psi_{i+2})\cos\mu_0^m\triangle - Z_{b'}^m(\psi_{i+2})]\cos\frac{m\pi y}{c}\cosh\lambda_0^m(1+z)
\end{aligned}
$$

From (2.11) and (3.3) , we have

$$
\begin{aligned}
Z_b^m(\psi_{i+2}) &= 0 \qquad 0 \le i \ne m \le M, &(3.7)\\
Z_{b'}^m(\psi_{i+2}) &= 0 \qquad 0 \le m \le M, 0 \le i \le M; &(3.8)
\end{aligned}
$$

Denote

$$\Gamma_0^{'} = \{(x,y,z)|b' \le x \le b, 0 \le y \le c, z = 0\}$$

Notice (3.4), (3.5), (3.7) and (3.8) , for any $\phi \in U$, a computation shows

$$
\begin{aligned}
0 &= -\int_{\Omega_2} \phi \triangle \psi_{i+2} dxdydz \\
&= \int_{\Omega_2} \triangledown\phi\cdot\triangledown\psi_{i+2}dxdydz - \int_{\Gamma_0'} \phi\frac{\partial\psi_{i+2}}{\partial z}dxdy - \int_{\Gamma_b}\phi\frac{\partial\psi_{i+2}}{\partial x}dydz + \int_{\Gamma_{b'}}\phi\frac{\partial\psi_{i+2}}{\partial x}dydz \\
&= \int_{\Omega_2} \triangledown\phi\cdot\triangledown\psi_{i+2}dxdydz + \mu^2\int_{\Gamma_0'}\phi\frac{\partial^2\psi_{i+2}}{\partial x^2}dxdy \\
&- \sum_{m=0}^{M}\Big\{W_m[Z_b^m(\psi_{i+2})\cos\mu_0^m\triangle - Z_{b'}^m(\psi_{i+2})]\int_{\Gamma_b}\cos\frac{m\pi y}{c}\cosh\lambda_0^m(1+z)\phi dydz\Big\} \\
&+ \sum_{m=0}^{M}\Big\{W_m[\frac{Z_b^m(\psi_{i+2})\cos\mu_0^m\triangle - Z_{b'}^m(\psi_{i+2})}{\cos\mu_0^i\triangle}]\int_{\Gamma_{b'}}\cos\frac{m\pi y}{c}\cosh\lambda_0^m(1+z)\phi dydz\Big\} \\
&= \int_{\Omega_2} \triangledown\phi\cdot\triangledown\psi_{i+2}dxdydz - \mu^2\int_{\Gamma_0'}\frac{\partial\phi}{\partial x}\frac{\partial\psi_{i+2}}{\partial x}dxdy \\
&+ \frac{1}{(\mu_0^i)^2}W_i[Z_b^i(\psi_{i+2})\cos\mu_0^i\triangle]Z_b^i(\phi) - \frac{1}{(\mu_0^i)^2}W_iZ_b^i(\psi_{i+2})Z_{b'}^i(\phi) \\
&= \int_{\Omega_2} \triangledown\phi\cdot\triangledown\psi_{i+2}dxdydz - \mu^2\int_{\Gamma_0}\frac{\partial\phi}{\partial x}\frac{\partial\psi_{i+2}}{\partial x}dxdy
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{m=0}^{M} \frac{1}{(\mu_0^m)^2} W_m Z_b^m(\psi_{i+2})[Z_b^m(\phi)\cos\mu_0^m\triangle - Z_{b'}^m(\phi)] \\
&= A_T(\phi,\psi_{i+2}) + A_0(\phi,\psi_{i+2}) + A_b^{M0}(\phi,\psi_{i+2}) \\
&= A_T(\phi,\psi_{i+2}) + A_0(\phi,\psi_{i+2}) + A_a^{MK}(\phi,\psi_{i+2}) + A_b^{MK}(\phi,\psi_{i+2}) \\
&= A(\phi,\psi_{i+2})
\end{aligned}
$$

Obviously $F(\psi_{i+2}) = \int_{\partial\Omega_i} \mu\cos\theta\psi_{i+2}ds = 0$, the equalities (3.6) follows directly for $2 \le i \le M+2$. The proof of lemma 3.1 is completed.

Let $V = V^* \oplus \{\psi_1, \psi_2, \cdots, \psi_{M+2}\}$. From the lemma 3.1 we know that

Theorem 3.2. *The boundary value problem (2.49)-(2.58) is equivalent to the following variational problem*

Find $\phi_{MK} \in U$, *such that*

$$A(\phi_{MK},\psi) = F(\psi), \qquad \forall\psi \in V^* \tag{3.9}$$

Suppose U_h and V_h^* are the finite element subspaces of U and V^*, then we obtain the finite element approximation of the problem (3.9) :

Find $\phi_h^{MK} \in U_h$, such that

$$A(\phi_h^{MK},\psi_h) = F(\psi_h), \qquad \forall\psi_h \in V_h^* \tag{3.10}$$

After solving the problem (3.10) we obtain the approximate solution ϕ_h^{MK} of the original problem (1.1)-(1.7) on the computational domain Ω_T

4. Numerical Results

In this section, we first present the numerical experiments which demonstrate the effectiveness of our global artificial boundary conditions, then we obtain the approximate solution of problem (1.1)-(1.7).

In following computations, the body Ω_i is defined by the domain

$$\Omega_i = \{(x,z) \in \mathbb{R}^2 : -0.2 < x < 0.2, 0.4 < y < 0.6, -0.6 < z < -0.4\}.$$

Then the bounded computational domain Ω_T is given by

$$\Omega_T = \{(x,z) \in \mathbb{R}^2 : a < x < b, 0 < y < c, -1 < z < 0\}\backslash\overline{\Omega}_i.$$

Three meshes were used in our computations. For a given mesh, the mesh size h is defined as the maximum of the lengths of all cubes in the mesh. We shall always take $b' = b - h$ in the following. The partition for mesh A consist of equal cubes with $h = 0.2$. Mesh B (h=0.1) is generated by dividing each cube in mesh A into eight equal smaller cubes. Mesh C (h=0.05) is obtained from mesh B in a similar way.

4.1. Numerical Results of the Test Problem

Let

$$\phi_0 = \cos\mu_0^1(x-a)\cos\frac{\pi y}{c}\cosh\lambda_0^1(1+z)$$

We consider following boundary value problem which is similar to problem (2.49)-(2.58) :

$$\triangle\phi = 0 \qquad \text{in } \Omega_T, \tag{4.1}$$

$$(\mu^2\phi_{xx} + \phi_z)|_{z=0} = 0 \qquad a < x < b,\ \ 0 < y < c, \tag{4.2}$$

$$\phi_z|_{z=-1} = 0 \qquad a < x < b,\ \ 0 < y < c, \tag{4.3}$$

$$\phi_y|_{y=0} = 0 \qquad a < x < b,\ \ -1 < z < 0, \tag{4.4}$$

$$\phi_y|_{y=c} = 0 \qquad a < x < b, \;\; -1 < z < 0, \tag{4.5}$$
$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi_0}{\partial n} \qquad \text{on } \partial\Omega_i, \tag{4.6}$$
$$R_a^m(\phi) = R_a^m(\phi_0) \qquad 0 \le m \le M, \tag{4.7}$$
$$\int_{\Gamma_a} \phi dydz = \int_{\Gamma_a} \phi_0 dydz, \tag{4.8}$$
$$\frac{\partial \phi}{\partial n}\Big|_{\Gamma_a} = U_a^{MK}(\phi) \qquad 0 < y < c, -1 < z < 0, \tag{4.9}$$
$$\frac{\partial \phi}{\partial n}\Big|_{\Gamma_b} = D_b^{MK}(\phi) \qquad 0 < y < c, -1 < z < 0; \tag{4.10}$$

Then ϕ_0 is the exact solution of this problem.

Problem (4.1)-(4.10) is equivalent to the following variational problem:

Find $\phi_{MK} \in U'$, such that

$$A_T(\phi_{MK}, \psi) + A_0(\phi_{MK}, \psi) + A_a^{MK}(\phi_{MK}, \psi) + A_b^{MK}(\phi_{MK}, \psi) = F'(\psi), \qquad \forall\, \psi \in V^* \tag{4.11}$$

where $V^*, A_T, A_0, A_a^{MK}, A_b^{MK}$ is described in section 3 and

$$U' = \{v | v \in V, R_a^m(v) = R_a^m(\phi_0), 0 \le m \le M, \text{ and } \int_{\Gamma_a} vdydz = \int_{\Gamma_a} \phi_0 dydz\}$$
$$F'(\psi) = \int_{\partial\Omega_i} \frac{\partial \phi_0}{\partial n} \psi ds$$

Take $a = -1, b = 1$ and $c = 1$, $M = 8, K = 20$, $\mu = 0.4$ in (4.11), after calculation with our finite element method, a numerical solution ϕ_h of variational problem (4.11) can be obtained. The relative errors of $\phi_h - \phi_0$ in L_∞-norm, L_2-norm and H^1-norm are given in the Table 1 for mesh A, B, C, respectively.

Table 1: comparison of ϕ_h with ϕ_0

Errors	$h = 0.2$	$h = 0.1$	$h = 0.05$
$\max\|\phi_h - \phi_0\|/ \max\|\phi_0\|$	13.41%	3.21%	0.85%
$\\|\phi_h - \phi_0\\|_{0,\Omega_T}/\\|\phi_0\\|_{0,\Omega_T}$	16.88%	4.28%	1.23%
$\\|\phi_h - \phi_0\\|_{1,\Omega_T}/\\|\phi_0\\|_{1,\Omega_T}$	40.28%	19.77%	10.51%

As shown in Table 1, ϕ_h tends to ϕ_0 when mesh size h decreases, the converge order of max $|\phi_h - \phi_0|$ and $\|\phi_h - \phi_0\|_{0,\Omega_T}$ is $O(h^2)$, $\|\phi_h - \phi_0\|_{1,\Omega_T}$ is $O(h)$. The results demonstrate our global artificial boundary conditions are very effective.

4.2. Approximate Solution of Problem (1.1)-(1.7)

We can obtain approximate solution ϕ_h^{MK} of problem (1.1)-(1.7) by computing problem (3.10). Take $a = -1, b = 1$ and $c = 1$. Figure 1-3 show $\phi_h^{8,20}$ on mesh B on surface of water when $\mu = 0.2, 0.4, 0.6$, respectively.

We shall test the effect of the terms M, K used in our global artificial boundary conditions. Let ϕ_h^∞ denote the finite element solution of the problem (3.10) when $M = M^*$ and $K = K^*$ is sufficiently large, so ϕ_h^∞ can be treated as numerical solution of problem (3.10) solved with exact boundary conditions. In our computation we take $M^* = 8$ and $K^* = 20$.

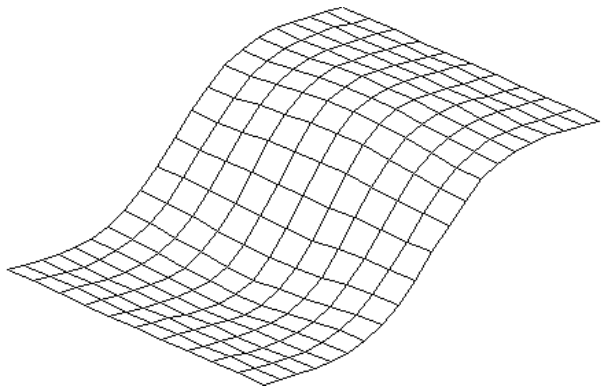

Figure 1 $\phi_h^{8,20}$ on mesh B for $\mu = 0.2$ when $z = 0$

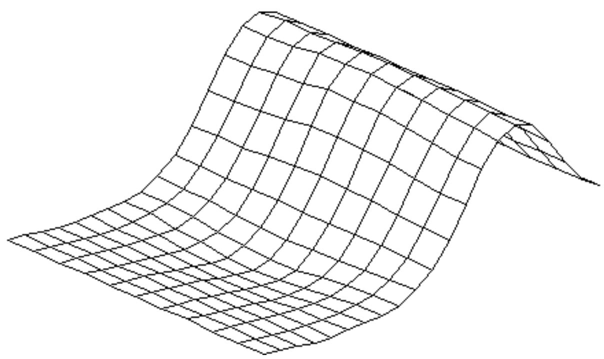

Figure 2 $\phi_h^{8,20}$ on mesh B for $\mu = 0.4$ when $z = 0$

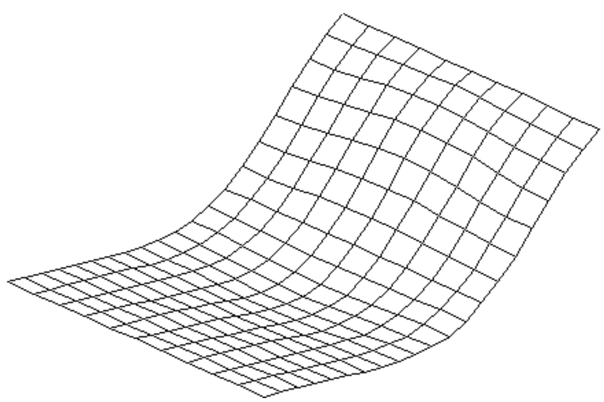

Figure 3 $\phi_h^{8,20}$ on mesh B for $\mu = 0.6$ when $z = 0$

Table 2: The effect of the artificial boundary conditions $\mu = 0.2$

Errors($K = K^*$)	$M = 1$	$M = 3$	$M = 5$	$M = 7$
$\max\|\phi_h^\infty - \phi_h^{MK}\|/\max\|\phi_h^\infty\|$	8.7219E-4	9.0096E-4	8.7077E-4	8.6417E-4
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{0,\Omega_T}/\\|\phi_h^\infty\\|_{0,\Omega_T}$	4.3053E-4	4.2573E-4	4.2437E-4	4.2214E-4
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{1,\Omega_T}/\\|\phi_h^\infty\\|_{1,\Omega_T}$	5.1055E-4	4.6860E-4	4.6413E-4	4.6253E-4

Table 3: The effect of the artificial boundary conditions $\mu = 0.2$

Errors($M = M^*$)	$K = 1$	$K = 5$	$K = 10$	$K = 15$
$\max\|\phi_h^\infty - \phi_h^{MK}\|/\max\|\phi_h^\infty\|$	3.2301E-4	2.8862E-5	1.8553E-6	9.3234E-7
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{0,\Omega_T}/\\|\phi_h^\infty\\|_{0,\Omega_T}$	1.3731E-4	6.2450E-6	7.0347E-7	2.4330E-7
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{1,\Omega_T}/\\|\phi_h^\infty\\|_{1,\Omega_T}$	2.5478E-4	1.45086E-5	1.4297E-6	5.2219E-7

Table 4: The effect of the artificial boundary conditions $\mu = 0.4$

Errors($K = K^*$)	$M = 1$	$M = 3$	$M = 5$	$M = 7$
$\max\|\phi_h^\infty - \phi_h^{MK}\|/\max\|\phi_h^\infty\|$	2.5389E-2	1.9617E-2	7.0296E-3	6.6776E-3
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{0,\Omega_T}/\\|\phi_h^\infty\\|_{0,\Omega_T}$	8.2597E-3	7.6989E-3	2.1021E-3	3.8877E-4
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{1,\Omega_T}/\\|\phi_h^\infty\\|_{1,\Omega_T}$	4.9758E-2	1.3734E-2	3.9729E-3	4.4147E-4

Table 5: The effect of the artificial boundary conditions $\mu = 0.4$

Errors($M = M^*$)	$K = 1$	$K = 5$	$K = 10$	$K = 15$
$\max\|\phi_h^\infty - \phi_h^{MK}\|/\max\|\phi_h^\infty\|$	6.0553E-3	4.1266E-3	4.2377E-4	1.7203E-4
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{0,\Omega_T}/\\|\phi_h^\infty\\|_{0,\Omega_T}$	4.6411E-3	2.5854E-3	2.7808E-4	1.1423E-4
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{1,\Omega_T}/\\|\phi_h^\infty\\|_{1,\Omega_T}$	6.8457E-3	3.6049E-3	3.9006E-4	1.5824E-4

Table 6: The effect of the artificial boundary conditions $\mu = 0.6$

Errors($K = K^*$)	$M = 1$	$M = 3$	$M = 5$	$M = 7$
$\max\|\phi_h^\infty - \phi_h^{MK}\|/\max\|\phi_h^\infty\|$	6.5388E-2	7.8977E-3	1.0496E-3	1.9887E-4
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{0,\Omega_T}/\\|\phi_h^\infty\\|_{0,\Omega_T}$	2.9089E-2	3.6697E-3	3.7359E-4	1.4443E-4
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{1,\Omega_T}/\\|\phi_h^\infty\\|_{1,\Omega_T}$	3.8602E-2	4.9476E-3	5.2436E-4	1.3514E-4

Table 7: The effect of the artificial boundary conditions $\mu = 0.6$

Errors($M = M^*$)	$K = 1$	$K = 5$	$K = 10$	$K = 15$
$\max\|\phi_h^\infty - \phi_h^{MK}\|/\max\|\phi_h^\infty\|$	2.7047E-3	2.3370E-3	3.1732E-4	1.2738E-4
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{0,\Omega_T}/\\|\phi_h^\infty\\|_{0,\Omega_T}$	1.8192E-3	1.3384E-3	1.8077E-4	8.1712E-5
$\\|\phi_h^\infty - \phi_h^{MK}\\|_{1,\Omega_T}/\\|\phi_h^\infty\\|_{1,\Omega_T}$	1.9768E-3	1.5399E-3	2.1147E-4	8.9919E-5

Tables 2-7 show the relative errors of $\phi_h^\infty - \phi_h^{MK}$ in L_∞-norm, L_2-norm and H^1-norm for mesh B with $\mu = 0.2$, 0.4 and 0.6, respectively. As shown in Tables 2-7, our global artificial boundary conditions are good approximate to exact boundary conditions for some M and small K, the error caused by using of global artificial boundary conditions is very small, therefore in the computation very few terms in the bilinear form $A_a^{MK}(\phi,\psi)$ and $A_b^{MK}(\phi,\psi)$ are only needed in order to get good accuracy.

Finally, we shall test the effect of the location of the artificial boundary Γ_b. We take $a = -1$ and $b = 0.6, 0.8, 1.0, 1.2, 1.4$ and 1.6, respectively. For each b, we use a corresponding mesh with the mesh size h=0.1. Let ϕ denote the "exact solution" which is the finite element solution of (3.10) when $b = 1.6$ and $M = M^*, K = K^*$. Table 8 show the relative errors of $\phi - \phi_h^\infty$ in

L_∞-norm, L_2-norm and H^1-norm for different location of the artificial boundary Γ_b, where Ω_0 is the bounded computational domain Ω_T when $b = 0.6$.

Table 8: The effect of the location of the artificial boundary Γ_b

Errors	$b = 0.6$	$b = 0.8$	$b = 1.0$	$b = 1.2$	$b = 1.4$
$\max\lvert\phi - \phi_h^\infty\rvert / \max\lvert\phi\rvert$	3.3529E-2	5.6646E-3	1.0857E-3	3.7762E-4	2.5664E-4
$\lVert\phi - \phi_h^\infty\rVert_{0,\Omega_0} / \lVert\phi\rVert_{0,\Omega_0}$	1.1060E-2	2.2828E-3	4.8799E-4	2.6128E-4	2.0300E-4
$\lVert\phi - \phi_h^\infty\rVert_{1,\Omega_0} / \lVert\phi\rVert_{1,\Omega_0}$	2.0457E-2	3.8641E-3	8.0621E-4	3.6427E-4	2.4676E-4

As shown in table 8, the influents caused by different location of artificial boundary Γ_b is very small. Therefore for a given accuracy, it is possible to use a small bounded computational domain. So the using of global artificial boundary conditions can save computational cost greatly.

5. Conclusions

A sequence of high-order global artificial boundary conditions at the downstream and upstream artificial boundaries are designed for the three-dimensional steady potential flow around a body moving in a liquid of finite constant depth at constant speed and distance below a free surface in a channel. Then the original problem is reduced to a problem defined on a finite computational domain, which is equivalent to a variational problem. The variational problem can be solved by finite element method. Then the numerical approximation for the original problem is obtained. Numerical examples show that our global artificial boundary conditions are very effective. Summarizing this paper, We can make some remarks:

- Our global artificial boundary conditions are very effective. With this method, the original problem is reduced to a problem defined on a bounded computational domain with high accuracy, and only a few terms in the global artificial boundary conditions are needed in computation.
- Introduction of V^* is necessary, because the variational problem (3.1) is not suitable for finite element method.
- Introduction of auxiliary artificial boundary $\Gamma_{b'}$ is necessary, because wave shape at downstream must be determined by wave values on two different boundaries.
- The numerical example shows that the convergence rate of the mesh size is consistent with the usual finite element error estimation for the problems in a bounded domain when using our artificial boundary conditions to solve a problem in an unbounded domain.
- The numerical example shows that the influents caused by different location of artificial boundary Γ_b is very small. Therefore we can choose a small bounded computational domain to get high accuracy.

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