Journal of Computational Mathematics, Vol.21, No.4, 2003, 481–490.

ON THE ERROR ESTIMATE OF NONCONFORMING FINITE ELEMENT APPROXIMATION TO THE OBSTACLE PROBLEM*1)

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Abstract

This paper is devoted to analysis of the nonconforming element approximation to the obstacle problem, and improvement and correction of the results in [11], [12].

Key words: Obstacle problem, Nonconforming finite element.

1. Introduction

For the conforming (i.e. C^0-) linear finite element approximation to the obstacle problem, the error bound $O(h)$ has been obtained by Falk [5] in homogeneous boundary data and Brezzi et.al., [3] in nonhomogeneous data and with lower order term. The author considered nonconforming (i.e. nonC^0-) finite element approximation to the obstacle problem in [10] and [11]. Later, [12] presented a rigorous proof of the error bound $O(h)$ and correction of the proof in [11] for nonconforming linear element approximation to the obstacle problem under the hypothesis that the free boundary has finite length as in [3].

In general, the length of the free boundary could be not finite, because there exist probably infinite simply connected coincidence sets, while the length of the boundary of each such coincidence set is finite for the smooth solution of the obstacle problem. In fact, one can constructe example of infinite simply connected sets in a bounded domain, with property that the total length of the boundaries of all the sets is infinite. Thus it makes sense to estimate the error bound of nonconforming linear element approximation to the obstacle problem without the hypothesis of finite length of the free boundary.

In this paper, by the similar way as [3], we obtaine the error bound $O(h)$ for the nonconforming linear element approximation to the obstacle problem without the hypothesis of finite length of the free boundary. And in [10], [11] the author also analyzed Wilson's element for the obstacle problem with an unnatural construction of the discrete convex set K_h. In this paper we consider Wilson's element approximation to the obstacle problem with a natural and simple construction of the discrete convex set K_h, and obtaine the same error bound $O(h)$ as in [10], [11].

Let Ω be a bounded convex domain in R^2 with smooth boundary $\partial\Omega$, and $f \in L^2(\Omega), \chi \in H^2(\Omega), g$ be the trace of a function in $H^2(\Omega)$ on $\partial\Omega$ and $\chi \leq g$ on $\partial\Omega$. Let us consider the following obstacle problem:

$$\begin{cases} to \ \ find \qquad u \in K, \ \ such \ \ that \\ a(u, v-u) \geq (f, v-u) \qquad \forall \ \ v \in K, \end{cases} \tag{1.1}$$

where

$$K = \{v \in H^1(\Omega) : v \geq \chi \ \ a.e.in \ \ \Omega, v = g \ \ on \ \ \partial\Omega\}, \tag{1.2}$$

* Received March 16, 2001.
1) The project was supported by the National Natural Science Foundation of China.

$$a(u,v)=\int_\Omega \nabla u\cdot\nabla v dx, \qquad (f,v)=\int_\Omega f\cdot v dx. \tag{1.3}$$

It is well known that (see [4]) problem (1.1) is equivalent to the following differential problem:

$$\begin{cases} -\Delta u = f \quad in \quad \Omega^+=\{x\in\Omega : u(x)>\chi(x)\}, \\ -\Delta u \geq f \quad in \quad \Omega^0=\{x\in\Omega : u(x)=\chi(x)\}, \\ u\geq \chi \quad in \quad \Omega, \quad and \quad u=g \quad on \quad \partial\Omega. \end{cases} \tag{1.4}$$

For the regularity of the solution of the obstacle problem (1.1), we now present a very impotant result by Brezis:

Lemma 1.1 (see [6], [7]). *If $f\in L^\infty(\Omega)\cap BV(\Omega), (g,\chi)\in C^3(\bar\Omega)$ with $\chi\leq g$ on $\partial\Omega$ and $\partial\Omega$ is sufficiently smooth, then the problem (1.1) has a solution*

$$u\in W^{s,p}(\Omega), \qquad 1<p<\infty, \quad s<2+\frac{1}{p}. \tag{1.5}$$

We now consider the finite element approximation to problem (1.1). Let $\mathcal{T}_h$ be a regular subdivision on Ω, $\Omega_h=\cup_{T\in\mathcal{T}_h}T$, with $T\in\mathcal{T}_h$ denoting the element, and let $V_h\subset L^2(\Omega_h)$ be the finite element space with norm $\|\cdot\|_h$, and $K_h\subset V_h$ be a closed convex subset as an approximation of K. Then the finite element approximate problem of (1.1) is the following:

$$\begin{cases} to \ \ find \qquad u_h\in K_h, \qquad such\ that \\ a_h(u_h,v_h-u_h)\geq (f,v_h-u_h) \qquad \forall \ \ v_h\in K_h. \end{cases} \tag{1.6}$$

where

$$a_h(u_h,v_h)=\sum_{T\in\mathcal{T}_h}\int_T \nabla u_h\cdot\nabla v_h dx. \tag{1.7}$$

2. The Nonconforming Linear Element Approximation

Let $\mathcal{T}_h$ be a regular triangulation of Ω, the vertices of the element T be denoted by $a_i^T, 1\leq i\leq 3$, and the midpoints of the edges of the element T be denoted by $m_i^T, 1\leq i\leq 3$. And let X_h denote the nonconforming linear element space with respect to the triangulation $\mathcal{T}_h$. Let (see Fig.2.1)

$$V_h=\{v_h\in X_h : v_h(m)=g(P_m) \ \ \forall \ \ nodes \ \ m\in\partial\Omega_h\}, \tag{2.1}$$

$$K_h=\{v_h\in V_h : v_h(m_i^T)\geq \chi(m_i^T) \ \ \forall \ \ T\in\mathcal{T}_h \ \ and \ \ m_i^T\notin\partial\Omega_h\}. \tag{2.2}$$

Let

$$V_h^0=\{v_h\in X_h : v_h(m)=0 \ \ \forall \ \ nodes \ \ m\in\partial\Omega_h\}, \tag{2.3}$$

then

$$\|w_h\|_h=a_h(w_h,w_h)^{\frac{1}{2}} \tag{2.4}$$

is a norm in V_h^0.

In order to estimate the error bound of the approximate problem (1.6), firstly we have the abstract error estimate:

Lamma 2.1. *Assume that u and u_h denote the solutions of the problems (1.1) and (1.6) respectively. Then $\forall v_h\in K_h$, the following inequalities hold:*

$$\|u-u_h\|_h\leq \|u-v_h\|_h+\|v_h-u_h\|_h, \tag{2.5}$$

and

$$\|u_h - v_h\|_h \le C\{\|u - v_h\|_h^2 + (w, v_h - u_h) + \sum_T \int_{\partial T} \partial_\nu u \cdot (v_h - u_h) ds\}, \tag{2.6}$$

with $C = Const. > 0$ *independent of* u *and* h, *where* $w = -(\Delta u + f)$.

Proof. Inequality (2.5) is just the triangle inequality. Inequality (2.6) can be proved as follows: Since u_h being the solution of (1.6), then $\forall v_h \in K_h$ we have

$$\begin{aligned}\|v_h - u_h\|_h^2 &= a_h(v_h - u_h, v_h - u_h)\\ &= a_h(v_h - u, v_h - u_h) + a_h(u, v_h - u_h) - a_h(u_h, v_h - u_h)\\ &\le \|u - v_h\|_h \cdot \|v_h - u_h\|_h + a_h(u, v_h - u_h) - (f, v_h - u_h),\end{aligned}$$

from which we have

$$\|v_h - u_h\|_h^2 \le 2\{\|v_h - u\|_h^2 + a_h(u, v_h - u_h) - (f, v_h - u_h)\},$$

and by Green's formula, we have

$$\begin{aligned}a_h(u, v_h - u_h) - (f, v_h - u_h) &= \sum_T \int_T \nabla u \cdot \nabla(v_h - u_h) dx - \int_\Omega f(v_h - u_h) dx\\ &= \int_\Omega (-\Delta u - f)(v_h - u_h) dx + \sum_T \int_T \partial_\nu u \cdot (v_h - u_h) ds,\end{aligned}$$

from which and previous inequality, the estimate (2.6) is proved.

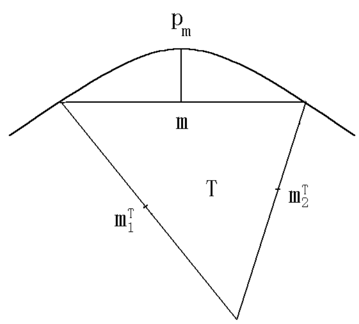

Fig. 2. 1.

For later use, we introduce the following

Lemma 2.2 (see [3]). *For all* $T \in \mathcal{T}_h$, *and* $v \in P_1(T)$ *such that* $v(x^T) = 0, x^T \in T$, *the following inequality holds:*

$$\|v\|_{0,T} \le h_T |v|_{1,T}. \tag{2.7}$$

Proof. In fact, we have

$$\begin{aligned}\|v\|_{0,T}^2 &= \int_T |v(x)|^2 dx = \int_T |v(x) - v(x^T)|^2 dx\\ &= \int_T |\nabla v|^2 \cdot |x - x^T|^2 dx \le h_T^2 |v|_{1,T}^2.\end{aligned}$$

The following error estimate using the nonconforming linear element approximation to the obstacle problem (1.1) holds:

Theorem 2.1. *Assume that the solution* u *of problem (1.1) possesses the regularity:* $u \in W^{s,p}(\Omega)\ \forall\ 1 < p < \infty, s < 2 + \frac{1}{p}$, *and the data are satisfying:* $f \in H^{0.5-\epsilon}(\Omega), \chi \in W^{2+\frac{1}{p}-\epsilon,p}(\Omega)$

for any $\epsilon > 0$ *and* $g \in H^2(\Omega), g \geq \chi$ *on* $\partial\Omega$, *then for the solution* u_h *of the nonconforming linear element approximation (1.6), the following error estimate holds:*

$$\|u - u_h\|_h \leq Ch, \tag{2.8}$$

with $C = Const. > 0$ *independent of* h.

3. The Proof of Theorem 2.1

We now prove Theorem 2.1 as follows:

(i) Let $\Pi_h : C^0(\bar{\Omega}) \longrightarrow X_h$ be the interpolate operator as follows, for any given $v \in C^0(\bar{\Omega}), \Pi_h v \in X_h$, such that

$$\Pi_h v(m_i^T) = v(m_i^T) \;\; \forall \;\; T \in \mathcal{T}_h, \;\;\; 1 \leq i \leq 3,$$

and let

$$\Pi_T v = \Pi_h v|_T.$$

Since $\Pi_h v(m) = v(m) \neq v(P_m) = g(P_m) \;\;\; \forall m \in \partial\Omega_h$ (see Fig.2.1), then $\Pi_h v \notin K_h$ for $v \in K$. So we should modify the interpolate operator Π_h as follows: Let $\tilde{\Pi}_h : C^0(\Omega_h) \longrightarrow X_h$ be defined as follows, for any given $v \in C^0(\Omega_h), \tilde{\Pi}_h v \in X_h$, such that

$$\tilde{\Pi}_T v = \Pi_T v \;\; \forall \;\; T \in \mathcal{T}_h^0 - -\textit{the set of the interior elements},$$

and for any $T \in \partial\mathcal{T}_h$ $--$ the set of the boundary elements, $m \in \partial\Omega_h$ – the midpoint of one edge of $T \in \partial\mathcal{T}_h$ and m_1^T, m_2^T– the midpoints of the other two edges of $T \in \partial\mathcal{T}_h$ (see Fig.2.1),

$$\tilde{\Pi}_T v(m_i^T) = v(m_i^T) \;\;\; i = 1, 2,$$

and

$$\tilde{\Pi}_T v(m) = g(P_m) = v(P_m).$$

Then for any $v \in K$, it can be seen that $\tilde{\Pi}_h v \in K_h$ and for any $T \in \partial\mathcal{T}_h$, we have

$$(\Pi_T v - \tilde{\Pi}_T v)(x) = (v(m) - v(P_m)) \cdot \mu_m(x) \;\; \forall \;\; x \in T,$$

where $\mu_m(x)$ denotes the basic function of the nonconforming linear interpolant: $\mu_m(x) \in P_1(T)$– the space of linear polynomials defined on T, and

$$\mu_m(m) = 1; \;\;\; \mu_m(m_i^T) = 0, \;\;\; i = 1, 2.$$

Under the assumption that the boundary $\partial\Omega$ is piecewise smooth, then $|\overrightarrow{P_m m}| \leq Ch_T^2$, and

$$|v(m) - v(P_m)| = |\nabla v(Q_m) \cdot \overrightarrow{P_m m}| \leq Ch_T^2 \cdot |v|_{1,\infty,\Omega}.$$

Thus we have $\forall \; T \in \partial\mathcal{T}_h$,

$$\|\Pi_T v - \tilde{\Pi}_T v\|_{0,T} \leq |v(m) - v(P_m)| \cdot \|\mu_m\|_{0,T} \leq Ch_T^3 |v|_{1,\infty,\Omega},$$

and

$$\begin{aligned} \|\Pi_h v - \tilde{\Pi}_h v\|_{0,\Omega_h} &= \{\sum_{T \in \partial\mathcal{T}_h} \|\Pi_T v - \tilde{\Pi}_T v\|_{0,T}^2\}^{\frac{1}{2}} \\ &\leq Ch^3 \{\sum_{T \in \partial\mathcal{T}_h} 1\}^{\frac{1}{2}} \cdot |v|_{1,\infty,\Omega} \leq Ch^{2.5} |v|_{1,\infty,\Omega}. \end{aligned} \tag{3.1}$$

Furthermore, $\forall \; T \in \partial\mathcal{T}_h$

$$|\Pi_T v - \tilde{\Pi}_T|_{1,T} \leq |v(m) - v(P_m)| \cdot |\mu_m|_{1,T} \leq Ch^2 |v|_{1,\infty,\Omega},$$

and

$$\|\Pi_h v-\tilde{\Pi}_h v\|_h=\{\sum_{T\in\partial\mathcal{T}_h}|\Pi_T v-\tilde{\Pi}_T v|_{1,T}^2|^2\}^{\frac{1}{2}}$$
$$\le Ch^2\{\sum_{T\in\partial\mathcal{T}_h}1\}^{\frac{1}{2}}|v|_{1,\infty,\Omega}\le Ch^{1.5}|v|_{1,\infty,\Omega}. \tag{3.2}$$

Now we let $v_h=\tilde{\Pi}_h u$ in (2.5) and (2.6), with u being the solution of problem (1.1). Then by the interpolation error estimate and (3.2) we have

$$\|u-\tilde{\Pi}_h u\|_h\le\|u-\Pi_h u\|_h+\|\Pi_h u-\tilde{\Pi}_h u\|_h$$
$$\le Ch(|u|_{2,\Omega}+h^{0.5}|u|_{1,\infty,\Omega}). \tag{3.3}$$

And by the standard error estimate for nonconforming linear finite element (see [9]), we have

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_\nu\cdot(\tilde{\Pi}_h u-u_h)ds\le Ch|u|_{2,\Omega}\cdot\|\tilde{\Pi}_h u-u_h\|_h. \tag{3.4}$$

Thus by Lemma 2.1, in order to estimate the error of the nonconforming linear element approximation to the obstacle problem, it is sufficient to estimate:

$$(w,\tilde{\Pi}_h u-u_h)$$
$$=(w,\tilde{\Pi}_h(u-\chi)-(u-\chi))+(w,u-\chi)+(w,\tilde{\Pi}_h\chi-u_h). \tag{3.5}$$

Since $(w,u-\chi)=0$ (see (1.4)), by using the interpolation error estimate and (3.1):

$$(w,\tilde{\Pi}_h(u-\chi)-(u-\chi))\le\|w\|_{0,\Omega}\cdot\|\tilde{\Pi}_h(u-\chi)-(u-\chi)\|_{0,\Omega_h}$$
$$\le Ch^2\|w\|_{0,\Omega}\{\|u-\chi\|_{2,\Omega}+h^{0.5}|u-\chi|_{1,\infty,\Omega}\}, \tag{3.6}$$

we have

$$(w,\tilde{\Pi}_h u-u_h)\le Ch^2(\|u-\chi\|_{2,\Omega}+h^{0.5}|u-\chi|_{1,\infty,\Omega})+(w,\tilde{\Pi}_h\chi-u_h). \tag{3.7}$$

(ii) By (3.1), we have

$$(w,\tilde{\Pi}_h\chi-u_h)=(w,\Pi_h\chi-u_h)+(w,\tilde{\Pi}_h\chi-\Pi_h\chi)$$
$$\le(w,\Pi_h\chi-u_h)+Ch^{2.5}\|w\|_{0,\Omega}\cdot|\chi|_{1,\infty,\Omega}. \tag{3.8}$$

In order to estimate the first term on the right hand side of (3.8), let

$$\begin{cases}\Omega_h^+=\{T\in\mathcal{T}_h:T\subset\Omega^+\},\\ \Omega_h^0=\{T\in\mathcal{T}_h:T\subset\Omega^0\},\\ \Omega_h^-=\{T\in\mathcal{T}_h:T\cap\Omega^+\neq\emptyset,T\cap\Omega^0\neq\emptyset\}.\end{cases} \tag{3.9}$$

Since $w(x)=0\ \ \forall\ \ x\in\Omega^+$, then

$$(w,\Pi_h\chi-u_h)=\sum_{T\in\Omega_h^0}\int_T w(\Pi_h\chi-u_h)dx+\sum_{T\in\Omega_h^-}\int_T w(\Pi_h\chi-u_h)dx. \tag{3.10}$$

We now firstly estimate the first term on the right hand side of (3.10). Since on $T\in\Omega_h^0, w\ge 0$ and $\chi=u$, if $\Pi_T\chi-u_h\le 0$ on $T\in\Omega_h^0$, then

$$\int_T w(\Pi_h\chi-u_h)dx\le 0. \tag{3.11}$$

Otherwise, there exists $x^0\in T$, such that $(\Pi_h\chi-u_h)(x^0)=0$, in this case, we have, by Lemma 2.2,

$$\int_T w(\Pi_h\chi-u_h)dx\le Ch\|w\|_{0,T}|\Pi_h\chi-u_h|_{1,T}$$
$$\le Ch\|w\|_{0,T}\{|\Pi_h\chi-\chi|_{1,T}+|\chi-u_h|_{1,T}\}$$
$$\le Ch\|w\|_{0,T}\{h|\chi|_{2,T}+|u-u_h|_{1,T}\}, \tag{3.12}$$

since $\chi = u$ on $T \in \Omega_h^0$. Then

$$\sum_{T\in\Omega_h^0} \int_T w(\Pi_h \chi - u_h)dx \le Ch\|w\|_{0,\Omega}\{h|\chi|_{2,\Omega} + \|u-u_h\|_h\}. \tag{3.13}$$

(iii) Next we estimate the second term on the right hand side of (3.10). On $T \in \Omega_h^-, w \ge 0$, but $\chi \ne u$. If $\Pi_T\chi - u_h \le 0$ on $T \in \Omega_h^-$, then

$$\int_T w(\Pi_T\chi - u_h)dx \le 0. \tag{3.14}$$

Thus we consider only the case where there exists $x^T \in T \in \Omega_h^-$, such that $(\Pi_T\chi - u_h)(x^T) = 0$. By the Lemma 2.2 and $(\Pi_T\chi - u_h)(m_i^T) \le 0, m_i^T$–the midpoints of the edges of T, $1 \le i \le 3$, we have

$$\begin{aligned}\int_T w(\Pi_T\chi - u_h)dx &\le \int_T (w - P_0^T w)(\Pi_T\chi - u_h)dx \\ &\le \|w - P_0^T w\|_{0,T}\|\Pi_T\chi - u_h\|_{0,T} \\ &\le Ch\|w - P_0^T w\|_{0,T}|\Pi_T\chi - u_h|_{1,T}.\end{aligned} \tag{3.15}$$

By the interpolation error estimate (see [4], [2]), we have

$$\|w - P_0^T w\|_{0,T} \le Ch^{0.5-\epsilon_1}\|w\|_{0.5-\epsilon_1,T},$$

then

$$\int_T w(\Pi_T\chi - u_h)dx \le Ch^{1.5-\epsilon_1}\|w\|_{0.5-\epsilon_1,T} \cdot |\Pi_T\chi - u_h|_{1,T} \tag{3.16}$$

for any $\epsilon_1 > 0$. By the triangle inequality as in [3], we have that

$$|\Pi_T\chi - u_h|_{1,T} \le |\Pi_T\chi - \chi|_{1,T} + |\chi - u|_{1,T} + |u - u_h|_{1,T}, \tag{3.17}$$

thus

$$\begin{aligned}&\sum_{T\in\Omega_h^-} \int_T w(\Pi_T\chi - u_h)dx \\ &\le Ch^{1.5-\epsilon_1} \sum_{T\in\Omega_h^-} \|w\|_{0.5-\epsilon_1,T}\{|\Pi_T\chi - \chi|_{1,T} + |\chi - u|_{1,T} + |u - u_h|_{1,T}\}.\end{aligned} \tag{3.18}$$

The first term on the right hand side of (3.18) can be estimated as follows: By the interpolation error estimate (see [4]),

$$\begin{aligned}&Ch^{1.5-\epsilon_1} \sum_{T\in\Omega_h^-} \|w\|_{0.5-\epsilon_1,T}|\Pi_T\chi - \chi|_{1,T} \\ &\le Ch^{2.5-\epsilon_1} \sum_{T\in\Omega_h^-} \|w\|_{0.5-\epsilon_1,T}|\chi|_{2,T} \le Ch^{2.5-\epsilon_1}\|w\|_{0.5-\epsilon_1,\Omega} \cdot |\chi|_{2,\Omega}.\end{aligned} \tag{3.19}$$

The third term on the right hand side of (3.18) can be estimated as follows:

$$\begin{aligned}&Ch^{1.5-\epsilon_1} \sum_{T\in\Omega_h^-} \|w\|_{0.5-\epsilon_2,T} \cdot |u - u_h|_{1,T} \\ &\le Ch^{1.5-\epsilon_1}\|w\|_{0.5-\epsilon_1,\Omega} \cdot \|u - u_h\|_h.\end{aligned} \tag{3.20}$$

Finally we estimate the second term on the right hand side of (3.18). By the assumptions of Theorem 3.1, we have that

$$(\chi, u) \in W^{2+\frac{1}{p}-\epsilon_2,p}(\Omega), \quad 1 < p < \infty, \quad \epsilon_2 > 0, \tag{3.21}$$

then (see [1], [7, §3.5, Theorem 5.1])

$$\nabla(\chi - u) \in W^{1+\frac{1}{p}-\epsilon_2,p}(\Omega) \hookrightarrow C^{0,\alpha}\Omega, \quad \alpha = 1 - \frac{1}{p} - \epsilon_2, \tag{3.22}$$

from which, and for such $T \in \Omega_h^-$, that there exists $x^T \in T$:

$$\nabla(\chi - u)(x^T) = 0,$$

we have $\forall \;\; x \in T$,

$$\begin{aligned} &|\nabla(\chi - u)(x)| = |\nabla(\chi - u)(x) - \nabla(\chi - u)(x^T)| \\ &\le C|x - x^T|^\alpha \|\chi - u\|_{2+\frac{1}{p}-\epsilon_2,p,\Omega} \\ &\le Ch^\alpha \|\chi - u\|_{2+\frac{1}{p}-\epsilon_2,p,\Omega}. \end{aligned}$$

Then

$$|\chi - u|_{1,T} = \{\int_T |\nabla(\chi - u)|^2 dx\}^{\frac{1}{2}} \le Ch^{\alpha+1}\|\chi - u\|_{2+\frac{1}{p}-\epsilon_2,p,\Omega}. \tag{3.23}$$

Thus taking into account that $\alpha = 1 - \frac{1}{p} - \epsilon_2$, the second term on the right hand side of (3.18) can be estimated as follows:

$$\begin{aligned} &Ch^{1.5-\epsilon_1} \sum_{T\in\Omega_h^-} \|w\|_{0.5-\epsilon_1,T} \cdot |\chi - u|_{1,T} \\ &\le Ch^{2.5+\alpha-\epsilon_1}\{\sum_{T\in\Omega_h^-} \|w\|_{0.5-\epsilon_1,T}\}\|\chi - u\|_{2+\frac{1}{p}-\epsilon_2,p,\Omega} \\ &\le Ch^{2.5+\alpha-\epsilon_1}(\sum_{T\in\Omega_h^-} 1)^{\frac{1}{2}} \cdot \|w\|_{0.5-\epsilon_1,\Omega} \cdot \|\chi - u\|_{2+\frac{1}{p}-\epsilon_2,p,\Omega} \\ &\le Ch^{2+\frac{1}{2}-(\frac{1}{p}+\epsilon_1+\epsilon_2)}\|w\|_{0.5-\epsilon_1,\Omega} \cdot \|\chi - u\|_{2+\frac{1}{p}-\epsilon_2,p,\Omega}, \end{aligned} \tag{3.24}$$

from which it can be seen that the second term on the right hand side of (3.18) is bounded by

$$Ch^{1.5-\epsilon_1} \sum_{T\in\Omega_h^-} \|w\|_{0.5-\epsilon_1,T} \cdot |\chi - u|_{1,T} = O(h^2), \tag{3.25}$$

if $\frac{1}{p} + \epsilon_1 + \epsilon_2 \le \frac{1}{2}$.

Thus the proof is completed.

4. Wilson's Element Approximation

In this section, we consider Wilson's element approximation to problem (1.1). Let Ω be a rectangular domain in R^2, $\mathcal{T}_h$ a regular subdivision of Ω, and $T \in \mathcal{T}_h$ be the rectangle element, whose vertices are denoted by $a_i, 1 \le i \le 4$ (see Fig.4.1). Let X_h denote Wilson's finite element space, i.e., for any $v_h \in X_h, v_h|_T \in P_2(T)-$ the space of polynomials, whose degree ≤ 2, with the expression:

$$v_h(x)|_T = \sum_{i=1}^4 v_h(a_i) \cdot p_i(x) + \sum_{j=1}^2 \phi_j(v_h) \cdot q_j(x), \tag{4.1}$$

where

$$\begin{cases} p_2(x) = \frac{1}{4}(1 + \frac{x_1-c_1}{h_1})(1 + \frac{x_2-c_2}{h_2}), \\ \cdots\cdots \\ p_4(x) = \frac{1}{4}(1 + \frac{x_1-c_1}{h_1})(1 - \frac{x_2-c_2}{h_2}); \end{cases} \tag{4.2}$$

and

$$\begin{cases} q_j(x) = \frac{1}{8}[(\frac{x_j - c_j}{h_j})^2 - 1], \\ \phi_j(v_h) = \frac{h_j^2}{h_1 h_2} \int_T \partial_{jj} v_h dx, \ 1 \le j \le 2, \end{cases} \tag{4.3}$$

$$c = \frac{1}{4} \sum_{i=1}^{4} a_i. \tag{4.4}$$

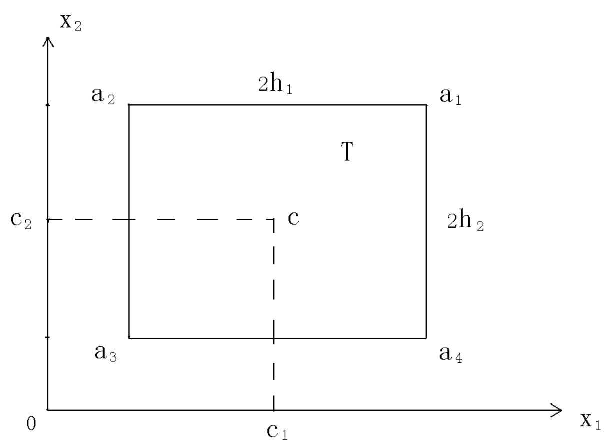

Fig. 4. 1.

In [10] and [11], the following Wilson's element approximation to the obstacle problem (1.1) has been considered:

$$\begin{cases} to\ find \quad u_h \in \tilde{K}_h, \quad such\ that \\ a_h(u_h, v_h - u_h) \ge (f, v_h - u_h) \quad \forall\ v_h \in \tilde{K}_h, \end{cases} \tag{4.5}$$

where

$$\begin{aligned} \tilde{K}_h = \quad & \{v_h \in X_h : v_h|_T(a_i) \ge \chi|_T(a_i), 1 \le i \le 4 \ and\ \phi_j(v_h) \le \phi_j(\chi), 1 \le j \le 2 \\ & \forall\ T \in \mathcal{T}_h; v_h(Q) = g(Q)\ \ \forall\ \ boundary\ nodes\ Q \in \partial\Omega\}. \end{aligned} \tag{4.6}$$

The reason of including the restrictions: $\phi_j(v_h) \le \phi_j(\chi), 1 \le j \le 2$, in $\tilde{K}_h$, is that if $v_h \in \tilde{K}_h$ then $v_h \ge \Pi_h \chi$, where $\Pi_h : H^2(\Omega) \longrightarrow X_h$ denotes the interpolation operator defined later (see (4.10)). But these restrictions are such that the construction of $\tilde{K}_h$ is unnatural, and the computation of problem (4.5) is uncoveniunt. We now consider a natural and simple formula of Wilson's element approximation to the obstacle problem (1.1) without the previous restrictions: $\phi_j(v_h) \le \phi_j(\chi), 1 \le j \le 2$, in $\tilde{K}_h$. Let

$$\begin{aligned} K_h = \quad & \{v_h \in X_h : v_h|_T(a_i) \ge \chi|_T(a_i), 1 \le i \le 4\ \ \forall\ \ T \in \mathcal{T}_h; \\ & v_h(Q) = g(Q)\ \ \forall\ \ boundary\ nodes\ Q \in \partial\Omega\}, \end{aligned} \tag{4.7}$$

and Wilson's element approximation to the obstacle problem (1.1) is as follows:

$$\begin{cases} to\ \ find \quad u_h \in K_h, \quad such\ that \\ a_h(u_h, v_h - u_h) \ge (f, v_h - u_h) \quad \forall\ v_h \in K_h. \end{cases} \tag{4.8}$$

We have the following error estimate

Theorem 4.1. *Assume that u and u_h are the solutions of the problems (1.1) and (4.8) respectively, and that $u \in H^2(\Omega), \chi \in H^2(\Omega), g \in H^2(\Omega), g \ge \chi$ on $\partial\Omega$ and the inverse hypothesis of the subdivision $\mathcal{T}_h$ is satisfied. Then the error estimate holds:*

$$\|u - u_h\|_h \le Ch\{|u|_{2,\Omega} + |\chi|_{2,\Omega} + \|w\|_{0,\Omega}\}, \tag{4.9}$$

where $w = -(\Delta u + f)$, and $C = Const. > 0$ independent of u and h.

Proof. (i) Let $\Pi_h : H^2(\Omega) \longrightarrow X_h$ denote the interpolation operator defined as follows: for any given $v \in H^2(\Omega)$,

$$\Pi_T v(x) = \sum_{i=1}^{4} v(a_i)p_i(x) + \sum_{j=1}^{2} \phi_j(v)q_j(x), \qquad \forall \; T \in \mathcal{T}_h, \tag{4.10}$$

and

$$\Pi_h v|_T = \Pi_T v \qquad \forall \; T \in \mathcal{T}_h. \tag{4.11}$$

Then we have $\Pi_h u \in K_h$. Thus by taking $v_h = \Pi_h v$ in (2.5) and (2.6) in the abstract error estimate Lemma 2.1., and the standard error estimate for Wilson's element (see [9]):

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \partial_\nu \cdot (\Pi_h u - u_h) ds \leq Ch|u|_{2,\Omega} \cdot \|\Pi_h u - u_h\|_h, \tag{4.12}$$

in order to obtain the error estimate (4.9), it is sufficient to estimate $(w, \Pi_h u - u_h)$.

(ii) by formula (1.4), then $(w, u - \chi) = 0$, thus we have

$$(w, \Pi_h u - u_h) = (w, \Pi_h(u - \chi) - (u - \chi)) + (w, \Pi_h \chi - u_h). \tag{4.13}$$

By the interpolation error estimate (see [4]), we have

$$(w, \Pi_h(u - \chi) - (u - \chi)) \leq Ch^2 \|w\|_{0,\Omega} |u - \chi|_{2,\Omega}. \tag{4.14}$$

Thus the remainder to be estimated is the term $(w, \Pi_h \chi - u_h)$. We now introduce another interpolation operator $\tilde{\Pi}_h : H^2(\Omega) \longrightarrow X_h$ defined as follows: for any given $v \in H^2(\Omega)$,

$$\tilde{\Pi}_T v(x) = \sum_{i=1}^{4} v(a_i)p_i(x) + \sum_{j=1}^{2} \tilde{\phi}_j(v, \chi)q_j(x) \qquad \forall \; T \in \mathcal{T}_h, \tag{4.15}$$

and

$$\tilde{\Pi}_h v|_T = \tilde{\Pi}_T v, \tag{4.16}$$

where

$$\tilde{\phi}_j(v, \chi) = min(\phi_j(v), \phi_j(\chi)). \tag{4.17}$$

Since

$$\phi_j(\tilde{\Pi}_T v) = \tilde{\phi}_j(v, \chi) \leq \phi_j(\chi), \quad \tilde{\Pi}_T v(a_i) = v(a_i), \tag{4.18}$$

it is easily seen that

$$(\Pi_h \chi - \tilde{\Pi}_h u_h)(x) \leq 0 \qquad \forall \; x \in \Omega, \tag{4.19}$$

from which we have, since $w \geq 0$ in Ω (see (1.4)),

$$\begin{aligned}
(w, \Pi_h \chi - u_h) \qquad &= (w, \Pi_h \chi - \tilde{\Pi}_h u_h) + (w, \tilde{\Pi}_h u_h - u_h) \\
&\leq (w, \tilde{\Pi}_h u_h - \Pi_h u_h) = \sum_T \int_T w(\tilde{\Pi}_T u_h - \Pi_T u_h) dx \\
&= \sum_T \int_T w \sum_{j=1}^{2} \{\tilde{\phi}_j(u_h, \chi) - \phi_j(u_h)\} \cdot q_j(x) dx \\
&\leq \sum_T \|w\|_{0,T} \sum_{j=1}^{2} |\phi_j(\chi) - \phi_j(u_h)| \cdot \|q_j\|_{0,T} \\
&\leq Ch \sum_T \|w\|_{0,T} \int_T |\partial_{jj}\chi - \partial_{jj} u_h| dx \\
&\leq Ch^2 \sum_T \|w\|_{0,T} \cdot |\chi - u_h|_{2,T}.
\end{aligned} \tag{4.20}$$

By the triangle inequality and the inverse inequality (see [4]), we have

$$\begin{aligned}
&|\chi - u_h|_{2,T} \le |\chi - u|_{2,T} + |u - \Pi_T u|_{2,T} + |\Pi_T u - u_h|_{2,T} \\
&\le C\{|\chi|_{2,T} + |u|_{2,T} + h^{-1}|\Pi_T u - u_h|_{1,T}\} \\
&\le C\{|\chi|_{2,T} + |u|_{2,T} + h^{-1}|\Pi_T u - u|_{1,T} + h^{-1}|u - u_h|_{1,T}\} \\
&\le C\{|\chi|_{2,T} + |u|_{2,T}\} + Ch^{-1}|u - u_h|_{1,T}.
\end{aligned} \tag{4.21}$$

From (4.20) and (4.21), we have

$$\begin{aligned}
&(w, \Pi_h\chi - u_h) \le Ch^2 \sum_T \|w\|_{0,T}(|\chi|_{2,T} + |u|_{2,T}) + Ch \sum_T \|w\|_{0,T}|u - u_h|_{1,T} \\
&\le Ch^2\|w\|_{0,\Omega}(|\chi|_{2,\Omega} + |u|_{2,\Omega}) + Ch\|w\|_{0,\Omega}\|u - u_h\|_h.
\end{aligned} \tag{4.22}$$

Thus the proof is completed.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods, Springer-Verlag, 1994.
[3] F. Brezzi, W.W. Hager and P.A. Raviart, Error estimates for the finite element solution of variational inequalities, *Numer. Math.,* **28** (1977), 431-443.
[4] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, New York, Oxford, 1978.
[5] R. Falk, Error estimates for the approximation of a class of variational inequalities, *Math.Comput.,* **28** (1974), 963-971.
[6] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.
[7] J.F. Rodrigues, Obstacle Problems in Mathematical Physics, North-Holland, Amsterdam, 1987.
[8] G. Strang, The finite element method- linear and nonlinear application, Proceedings of the international Congress of Mathematicians, Vancouver, Canada, 1974.
[9] F. Stummel, The generalized patch test, *SIAM J. Numer.Anal.,* **16** (1979), 449-471.
[10] Lie-heng Wang, Error estimate of Wilson's finite element method for the obstacle problem, Proc. of the 1984 Beijing Symposium on Diff. Geometry and Diff. Eqs.- Comp. of PDEs., Chief Editor, Feng Kang, Science Press, Beijing, China, 307-313, 1985.
[11] Lie-heng Wang, Error estimates of two nonconforming finite elements for the obstacle problem, *Journal Comp. Math.,* **4**:1 (1986), 11-20.
[12] Qingping Deng, On a non-internal approximation method of the obstacle problem (in Chinese), *Numer. Math. A Journal of Chinese Universities,* **11**:3 (1989), 218-226.