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TRAPEZOIDAL PLATE BENDING ELEMENT WITH DOUBLE SET PARAMETERS *1)

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Abstract

Using double set parameter method, a 12-parameter trapezoidal plate bending element is presented. The first set of degrees of freedom, which make the element convergent, are the values at the four vertices and the middle points of the four sides together with the mean values of the outer normal derivatives along four sides. The second set of degree of freedom, which make the number of unknowns in the resulting discrete system small and computation convenient are values and the first derivatives at the four vertices of the element. The convergence of the element is proved.

Key words: Plate bending problem, Nonconforming finite element, Double set parameter method.

1. Introduction

The plate bending problem is an fourth order elliptic problem. The conforming finite element space for this kind of problems should has C1 continuity. It is difficult to satisfy this condition, so nonconforming finite element methods for plate problem are developed. For the analysis of the nonconforming finite element, Irons et al proposed Patch-Test[1]. In the terms of mathematics, Stummel proposed GPT (Generalized Patch -Test)[2], it is a necessary and sufficient condition for the convergent of nonconforming finite element. Based on GPT Shi Zhongci proposed F-E-M-Test[3], and it is easy to be applied, because it checks only the local properties of the shape functions along each interface or on each element. Choosing simple nodal parameters such as values and some derivatives at the vertices of the element usually can not pass GPT. To pass GPT , node parameters should be chosen some complicated forms that makes the computation very expensive. In order to overcome this difficulty, we presented double set parameter method[4], DSP for short. The essential point is to separate the two requirements by taking two sets of nodal parameters. DSP method has some advantages: two sets of node parameters can be chosen independently, one set is chosen simple such that the total unknowns of the discrete system is small, while another set is chosen to pass GPT or F-E-M-Test. DSP method can also better solve the matching problem between the shape functions and nodal parameters. Moreover, DPS method can be put into the general framework of finite element package.

Many papers have studied triangular and rectangular plate elements, such as Morly's element[5], Zienkiewicz's element [6], Vebeuke's elements[7], Bergan's elements[8], Adini's element[7], Quasi-conforming elements[9] Generalized conforming elements[10], Specht's element[11], Double set parameter elements[4,11] and so on. But seldom papers study arbitrary quadrilateral plate element. In this case it is difficult to pass GPT because the transformation from the reference element to a general element is nonlinear.

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In this paper, we construct a trapezoidal plate element with double set parameter method, which can be applied to polygonal domain, because a triangle can be divided into three trapezoids. The convergence of this element is proved.

2. Double Set Parameter Method

For element K , let the initial shape function space be

$$\overline{P}(K) = Span\{\varphi_1, \varphi_2, ..., \varphi_m\} \tag{2.1}$$

where $\varphi_1, \varphi_2, ..., \varphi_m$ are linearly independent polynomials, and the degrees of freedom are

$$D(v) = (d_1(v), d_2(v), ..., d_m(v))^T \tag{2.2}$$

with $d_1(v), d_2(v), ..., d_m(v)$ the linear functionals on $H^k(K), k \geq 1$.

$\forall v \in \overline{P}(K), v$ can be represented as

$$v = \beta_1\varphi_1 + \beta_2\varphi_2 + \cdots + \beta_m\varphi_m \tag{2.3}$$

Substituting this formula in the functional d_i,we get a relation between β_i, and

$$Cb = D(v) \tag{2.4}$$

where $b = (\beta_1, \beta_2, \cdots, \beta_m)^T, C = (d_i(\varphi_j))_{m\times m}$ is a matrix . To determine b by $D(v)$, we assume the well posed condition

$$\det(C) \neq 0 \tag{2.5}$$

holds.

Being the unknowns of the discrete system, $D(v)$ should be chosen simple, and on the other hand , they must make the shape functions have certain continuity across interelement boundaries to ensure the convergence. Those two goals are not easy to be satisfied simultaneously.

To meet these two requirements, DSP method takes another set of nodal parameters:

$$Q(v) = (q_1(v), q_2(v), \cdots, q_l(v))^T \tag{2.6}$$

Then one can use linear combinations of nodal parameter (2.6) to discretize the degrees of freedom (2.3), which yields

$$D(v) = GQ(v) + \varepsilon(v) \tag{2.7}$$

where $\varepsilon(v)$is the remainder term of discretization, G is the discretizing matrix. Neglecting $\varepsilon(v)$in (2.7), we define real shape function space on K as follows

$$P(K) = \left\{ p \in \overline{P}(k) | p = \sum_{i=1}^{m} \beta_i\varphi_i, b = C^{-1}GQ, \forall Q \in R^l \right\} \tag{2.8}$$

$P(K)$is a subspace of $\overline{P}(k)$ with the dimension $\leq l$, and the final set of unknowns is $Q(v)$. The convergence of DSP elements is given by the following theorem.

Theorem 1[4]**.** *Let the DSP element satisfies:*
1) the well posed condition (2.5)holds;
2) $|\cdot|_{2,h} = (\sum_K |.|_{2,K}^2)^{1/2}$*is a norm on finite element space* V_h*;*
3) polynomial space $P_2(K) \subset P(K)$ *and* $\varepsilon(v)$ *in (2.7) vanishes for all* $v \in P_2(K)$*;*
4) the degrees of freedom $D(v)$ *with discretization (2.7) pass the F-E-M-Test.*
Then the DSP element is convergent for plate problems.

3. The Plate bending Problem

Consider the plate bending problem with clamped boundary conditions, that is , to find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = f(v) \quad \forall v \in H_0^2(\Omega) \tag{3.1}$$

where

$$a(u,v) = \int_\Omega A(u,v)dxdy,$$
$$A(u,v) = \Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - v_{xx}u_{yy}),$$
$$f(v) = \int_\Omega fvdxdy,$$

$0 < \sigma \le 0.5$ is the poisson ratio, $u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$, etc., Ω is convex polygon in $R^2, f \in L^2(\Omega)$.

Dividing Ω into a regular family of trapezoids K with diameter $h_K \le h \forall K \in T_h$ and defining on each trapezoid K a shape function space, we obtain the finite element space V_h . The finite element approximation of (3.1) is to find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h \tag{3.2}$$

where

$$a_h(u,v) = \sum_K \int_K A(u,v)dxdy. \tag{3.3}$$

4. 12-parameter Trapezoid Plate Element

Let K be a general trapezoid element on the x-y plane, $a_i(x_i, y_i), (i = 1,2,3,4)$ are four node of K,$a_5 = \frac{1}{2}(a_1 + a_2), a_6 = \frac{1}{2}(a_2 + a_3), a_7 = \frac{1}{2}(a_3 + a_4), a_8 = \frac{1}{2}(a_1 + a_4)$, are the middle points of four sides. $x_{ij} = x_i - x_j, y_{ij} = y_i - y_j, F_{ij} = a_i a_j, l_{ij} = |F_{ij}|, 1 \le i \le 4$, and $a_1 a_2$ parallels to$a_3 a_4$, $\widehat{K}$ is the reference element on the $\xi - \eta$ plane,$\widehat{a}_i(\xi_i, \zeta_i), (i = 1,2,3,4)$ are four nodes of $\widehat{K}$,$(\xi_1, \xi_2, \xi_3, \xi_4) = (-1,1,1,-1), (\eta_1, \eta_2, \eta_3, \eta_4) = (-1,-1,1,1)$, $\widehat{K}$is a square with center (0,0). The mapping $F_K : \widehat{K} \to K$ is

$$x = \sum_{i=1}^{4} \varphi_i(\xi,\eta) x_i, \quad y = \sum_{i=1}^{4} \varphi_i(\xi,\eta) y_i$$

where $\varphi_i(\xi,\eta) = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta), (i = 1,2,3,4)$.

Let J_K be Jacobian of F_k, $|J_k|$ be the determinant of J_K.

$$J_K = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}, J_K^{-1} = \frac{1}{|J_k|} \begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{pmatrix}$$

Since $a_1 a_2$ parallels to $a_3 a_4$, so $y_{21} x_{34} = x_{21} y_{34}$, we have

$$|J_K| = \frac{1}{4}(s_0 + s_1 \eta) \tag{4.1}$$

where s_0, s_1 denote the areas of K and $(\triangle a_1 a_3 a_4 - \triangle a_1 a_2 a_4)$ respectively, the unit out-normal vector of $F_{i,i+1}$ is $n_{i,i+1} = (\frac{y_{i,i+1}}{l_{i,i+1}}, -\frac{x_{i,i+1}}{l_{i,i+1}}), 1 \le i \le 4$. Shape function space is

$$\overline{P}(K) = Span\{p_1, p_2, \cdots, p_{12}\} \tag{4.2}$$

where

$$\begin{aligned} p_1 &= \frac{1}{4}(1-\xi)(1-\eta)\xi\eta, & p_2 &= -\frac{1}{4}(1+\xi)(1-\eta)\xi\eta, \\ p_3 &= \frac{1}{4}(1+\xi)(1+\eta)\xi\eta, & p_4 &= -\frac{1}{4}(1-\xi)(1+\eta)\xi\eta, \\ p_5 &= -\frac{1}{2}(1-\xi^2)(1-\eta)\eta, & p_6 &= \frac{1}{2}(1+\xi)(1-\eta^2)\xi, \\ p_7 &= \frac{1}{2}(1-\xi^2)(1+\eta)\eta, & p_8 &= -\tfrac{1}{2}(1-\xi)(1-\eta^2)\xi, \\ p_9 &= (1-\xi^2)(1-\eta^2), & p_{10} &= (1-\eta^2)\eta, \\ p_{11} &= (1-\xi^2)\xi, & p_{12} &= (1-\xi^2)(1-\eta^2)^2. \end{aligned}$$

The degrees of freedom are

$$D(v) = (d_1(v), d_2(v), \cdots, d_{12}(v))^T \tag{4.3}$$

where

$$d_i(v) = v(a_i) = v_i, 1 \le i \le 8, d_9(v) = \int_{F_{12}} \frac{\partial v}{\partial n} ds,$$
$$d_{10}(v) = \int_{F_{34}} \frac{\partial v}{\partial n} ds, d_{11}(v) = \int_{F_{23}} \frac{\partial v}{\partial n} ds, d_{12}(v) = \int_{F_{41}} \frac{\partial v}{\partial n} ds$$

$\forall v \in \overline{P}(K), v$ can be represented as

$$v = \beta_1 p_1 + \beta_2 p_2 + \cdots, \beta_{12} p_{12} \tag{4.4}$$

where

$$D(v) = Cb \tag{4.5}$$

$$b = (\beta_1, \beta_1, \cdots, \beta_{12})^T,$$

$$C = \begin{pmatrix} C_{11} & 0 & 0 \\ C_{21} & C_{22} & 0 \\ C_{31} & C_{32} & C_{33} \end{pmatrix},$$

where C_{11} is unit 8×8 matrix, the nonsingularity of C is independent of C_{21}, C_{31}, C_{32}, so they are omitted here.

$$C_{22} = \begin{pmatrix} -\frac{4l_{12}^2}{3S_{123}} & \frac{2l_{12}^2}{S_{123}} \\ -\frac{4l_{34}^2}{3S_{234}} & -\frac{2l_{34}^2}{S_{234}} \end{pmatrix}, C_{33} = \begin{pmatrix} -\frac{4l_{23}^2}{2} D_{23} & -\frac{l_{23}^2}{2} E_{23} \\ \frac{l_{41}^2}{2} D_{41} & -\frac{l_{41}^2}{2} E_{41} \end{pmatrix},$$

where

$$D_{41} = D_{23} = \int_{-1}^{1} \frac{1}{|J_K|(\eta)} d\eta, E_{41} = E_{23} = \int_{-1}^{1} \frac{(1-\eta^2)^2}{|J_K|(\eta)} d\eta,$$

S_{ijk} denotes the area of triangle $\triangle a_i a_j a_k$. It is easy to see $D_{41} = D_{23} > 0, E_{41} = E_{23} > 0$, and that

$$\det(C) = \frac{4l_{12}^2 l_{34}^2 l_{23}^2 l_{41}^2}{3S_{123} S_{234}} (D_{23} E_{41} + D_{41} E_{23}) > 0. \tag{4.6}$$

So the function of $\overline{P}(K)$can be uniquely determined by the degrees of freedom $D(v)$.

Let nodal parameters are

$$Q(v) = (v_1, v_{1x}, v_{1y}, \cdots, v_4, v_{4x}, v_{4y})^T \tag{4.7}$$

where

$$v_i = v(a_i), v_{ix} = \frac{\partial v}{\partial y}(a_i), v_{iy} = \frac{\partial v}{\partial y}(a_i), 1 \le i \le 4.$$

Using the following method to discretize the degrees of freedom (4.3) as the linear combinations of nodal parameters (4.7):

$$v_i = v_i \quad 1 \le i \le 4.$$

Let $q_{ij} \in P_3(F_{ij})$ be the Hermite interpolation polynomial of v on F_{ij}, then let $v_5 = q_{12}(a_5), v_6 = q_{23}(a_6), v_7 = q_{34}(a_7), v_8 = q_{41}(a_8)$, the trapezoid formula of numerical integration is used for $d_i(v)$ $9 \le i \le 12$. We have

$$D(v) = GQ(v) + \varepsilon(v) \tag{4.8}$$

where

$$\varepsilon(v) = (0,0,0,0,\varepsilon_0,\varepsilon_0,\cdots,\varepsilon_0)^T, \qquad \varepsilon_0 = O(h_k^3|v|_{3,\infty,K}).$$

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \\ G_{31} & G_{32} \end{pmatrix},$$

$$G_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad G_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$G_{21} = \begin{pmatrix} \frac{1}{2} & \frac{x_{21}}{8} & \frac{y_{21}}{8} & \frac{1}{2} & \frac{x_{12}}{8} & \frac{y_{12}}{8} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{x_{32}}{8} & \frac{y_{32}}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{x_{41}}{8} & \frac{y_{41}}{8} & 0 & 0 & 0 \end{pmatrix}, \quad G_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{x_{23}}{8} & \frac{y_{23}}{8} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{x_{43}}{8} & \frac{y_{43}}{8} & \frac{1}{2} & \frac{x_{34}}{8} & \frac{y_{34}}{8} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{x_{14}}{8} & \frac{y_{14}}{8} \end{pmatrix},$$

$$G_{31} = \begin{pmatrix} 0 & \frac{y_{21}}{2} & \frac{x_{12}}{2} & 0 & \frac{y_{21}}{2} & \frac{x_{12}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{y_{32}}{2} & \frac{x_{23}}{2} \\ 0 & \frac{y_{14}}{2} & \frac{x_{41}}{2} & 0 & 0 & 0 \end{pmatrix}, \quad G_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{y_{43}}{2} & \frac{x_{34}}{2} & 0 & \frac{y_{43}}{2} & \frac{x_{34}}{2} \\ 0 & \frac{y_{32}}{2} & \frac{x_{23}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{y_{14}}{2} & \frac{x_{41}}{2} \end{pmatrix}.$$

5. Convergence Analysis

Theorem 2. *The 12-parameter trapezoidal DSP element constructed in above section is convergent for plate bending problem.*

Proof. Only need to check the conditions of Theorem 1:

1) From (4.6) the well posed condition (2.5) holds.

2) It is easy to check that $|\cdot|_{2,h}$ is a norm on V_h. In fact, it only needs to check that if $v \in V_h, |v|_{2,h} = 0$ then $v = 0$. Suppose $v \in V_h, |v|_{2,h} = 0$ then $\forall K, \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = const$ on each K. As $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous at nodes of elements and vanish at boundary nodes of Ω , we have $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} = 0$ on Ω , so $v = const$ on each K. As v is continuous at nodes of elements and vanish at boundary nodes of Ω, we have $v = 0$ on Ω.

3)Since $P_2(K) \subset Q_2(\hat{K}) = Span\{p_1, p_2, \cdots, p_9\} \subset \overline{P}(K), \varepsilon(v) = 0, \forall v \in P_2(K)$, so $P_2(K) \subset \overline{P}(K)$[4].

4)Obviously $d_i(v) = v_i (1 \le i \le 4)$ is continuous. Only $v_i, v_{ix}, v_{iy}, (i = 1, 2)$ which are continuous between elements, and geometry quantities of F_{12} are used to discretize v_5 by Hermite interpolation of v on F_{12}, so $d_5(v) = v_5$ is continuous across interelement boundaries. Similarly, only $v_{ix}, v_{iy} (i = 1, 2)$ and geometry quantities of F_{12} are used to descretize $\int_{F_{12}} \frac{\partial v}{\partial n} ds$ by trapezoidal formula of numerical integration, so $d_9(v) = \int_{F_{12}} \frac{\partial v}{\partial n} ds$ is continuous across interelement boundaries. In the same way we know that $d_5(v), \cdots, d_{12}(v)$ are continuous. Therefore, for any side F of every element, we have

$$[v_i] = \int_F \left[\frac{\partial v}{\partial s}\right] ds = \int_F \left[\frac{\partial v}{\partial n}\right] ds = 0 \qquad 1 \le i \le 4, \tag{5.1}$$

where $[w]$ is the jump of w across F. Hence v_h passes F-E-M-Test [3]. Then 12-parameter trapezoidal DSP element is convergent for plate problems by Theorem 1.

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