Journal of Computational Mathematics, Vol.21, No.4, 2003, 519–534.

NUMERICAL DISSIPATION FOR THREE-POINT DIFFERENCE SCHEMES TO HYPERBOLIC EQUATIONS WITH UNEVEN MESHES *1)

Zi-niu Wu
(*Department of Engineering Mechanics, Tsinghua University, Beijing 100084, China*)

Abstract

The widely used locally adaptive Cartesian grid methods involve a series of abruptly refined interfaces. The numerical dissipation due to these interfaces is studied here for three-point difference approximations of a hyperbolic equation. It will be shown that if the wave moves in the fine-to-coarse direction then the dissipation is positive (stabilizing), and if the wave moves in the coarse-to-fine direction then the dissipation is negative (destabilizing).

Key words: Refined interfaces, Numerical dissipation, Three-point difference approximation, Hyperbolic equation.

1. Introduction

In the adaptive Cartesian grid method [1, 2, 3, 4, 5, 9, 10, 11, 13, 18, 20, 25], the entire grid is composed of divided zones (which will be called subgrids for convenience) each having a uniform mesh size and with abrupt mesh refinement at the interfaces. The adaptive Cartesian grid method shares some common feature with the multilevel methods originally proposed by Brandt and then evolved to the well-known multigrid method for convergence acceleration, see, e.g., [6, 17, 19].

A particular feature of the abrupt refinement method is the existence of multiple refinement interfaces which are separated by subgrids of uniform mesh size. Little attention has been paid to the stability and accuracy of the abrupt method with multiple interfaces, though the single interface problem was addressed long before, see, e.g., [2, 7]. In this paper we will address the question of mesh refinement induced dissipation, which is closely related to the stability of the difference approximation. Some good dissipation analysis can be found in [12, 21, 23]. Normally the influence of the abrupt interfaces is coupled with the treatment of the exterior boundaries. But in this paper we will ignore the influence of the treatment of the exterior boundaries. This simplification will be stated throughout the paper in its suitable form whenever needed.

Here we only consider three-point difference equations with conservative treatment everywhere (inside subgrid and at interface). The difference approximations on both smoothly refined grid and abruptly refined grid, based on the same discretization procedure, are presented in Section 2. Section 3 is devoted to the study of mesh refinement induced dissipation under the framework of semidiscrete scheme. In Section 4 we will consider the fully discrete scheme in order to analyze how the mesh refinement influences the total dissipation. To this end we first we use an energy method to study the necessary condition for energy decreasing (or stability). Then we perform an eigenvalue analysis (including a von Neumann stability analysis for a particular case) in order to study sufficient stability conditions for the Lax-Wendroff scheme.

The conclusions of this paper are summrized in the following two Theorems.

* Received April 4, 2001; final revised June 14, 2001.
1) This work was supported by China NKBRSF Project (2001CB409600) and by China National Natural Science Foundations (10025210).

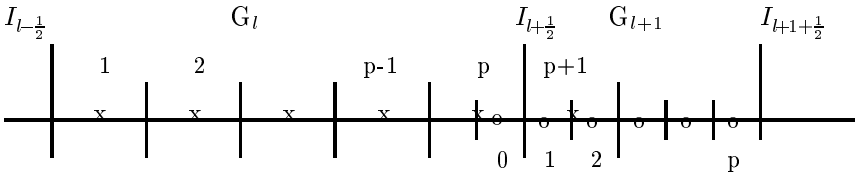

Figure 1: Abrupt refinement grid.

Theorem 1.1. *For a general semi-discrete three-point difference approximation with uneven mesh spacing, if the wave moves in the fine-to-coarse direction then the dissipation is positive (stabilizing) , and if the wave moves in the coarse-to-fine direction then the dissipation is negative (de-stabilizing). Moreover, the amount of dissipation is insenstive to the subgrid width if the total refinement degree is fixed.*

Theorem 1.2. *For the fully discrete Lax-Wendroff scheme, the lower bound of the stability region is increased by mesh refinement, while the upper bound is reduced.*

2. Three-point Difference Approximations on Smooth and Abrupt Refinement Grids

2.1. Smooth Refinement Grid and Abrupt Refinement Grid

On a smoothly refined grid , let h_l be the size of mesh l with $0 \le l \le L$. Conventionally, if $r = \frac{h_l}{h_{l-1}}$ is constant independent of l, then the refinement is said to be geometric. In practice we may have $r = \frac{h_l}{h_{l-1}}$ depending on l, but such irregular situations have rarely been investigated theoretically.

On an abruptly refined grid, as displayed in Fig.1, the entire grid is composed of a certain number of subgrids with different mesh sizes. Let h_l be the mesh size on subgrid G_l with $0 \le l \le L$. We assume $h_l = h_0 r^l$ with $r < 1$ and that the number of grid points in each subgrid is constant and equal to p, which we will call the subgrid width. For convenience, let us define the total refinement degree by $r_T = h_L/h_0$. For geometrical refinement, the local refinement degree $r = h_{l+1}/h_l$ is related to r_T by

$$r = r_T^{\frac{1}{L}} \tag{2.1}$$

In this paper we will only consider geometrical refinement. Note that the smooth refinement method can be considered as a particular case of the abrupt mesh refinement method with $p = 1$ and $r \to 1$. In deriving the difference equations we also require that the difference equation on an abrupt grid reduces to that on the smooth refinement grid when $p = 1$.

For purpose of studying the numerical dissipation, it is sufficient here to consider the following scalar equation:

$$u_t + a u_x = 0 \tag{2.2}$$

approximated by a three-point difference scheme. In order for the results to be useful for nonlinear problems, we require the treatment to be conservative. Also, for the comparison to be meaningful, the difference approximations should be designed from the same discretization methodology for both of the smooth and abrupt refinement methods so that they each reduce to the other when the grid is uniform.

2.2. Semi-discrete Difference Approximations

On a smoothly refined grid, let the numerical solution at the point l be v_l. Then the most accurate three-point conservative scheme, in the semidiscrete case, can be written as:

$$\frac{dv_l}{dt} = -\frac{1}{h_j}(g_{l+\frac{1}{2}} - g_{l-\frac{1}{2}}), \quad l = 0, 1, ..., L, \tag{2.3}$$

where the numerical flux $g_{l+\frac{1}{2}}$ is defined by linear interpolation:

$$g_{l+\frac{1}{2}} = \frac{r}{r+1} a v_l + \frac{1}{r+1} a v_{l+1}, \; r = \frac{h_{l+1}}{h_l}. \tag{2.4}$$

On an abruptly refined grid, let the numerical solution at grid j of subgrid G_l be $u_{l,j}$. The most accurate three-point scheme reads

$$\frac{du_{l,j}}{dt} = -\frac{a}{2h_l}(u_{l,j+1} - u_{l,j-1}), \quad j = 1, 2, ..., p, \; l = 0, 1, ..., L \tag{2.5}$$

For the interface $I_{l+\frac{1}{2}}$ separating the subgrids l and $l+1$, we have two interface unknowns $u_{l-1,p+1}$and $u_{l,0}$ which are used in but not provided by (2.5). They are here determined by the following interface condition which represents a linear interpolation and which was originally proposed in [7]:

$$\frac{u_{l-1,p} + u_{l-1,p+1}}{2} = \frac{u_{l,0} + u_{l,1}}{2}, \quad \frac{u_{l-1,p+1} - u_{l-1,p}}{h_{l-1}} = \frac{u_{l,1} - u_{l,0}}{h_l}. \tag{2.6}$$

Proposition 2.1. *The difference approximation (2.5), supplemented by the interface condition (2.6), is conservative and is equivalent with the scheme (2.3)-(2.4) on the same grid.*

Proof. It is obvious that (2.6) and (2.3) are equivalent when the grid is uniform. We now consider the nonuniform case. At the refinement interface $I_{l+\frac{1}{2}}$, the numerical flux for the left subgrid (l) is given by $f_{l,j+\frac{1}{2}} = \frac{1}{2}a(u_{l,p} + u_{l,p+1})$ which, with the help of (2.5), can be rewritten as

$$f_{l,p+\frac{1}{2}} = \frac{r}{r+1} a u_{l,p} + \frac{1}{r+1} a u_{l+1,1} \tag{2.7}$$

and the numerical flux for the right subgrid $(l+1)$ is similarly found to be

$$f_{l+1,\frac{1}{2}} = \frac{r}{r+1} a u_{l,p} + \frac{1}{r+1} a u_{l+1,1} \tag{2.8}$$

Since $f_{l,p+\frac{1}{2}} = f_{l+1,\frac{1}{2}}$by (2.7) and (2.8), conservation is clearly guaranteed. Also, (2.7) is equivalent to (2.4) so that the scheme on the abruptly refined grid is totally equivalent with the scheme on the smooth refinement grid.

Thus, in the following, the smooth refinement method will be considered as a particular case of abrupt refinement with $p = 1$.

For nonlinear problems, one can easily ensure conservation by using a nonlinear equivalent of the interface condition (2.6). Similarly one can do so for the fully discrete case.

2.3. Fully-discrete Difference Approximations

A general three-point conservative difference approximation can be written in the following viscous form:

$$u_{l,j}^{n+1} = u_{l,j}^n - \frac{1}{2}\lambda_l(u_{l,j+1}^n - u_{l,j-1}^n) + \frac{1}{2}\lambda_l Q_l^{(num)}(u_{l,j+1}^n - 2u_{l,j}^n + u_{l,j-1}^n) \tag{2.9}$$

where $l \in \mathcal{L}$ and $j \in \mathcal{J}$, with $\mathcal{L} = \{0, 1, ..., L\}$ and $\mathcal{J} = \{1, 2, ..., p\}$. Here $\lambda_l = a\sigma_l$, with σ_l denoting the ratio of the time step k_l and the mesh size h_l, and $Q_l^{(num)}$ is the numerical viscosity coefficient. The scheme (2.9) represents all multipoint schemes which can be written

in conservative form, with the influence of points outside those of a three-point scheme factored into the numerical viscosity coefficient $Q_l^{(num)}$. See for instance [22]. For example, if we take $Q_l^{(num)} = sgn(a)$, then we obtain the usual first-order upwind scheme. If we take $Q_l^{(num)}$ to be a small constant value, then we obtain a usual three-point centered scheme with artificial dissipation. If we take $Q_l^{(num)} = \lambda_l$, then we obtain the well-known Lax-Wendroff scheme[15] which can now be written as:

$$u_{l,j}^{n+1} = u_{l,j}^n - \frac{1}{2}\lambda_l(u_{l,j+1}^n - u_{l,j-1}^n) + \frac{1}{2}\lambda_l^2(u_{l,j+1}^n - 2u_{l,j}^n + u_{l,j-1}^n) \tag{2.10}$$

The interface condition (2.6) for the fully discrete case becomes

$$\frac{u_{l-1,p}^n + u_{l-1,p+1}^n}{2} = \frac{u_{l,0}^n + u_{l,1}^n}{2}, \quad \frac{u_{l-1,p+1}^n - u_{l-1,p}^n}{h_{l-1}} = \frac{u_{l,1}^n - u_{l,0}^n}{h_l} \tag{2.11}$$

where $l = 1, 2, ..., L$. The interface condition (2.11) can be solved for $u_{l-1,p+1}^n$ and $u_{l,0}^n$:

$$u_{l-1,p+1}^n = \frac{r-1}{r+1}u_{l-1,p}^n + \frac{2}{r+1}u_{l,1}^n, \quad u_{l,0}^n = -\frac{r-1}{r+1}u_{l,1}^n + \frac{2r}{r+1}u_{l-1,p}^n \tag{2.12}$$

For steady state computation, one usually uses local time stepping to accelerate convergence. Since using local time-stepping amounts to using larger time steps in refined regions, a numerical scheme using local time-stepping is generally less stable than a scheme using a uniform time step.

3. Positive and Negative Dissipation due to Mesh Refinement for a Semidiscrete Scheme

3.1. Smooth Refinement

For convenience, we rewrite the semidiscrete scheme (2.3) as

$$\frac{dv_l}{dt} = \sum_{m=-1}^{m=1} c_{l,m}v_{l+m} \tag{3.1}$$

with

$$c_{l,-1} = \frac{r}{r+1}\frac{a}{h_l}, \quad c_{l,0} = \left(\frac{1}{r+1} - \frac{r}{r+1}\right)\frac{a}{h_l}, \quad c_{l,1} = -\frac{1}{r+1}\frac{a}{h_l}$$

It is quite useful to introduce the parameter h_a defined by:

$$h_a = \frac{a}{c_{l,-1} - c_{l,1}} = \frac{(h_{l-1} + h_l)(h_l + h_{l+1})}{h_{l-1} + 2h_l + h_{l+1}}$$

This is *an average of the local mesh sizes* and reduces to the normal mesh size on an uniform grid. With this definition, the scheme (3.1) can be greatly simplified and reduces to the form of a standard difference scheme on a uniform grid:

$$\frac{dv_l}{dt} = -\frac{1}{2}\frac{a}{h_a}(v_{l+1} - v_{l-1}) + \frac{1}{2}Q_l^{(r)}(v_{l+1} - 2v_l + v_{l-1}) \tag{3.2}$$

where

$$Q_l^{(r)} = c_{l,-1} + c_{l,1} = \frac{(h_{l+1} - h_{l-1})a}{(h_l + h_{l-1})(h_l + h_{l+1})} \tag{3.3}$$

In comparison with the case of a uniform grid, a dissipation-like term is created by mesh refinement. The coefficient $Q_l^{(r)}$ defined by (3.3) is similar to the numerical viscosity coefficient in the usual sense. Thus we make

Definition 3.1. *The coefficient $Q_l^{(r)}$ is called the mesh refinement induced numerical viscosity.*

As usual, the numerical viscosity coefficient must always be positive in order that the scheme be stable. However, the coefficient $Q_l^{(r)}$ may be positive or negative depending on the sign of the wave speed a. The situation is more clearly stated in the following proposition:

Proposition 3.2. *For the smooth refinement method,*

a) the mesh refinement numerical viscosity $Q_l^{(r)}$ is positive if the wave travels in the coarsening direction (direction from the finer mesh to the coarser mesh), i.e., if $(h_{l+1} - h_{l-1})a > 0$;

b) the mesh refinement numerical viscosity $Q_l^{(r)}$ is negative if the wave travels in the refining direction (direction from the coarser mesh to the finer mesh), i.e., if $(h_{l+1} - h_{l-1})a < 0$.

For a system of equations, like the Euler equations in gas dynamics for a subsonic flow, we have waves traveling in both directions so that we have at least one characteristic component which contains a negative dissipation due to mesh refinement. When the wave travels in the refining grid so that there exists negative dissipation, the difference equation will be unstable unless an artificial dissipation is included. Some difference equations, such as the well-known Lax-Wendroff scheme and characteristic based upwind schemes, also contain internal dissipation which may balance the negative dissipation due to mesh refinement.

For geometrically refined grids with $h_l = rh_{l-1}$, the mesh refinement induced viscosity can be more conveniently written as:

$$Q_l^{(r)} = \frac{r-1}{r+1}\frac{a}{h_a} \tag{3.4}$$

If the time is scaled by the average mesh size h_a, as would occur in local time-stepping for convergence acceleration of steady state computation, then (3.2) reduces to a scheme on an uniform mesh with a dissipation coefficient given by (3.4).

3.2. Abrupt Refinement

For abrupt refinement, it makes no sense to define a local mesh refinement induced viscosity as was done for a smooth refinement, because the grid is uniform in each subgrid level. It is more convenient to define a global mesh refinement induced viscosity by considering the time evolution of the l_2-energy defined as:

$$\|u\|^2 = \sum_{l=0}^{L}\sum_{j=1}^{p} h_l u_{l,j}^2$$

Multiplying the difference equation (2.5) by $h_l u_{l,j}$, and summing the resulting equations over all j and all l, we obtain the following equation for the l_2-energy:

$$\frac{d\,\|u\|^2}{dt} = B_b + B_r \tag{3.5}$$

where B_b is a term due to exterior boundary treatment which we do not consider here, and B_l, which is due to mesh refinement, is given by

$$B_r = \sum_{l=1}^{L} B_l, \quad B_l = (au_{l,0}u_{l,1} - au_{l-1,p}u_{l-1,p+1}) \tag{3.6}$$

Introducing the interface condition (2.6) into (3.6) leads to the following more useful expression for B_l:

$$B_l = \frac{(h_{l-1} - h_l)a}{h_{l-1} + h_l}(u_{l-1,p} - u_{l,1})^2 \tag{3.7}$$

The term B_r is a term which dissipates or increases the energy E depending on its sign. Obviously B_r vanishes on a globally uniform grid. Thus we will call B_r the mesh refinement induced dissipation. We have the following proposition, which is similar to the smooth refinement method,

Proposition 3.3. *For the abrupt refinement method,*

a) the mesh refinement numerical dissipation is positive if the wave travels in the coarsening direction (direction from the finer mesh to the coarser mesh), i.e., if $(h_l - h_{l-1})a > 0$*;*

b) the mesh refinement numerical dissipation is negative if the wave travels in the refining direction (direction from the coarser mesh to the finer mesh), i.e., if $(h_l - h_{l-1})a < 0$.

3.3. Dependence of the Dissipation on the Subgrid Width for the Shortest Wavelength

Here let us study the influence of the subgrid width on the dissipation just for the shortest wavelength (wave number equal to π).

In order for the study of the subgrid width influence to be meaningful, the domain size S, the starting mesh size h_0, and the total refinement degree r_T (or h_L), should be kept fixed. Now we want to relate the number of levels L to the subgrid width p when S, h_0, and r_T are fixed.

By definition, $S = \sum_{l=0}^{L} ph_l = \sum_{l=0}^{L} pr^l h_0$. Thus

$$S = \frac{1 - r^{L+1}}{1 - r} p h_0$$

Noting that $r = r_T^{\frac{1}{L}}$, the above equation can be directly solved to yield the following two relations:

$$r_T^{\frac{1}{L}} = \frac{M - p}{M - p r_T}, \quad L = \frac{\ln r_T}{\ln \frac{M-p}{M-pr_T}}, \quad M = \frac{S}{h_0} \tag{3.8}$$

According to Propositions 3.2-3.3, the dissipation is negative only if $(h_l - h_{l-1})a < 0$. The worsest case occurs when the wave number is equal to π. Let A be the amplitude of the wave with wave number π. For such a wave the total dissipation (3.7) can be estimated by

$$B_r = A^2 \sum_{l=0}^{L} \frac{a(h_{l-1} - h_l)}{h_{l-1} + h_l} = -\left| \frac{L(1 - r_T^{\frac{1}{L}})}{1 + r_T^{\frac{1}{L}}} \right| C, \quad C = A^2 |a| \tag{3.9}$$

Introducing the relations in (3.8) into (3.9) leads to the following expression for the total dissipation:

$$B_r = -C \frac{p(1 - r_T)}{2M - p(1 + r_T)} \left(\ln \frac{M - p}{M - p r_T} \right)^{-1} \ln r_T \tag{3.10}$$

Let $r_T = 0.01$ and $M = 1000$, the dependence of the dissipation on the subgrid width p is given in Table 1:

Table 1: Dependence of the dissipation on the subgrid width p

p	1	2	5	10	10^2	10^3
B_r	-2.303	-2.303	-2.303	-2.303	-2.301	-2.216

We have the following important remarks:

Table 2: Dependence of the dissipation on the subgrid width p

p	1	2	5	10	50
B_r	$-1.151\,3$	$-1.151\,3$	$-1.151\,1$	$-1.150\,4$	$-1.113\,3$

1) The negative dissipation due to mesh refinement is insensitive to the subgrid width, provided the total refinement degree r_T and the ratio M be fixed.

2) The absolute value of the negative dissipation is an increasing function of the subgrid width. Since the smooth refinement method, in the case of geometrical refinement, is a particular case of the abrupt refinement method with $p = 1$, we remark that, with r_T and M being fixed, the abrupt method (the interface is more abrupt if the subgrid becomes wider) contains less negative dissipation than the smooth refinement method and should face less instability trouble than the latter. This is in contrast with what one would imagine normally. It is commonly believed that, in order a numerical method on an irregular grid to be stable, one should avoid using abrupt refinement and make the refinement as smooth as possible. The present analysis clearly shows that abrupt refinement is more favorable for stability, provided the parameters r_T and M be kept fixed.

With other choices of r_T and M we obtain similar results. For exmaple, if we take $r_T = 0.1$ and $M = 100$, the dependence of the dissipation on the subgrid width p is given in Table 2.

Thus we restate this important conclusion in the following theorem:

Proposition 3.4. *Let the total refinement degree r_T and the ratio $M = \frac{S}{h_0}$ be fixed. Then for the shortest wave length (wave number equal to π),*

a) the negative dissipation, in absolute value, is a decreasing function of the subgrid width;

b) the abrupt mesh refinement method produces less negative dissipation than the smooth refinement.

4. Balance between Numerical Dissipation and Refinement Dissipation for a Fully Discrete Scheme

In the previous section, it is shown that mesh refinement creates negative or positive dissipation. A fully discrete scheme may contain internal or artificial dissipations with a positive numerical viscosity coefficient $Q_l^{(num)}$. If the numerical viscosity is more important than the refinement induced negative dissipation, then the difference approximation, stable for a uniform grid, would remain stable for a mesh refinement problem. We know that (2.11) is stable, in the sense of GKS [14], for a large class of dissipative and nondissipative schemes if only one interface exists [7, 15]. But a stable single interface problem might become unstable when several interfaces are put together [24].

4.1. Analysis of Dissipation

Let us begin with the following lemma.

Lemma 4.1. *For the fully discrete difference scheme (2.9), the l_2-norm of the solution satisfies the following inequality:*

$$\left\|u^{n+1}\right\|^2 - \|u^n\|^2 \geq \sum_{l=0}^{L} k_l(\overline{D}_l^n + \overline{I}_l^n + \overline{B}_l^n)$$

where

$$\begin{aligned}
\overline{D}_l &= -Q_l^{(num)}\sum_{j=1}^{p-1}(u_{l,j}-u_{l,j+1})^2 \\
\overline{I}_l &= \begin{cases} \frac{2Q_l^{(num)}}{1+r}(u_{l,p}u_{l+1,1}-u_{l,p}^2)+Q_l^{(num)}(u_{l,0}u_{l,1}-u_{l,1}^2) & l=0 \\ \frac{2Q_l^{(num)}}{1+r}[r(u_{l,1}u_{l-1,p}-u_{l,1}^2)+u_{l,p}u_{l+1,1}-u_{l,p}^2] & 0<l<L \\ \frac{2Q_l^{(num)}}{1+r}[r(u_{l,1}u_{l-1,p}-u_{l,1}^2)]+Q_l^{(num)}(u_{l,p}u_{l,p+1}-u_{l,p}^2) & l=L \end{cases} \\
\overline{B}_l &= \begin{cases} \frac{a}{1+r}[(1-r)u_{l,p}^2-2u_{l,p}u_{l+1,1}]+au_{l,0}u_{l,1} & l=0 \\ \frac{a}{1+r}[(1-r)(u_{l,p}^2+u_{l,1}^2)+2ru_{l,1}u_{l-1,p}-2u_{l,p}u_{l+1,1}] & 0<l<L \\ \frac{a}{1+r}[(1-r)u_{l,1}^2+2ru_{l,1}u_{l-1,p}]-au_{l,p}u_{l,p+1} & l=L \end{cases}
\end{aligned}$$

Proof. Let us rewrite the scheme(2.9) as:

$$u_{l,j}^{n+1}=u_{l,j}^n-\frac{1}{2}\lambda_l F_{l,j}^n+\frac{1}{2}\lambda_l Q_l^{(num)}G_{l,j}^n \tag{4.1}$$

with

$$F_{l,j}=u_{l,j+1}-u_{l,j-1},\quad G_{l,j}=u_{l,j+1}-2u_{l,j}+u_{l,j-1}$$

Multiply (4.1) by $2h_l u_{l,j}^n$, sum the resulting equations over all l and all j, and using the obvious inequality

$$\sum h_l u_{l,j}^n u_{l,j}^{n+1}\le\frac{1}{2}(\left\|u^{n+1}\right\|^2+\|u^n\|^2)$$

we obtain

$$\left\|u^{n+1}\right\|^2-\|u^n\|^2\ge\sum_{l=0}^{L}k_l(R_l^n+S_l^n)$$

where

$$R_l^n=\sum_{j=1}^{p}au_{l,j}^nF_{l,j}^n,\quad S_l^n=\sum_{j=1}^{p}aQ_l^{(num)}u_{l,j}^nG_{l,j}^n$$

It is quite straightforward (summation by parts) to obtain the following relations

$$R_l=a(u_{l,p}u_{l,p+1}-u_{l,0}u_{l,1}),\quad S_l=aQ_l^{(num)}[H_l-\sum_{j=1}^{p-1}(u_{l,j}-u_{l,j+1})^2]$$

where

$$H_l=u_{l,0}u_{l,1}+u_{l,p}u_{l,p+1}-u_{l,1}^2-u_{l,p}^2$$

Eliminating $u_{l,p+1}^n$ and $u_{l,0}^n$ by using the interface condition (2.12), we obtain for $0<l<L$:

$$\begin{aligned}
R_l &= \frac{a}{1+r}[(1-r)(u_{l,p}^2+u_{l,1}^2)+ru_{l,1}u_{l-1,p}-u_{l,p}u_{l+1,1}] \\
H_l &= \frac{2r}{r+1}(u_{l-1,p}-u_{l,1})u_{l,1}+\frac{2}{r+1}(u_{l+1,1}-u_{l,p})u_{l,p}
\end{aligned}$$

The lemma then follows without difficulty.

Lemma 4.2. *Let $k_l=k$ and $Q_l^{(num)}=Q^{(num)}$ be constant independent of l. Then for the difference scheme (2.9), $\|u\|^2$ satisfies the following inequality:*

$$\left\|u^{n+1}\right\|^2-\|u^n\|^2\ge k(D^n+I^n+B^n+E^n) \tag{4.2}$$

where

$$D = -Q^{(num)} \sum_{l=0}^{L} \sum_{j=1}^{p-1} (u_{l,j} - u_{l,j+1})^2 \tag{4.3}$$

$$I = -Q^{(num)} \sum_{l=1}^{L} (u_{l-1,p} - u_{l,1})^2 \tag{4.4}$$

$$B = \frac{(1-r)}{1+r} \sum_{l=1}^{L} \Big(a(u_{l-1,p}^n - u_{l,1}^n)^2 - Q^{(num)}(u_{l-1,p}^2 - u_{l,1}^2) \Big) \tag{4.5}$$

$$E = (Q^{(num)} + a) u_{0,0}^n u_{0,1}^n + (Q^{(num)} - a) u_{L,p}^n u_{L,p+1}^n - Q^{(num)}(u_{0,1}^2 + u_{L,p}^2) \tag{4.6}$$

Proof. This follows directly from Lemma 4.1.

Remark 4.3. The four terms in (4.2) result from different sources:

- The term D given by (4.3)comes from internal dissipation inside each block and vanishes for $p = 1$
- The term I given by (4.4)comes from internal dissipation at the refinement interfaces
- The term B given by (4.5)comes from mesh refinement
- The term E given by (4.6) represents the influence of the ending points ($l = 0$, $j = 0, 1$) and ($l = L, j = p, p + 1$), including the boundary conditions.

Using Lemma 4.2, we have

Proposition 4.4. *Let L be large enough so that E can be neglected. If the internal dissipation $Q^{(num)}$ of the fully discrete scheme vanishes, then the energy is increasing for $(1 - r)a > 0$.*

For $p = 1$, we can derive a necessary condition for energy decreasing (or stability).

Proposition 4.5. *Let $k_l = k$ and $Q_l^{(num)} = Q^{(num)}$ be constant independent of l. Furhermore, let L be large enough so that the ending point influence E can be neglected. . Then for the smooth refinement method with $p = 1$, a necessary condition for energy decreasing (stability) is*

$$Q^{(num)} \geq \frac{a(1-r)}{1+r} \tag{4.7}$$

Proof. Using Lemma 4.2 and with E being neglected for L very large, we see that for $p = 1$:

$$\begin{aligned} D &= 0 \\ I + B &= -\left(Q^{(num)} - \frac{a(1-r)}{1+r} \right) \sum_{l=1}^{L} (u_{l-1,p} - u_{l,1})^2 \end{aligned}$$

Thus $D + I + B > 0$ if (4.7) is not satisfied.

The situation of $p > 1$ is more difficult to analyse. An informal analysis is given in Appendix.

4.2. Sufficient Stability Analysis

For the smooth refinement method which can be considered as a special case of the abrupt method with $p = 1$, the difference equation on the refined grid can be put into a special form for which the sufficient stability condition can be found through the classical von Neumann analysis. For the case $p > 1$, we have to use an eigenvalue analysis for finding the sufficient stability condition.

von Neumann analysis for p=1.

For the special case of $p = 1$ with local time-stepping ($\lambda_l = \lambda$) and with constant numerical viscosity ($Q_l^{(num)} = Q^{(num)}$), we can perform a von Neumann analysis for stability. This, for a scalar equation, allows us to obtain a necessary and sufficient stability condition.

Lemma 4.6. *For $p = 1$, $\lambda_l = \lambda$ and $Q_l^{(num)} = Q^{(num)}$, the difference equation (2.9) together with the interface condition (2.11) is equivalent to the following difference equation*

$$v_l^{n+1} = v_l^n - \frac{1}{2}\lambda A(v_{l+1}^n - v_{l-1}^n) + \frac{1}{2}\lambda B(v_{l+1}^n - 2v_l^n + v_{l-1}^n) \tag{4.8}$$

where

$$A = 1 + \frac{r-1}{r+1}Q^{(num)}, \quad B = Q^{(num)} - \frac{1-r}{1+r}$$

and $v_l = u_{l,1}$.

Proof. For $p = 1$, (2.9) takes the following form:

$$u_{l,1}^{n+1} = u_{l,1}^n - \frac{1}{2}\lambda(u_{l,2}^n - u_{l,0}^n) + \frac{1}{2}\lambda Q^{(num)}(u_{l,2}^n - 2u_{l,1}^n + u_{l,0}^n)$$

Introducing the interface relation (2.12) into the above equation leads to

$$\begin{aligned} u_{l,1}^{n+1} &= u_{l,1}^n - \frac{1}{2}\lambda(\frac{2}{r+1}u_{l+1,1}^n + \frac{2(r-1)}{r+1}u_{l,1}^n - \frac{2r}{r+1}u_{l-1,1}^n) \\ &\quad + \frac{1}{2}\lambda Q^{(num)}(\frac{2}{r+1}u_{l+1,1}^n - 2u_{l,1}^n + \frac{2r}{r+1}u_{l-1,1}^n) \end{aligned}$$

which, when setting $v_l = u_{l,1}$, can be rearranged to give (4.8).

Now we can do a direct von Neumann analysis on the difference equation (4.8). The following proposition follows directly from a classical von Neumann analysis, for which the details are ommitted:

Theorem 4.7. *The difference approximation (4.8) is stable if and if the following condition is satisfied:*

$$\begin{cases} B \geq 0 \\ \lambda \leq \min\left(\frac{B}{A^2}, \frac{1}{B}\right) \end{cases} \tag{4.9}$$

When the grid is not refined so that $r = 1$, then the above condition is reduced to

$$Q^{(num)} \geq 0, \quad \lambda \leq \min(Q^{(num)}, \frac{1}{Q^{(num)}}) \leq 1 \tag{4.10}$$

For convenience, let us denote $g(r) = \frac{1-r}{1+r}$ so that

$$\frac{B}{A^2} = \frac{Q^{(num)} - g(r)}{[1 + g(r)Q^{(num)}]^2}, \quad \frac{1}{B} = \frac{1}{Q^{(num)} - g(r)}$$

For $r \to 1$ and $r < 1$ we have $g(r) > 0$ and $g(r) \to 0$ and the following simplifications for $\frac{B}{A^2}$ and $\frac{1}{B}$:

$$\begin{aligned} \frac{B}{A^2} &\to Q^{(num)} - (1 + 2Q^{(num)}Q^{(num)})g(r) < Q^{(num)} \\ \frac{1}{B} &\to \frac{1}{Q^{(num)} - g(r)} > \frac{1}{Q^{(num)}} \end{aligned}$$

so that the second condition in (4.9) can be restated as:

$$\lambda \leq \min(\frac{B}{A^2}, \frac{1}{B}) < \min(Q^{(num)}, \frac{1}{Q^{(num)}})$$

This leads to the following two remarks:

Remark 4.8. The upper bound of the stability range is reduced, compared with (4.10).

Eigenvalue analysis for $p > 1$. Here by using an eigenvalue analysis suitable for obtaining sufficient stability condition, we will show that for the Lax-Wendroff scheme, mesh refinement not only increases the lower bound of the stability range, but also reduces the upper bound of the stability range. The eigenvalue analysis yields rigorous information once the operator is normal. Here the operator is not normal, but the information is still useful.

Let $u_{j,l}^n = z^n \phi_{j,l}$ where $z \in \mathbf{C}$. Introduce this solution into the scheme (2.9) and the interface condition (2.11) for all j and all l, we obtain the following system

$$zM_1 Y = M_2 Y \tag{4.11}$$

where M_1 and M_2 are two real matrices, Y is a column-vector with components $\phi_{j,l}$, $1 \le l \le L$, $1 \le j \le p$. System (4.11) is closed by a Dirichlet condition at the inflow boundary and a first-order extrapolation condition at the outflow boundary.

System (4.11) has in total $(p+1)L$ eigenvalues z_σ. The spectral radius $\rho = \max(\mid z_\sigma \mid)$ characterises the stability for long time integration. If $\rho \le 1$, then the solution will be bounded; if $\rho > 1$, then the solution will be unbounded after long time integration; if $\rho < 1$, then the solution will converge to a steady state.

Here we only display the results for the particular case with $r = \frac{1}{2}$ and $L = 10$ and for the Lax-Wendroff scheme with local time-stepping for which $Q^{(num)} = a\sigma$ where σ is the ratio between the time step and the mesh size. The results are displayed in Table 3.

Table 3: The spectral radii for $\sigma = 0.1$ and $\sigma = 0.2$

p	1	2	3	4	5	6	7	8
$\rho(\sigma = 0.1)$	1.02	1.008	1.008	1.003	1.002	1	0.999	0.997
$\rho(\sigma = 0.2)$	1.02	0.994	0.991	0.984	0.982	0.978	0.978	0.978

We see that the stability also depends on the subgrid width p. When $\sigma = 0.1$, the problem becomes stable for $p > 6$. When $\sigma = 0.2$, the problem becomes stable for $p > 1$.

It is interesting to compare the sufficient condition with the necessary conditions (A.4)-(A.6). Let $p = 2$, then the necessary conditions require $\sigma \ge \max(0.16, 0.12)$ for stability. While the eigenvalue analysis shows that σ should ly between 0.1 and 0.2 for stability. Let $p = 6$, then the necessary conditions require $\sigma \ge \max(0.05128, 0.0337)$ and the sufficient condition is $\sigma = 0.1$. In consequence, the necessary conditions (A.4)-(A.6) give results very close to the sufficient conditions. But they should not be used as a sufficient condition for stability.

Now let us give the spectral radii as a function of σ. For $p = 2, 5, 8$, the corresponding spectrial radii are given in Table 4.

Table 4: The spectral radii for $p = 2, 5, 8$

σ	0.05	0.1	0.2	0.5	0.7	0.8	0.85	0.9	1.
$\rho(p=2)$	1.009	1.008	0.994	0.825	0.6	0.55	0.74	0.921	1.32
$\rho(p=5)$	1.003	1.002	0.982	0.835	0.675	0.55	0.735	0.92	1.31
$\rho(p=8)$	0.999	0.997	0.978	0.85	0.7	0.62	0.734	0.91	1.31

From the above table we draw some important conclusions.

Remark 4.9. The lower bound of the stability range is increased (here near $\sigma = 0.1$ instead of $\sigma = 0$ for a uniform mesh).

Table 5: Convergence histories

Iterations	$\sigma = 0.5$,p=1	$\sigma = 0.5, p = 4$	$\sigma = 0.1, p = 1$	$\sigma = 0.1, p = 4$
100	1.39×10^{-2}	1.54×10^{-2}	1.22	1.21
200	4.40×10^{-5}	1.23×10^{-5}	1.93	2.40
300	2.73×10^{-8}	1.94×10^{-9}	2.38	2.48
400	1.40×10^{-11}	2.35×10^{-13}	3.75	4.15
500	6.33×10^{-15}		5.68	6.43
1000			$8.33\times10^{+1}$	$7.75\times10^{+1}$
2000			$5.19\times10^{+4}$	$3.11\times10^{+4}$
5000			$8.33\times10^{+11}$	$2.28\times10^{+12}$
∞			∞	∞

Remark 4.10. The upper bound of the stability range is reduced (here near 0.9 instead of $\sigma = 1$ for a uniform mesh).

Remark 4.11. The minimal spectral radius, thus maximum convergence rate, occurs near $\sigma = 0.8$ for almost all p.

4.3. Numerical Test

Now let us solve the transport equation $u_t + u_x = 0$ with $0 < x < 1$. At $x = 0$ we set $u(x) = 0$. The initial data are provided by random variables uniformly distributed in $(-\frac{1}{2}, \frac{1}{2})$. At steady state the analytical solution is zero everywhere.

The above problem is approximated by the Lax-Wendroff scheme with local time-stepping. The grid is refined from left to right with a total refinement degree fixed to be $r_T = 1.66 \times 10^{-4}$. For the smooth refinement method the number of levels is $L = 40$ so that $r = r_T^{\frac{1}{L}} = 0.80446$. For the abrupt refinement method the subgrid width is fixed to be $p = 4$ while the number of levels (subgrids) is $L = 10$.

Now we compute the problem with $\sigma = 0.1$ or $\sigma = 0.5$, which corresponds to instability or stability according to the above stability analysis. Now the evolution of the l_2 energy as a function of the time (iterations n) is displayed in Table 5.

The above table confirms the previous analysis which shows that the problem becomes unstable for σ as small as 0.1 and stability is recovered for σ as high as 0.5.

Appendix A. Analysis of Dissipation by a Statistical Method for $p > 1$ and for a Fully Discrete Scheme

Since we are considering only necessary conditions just to show that mesh refinement may induce energy increasing, we will consider two subsets in the solution space. If the energy is increasing for these subsets, then the energy is increasing in the solution space. This allows us to derive necessary conditions.

The first subset $\mathbf{Z}$ is an oscillating solution (corresponding to a wave number equal to π) defined by:

$$\mathbf{Z} \stackrel{def}{=} \{u_{l,j} : \ u_{l,j} = \frac{1}{2}(-1)^{lp+j}, \forall l, \forall j\} \tag{A.1}$$

The second subset $\mathbf{U}$ is defined in a statistical way. First we introduce the following definitions

$$\Phi_{l,j} = \overline{(u_{l,j} - u_{l,j+1})}^2, \quad \psi_l = \overline{(u_{l,1} - u_{l-1,p})}^2 \tag{A.2}$$

where the overline defines an average with respect to a short time. The subset $\mathbf{U}$ is defined by solutions which may take random values but for which $\overline{\Phi}_{l,j}$ and $\overline{u_{l,j}^2}$ are constant in each subgrid, that is,

$$\mathbf{U} \stackrel{def}{=} \{u_{l,j} : \ \overline{\Phi}_{l,j} = \overline{\phi}_l, \ \overline{u_{l,j}^2} = \overline{v}_l, \quad \forall j\} \tag{A.3}$$

and ψ_l does not depend on j.

The main advantage of considering these two subsets is that they allow for the derivation of useful analytical results. Besides, they represent two extremes:

1) the first subset represents the case of shortest wavelength that can be resolved on a grid, which is often the most dangerous wavelength for instability.

2) the second subset represents the case of uniform spatial increments of solutions in each subgrid, in the averaged sense, with sharp discontinuity of such increments only at the interfaces. When instability due to mesh refinement occurs, the solution variation (in space) takes its largest value at the interfaces.

But we should keep in mind that we should not use these two subsets to derive sufficient stability conditions, though the necessary condition derived from these two subsets is very close to the sufficient condition derived by eigenvalue analysis.

Proposition A.1. (Necessary condition based on subset $\mathbf{Z}$). Let $k_l = k$ and $Q_l^{(num)} = Q^{(num)}$ be constant independent of l. Furthermore, let L be large enough so that the end point influence E can be neglected. . Then a necessary condition for energy decreasing (stability) is

$$Q^{(num)} \geq \frac{L}{(L+1)(p-1)+L} \frac{a(1-r)}{1+r} \tag{A.4}$$

where L and $r = r_T^{\frac{1}{T}}$ are given by (3.8).

Proof. Let $u_{l,j} \in \mathbf{Z}$. Then

$$\begin{aligned} D &= -Q^{(num)} \sum_{l=0}^{L} \sum_{j=1}^{p-1} (u_{l,j} - u_{l,j+1})^2 = -Q^{(num)}(L+1)(p-1) \\ I &= -Q^{(num)} \sum_{l=1}^{L} (u_{l-1,p} - u_{l,1})^2 = -Q^{(num)} L \\ B &= \frac{(1-r)}{1+r} \sum_{l=1}^{L} \left(a(u_{l-1,p}^n - u_{l,1}^n)^2 - Q^{(num)}(u_{l-1,p}^2 - u_{l,1}^2) \right) = \frac{a(1-r)}{1+r} L \end{aligned}$$

Thus $D + I + B > 0$ if (A.4) is not satisfied.

Lemma A.2. *For subset* $\mathbf{U}$*, the functions* $\overline{\psi}_l$ *and* $\overline{\phi}_l$ *satisfy the following recursive relations:*

$$\overline{\psi}_l = \frac{(1+r)^2}{4} \overline{\phi}_{l-1} = \frac{(1+r)^2}{4r^2} \overline{\phi}_l. \tag{A.5}$$

Proof. The second equation in the interface condtion (2.11) can be rewritten as

$$\begin{aligned} u_{l-1,p+1}^n - u_{l-1,p}^n &= \frac{u_{l,1}^n - u_{l,0}^n}{r} \\ &\Longrightarrow \quad (u_{l-1,p+1}^n - u_{l-1,p}^n)^2 = \frac{(u_{l,1}^n - u_{l,0}^n)^2}{r^2} \\ &\Longrightarrow \quad \overline{(u_{l-1,p+1}^n - u_{l-1,p}^n)^2} = \frac{\overline{(u_{l,1}^n - u_{l,0}^n)^2}}{r^2} \\ &\Longrightarrow \quad \overline{\phi}_{l-1} = \frac{1}{r^2} \overline{\phi}_l \end{aligned}$$

Besides, using the interface condition (2.12) , we have

$$\begin{aligned} u^n_{l-1,p} - u^n_{l-1,p+1} &= (1-\frac{r-1}{r+1})u^n_{l-1,p} - \frac{2}{r+1}u^n_{l,1} \\ &= \frac{2}{r+1}(u^n_{l-1,p} - u^n_{l,1}) \\ &\Longrightarrow \quad (\ u^n_{l-1,p+1} - u^n_{l-1,p})^2 = \frac{4}{(r+1)^2}(u^n_{l-1,p} - u^n_{l,1})^2 \\ &\Longrightarrow \quad \overline{(\ u^n_{l-1,p+1} - u^n_{l-1,p})^2} = \frac{4}{(r+1)^2}\overline{(u^n_{l-1,p} - u^n_{l,1})^2} \\ &\Longrightarrow \quad \overline{\phi}_{l-1} = \frac{4}{(r+1)^2}\overline{\psi}_l \end{aligned}$$

Thus (A.5) holds.

Proposition A.3. (Necessary condition based on subset **U**).Let $k_l = k$ and $Q_l^{(num)} = Q^{(num)}$be constant independent of l. Furhermore, let L be large enough so that the end point influence E can be neglected. Then a necessary condition for energy decreasing (stability) is

$$Q^{(num)} > \frac{|1-r|}{1+r}\frac{(1+r)^2}{4r^2}\left(p-1+\frac{(1+r)^2}{4r^2}\right)^{-1} \tag{A.6}$$

where $r = r_T^{\frac{1}{L}}$ is given by (3.8).

Proof. Note that the linear decomposition $\overline{D+I+B} = \overline{D}+\overline{I}+\overline{B}$ holds. Now let $u_{l,j} \in \mathbf{U}$ so that the relations (A.3) and (A.5) can be used in computing $\overline{D}$, $\overline{I}$, and $\overline{B}$. First compute $\overline{D}$ by (4.3):

$$\begin{aligned} \overline{D} &= -Q^{(num)}\sum_{l=0}^{L}\sum_{j=1}^{p-1}\overline{(u^n_{l,j} - u^n_{l,j+1})^2} = -Q^{(num)}\sum_{l=0}^{L}\sum_{j=1}^{p-1}\Phi_{l,j} \\ &= -Q^{(num)}\sum_{l=0}^{L}\sum_{j=1}^{p-1}\overline{\phi}_l = -Q^{(num)}\sum_{l=0}^{L}(p-1)\overline{\phi}_l \end{aligned} \tag{A.7}$$

where we have used the definition (A.2).

From (4.4) we have

$$\begin{aligned} \overline{I} &= -Q^{(num)}\sum_{l=1}^{L}\overline{(u^n_{l-1,p} - u^n_{l,1})^2} \\ &= -Q^{(num)}\sum_{l=1}^{L}\psi_l = -Q^{(num)}\sum_{l=1}^{L}\psi_l \end{aligned} \tag{A.8}$$

where we have used the definition (A.2).

Finally from (4.5) we have

$$\overline{B} = \sum_{l=1}^{L}\frac{a(1-r)}{1+r}\overline{(u^n_{l-1,p} - u^n_{l,1})^2} = \frac{a(1-r)}{1+r}\sum_{l=1}^{L}\psi_l \tag{A.9}$$

where we have used the definition (A.2).

Using (A.7)- (A.9) and (A.5), we obtain the following:

$$\begin{aligned}
\overline{I_1 + I_2 + B} &= -Q^{(num)} \sum_{l=0}^{L} (p-1)\overline{\phi}_l - Q^{(num)} \sum_{l=1}^{L} \psi_l + \frac{a(1-r)}{1+r} \sum_{l=1}^{L} \psi_l \\
&= -Q^{(num)} \sum_{l=0}^{L} (p-1)\overline{\phi}_l + \left(\frac{a(1-r)}{1+r} - Q^{(num)} \right) \sum_{l=1}^{L} \psi_l \\
&= -Q^{(num)} \sum_{l=0}^{L} (p-1)\overline{\phi}_l + \left(\frac{a(1-r)}{1+r} - Q^{(num)} \right) \frac{(1+r)^2}{4r^2} \sum_{l=1}^{L} \overline{\phi}_l \\
&= \sum_{l=1}^{L} C_l \overline{\phi}_l
\end{aligned}$$

where

$$C_l = \frac{a(1-r)}{1+r} \frac{(1+r)^2}{4r^2} - Q^{(num)} \left(p - 1 + \frac{(1+r)^2}{4r^2} \right)$$

Thus $\overline{D + I + B} < 0$ if $C_l < 0$ which, by the above expression for C_l, is equivalent to (A.6).This completes the proof.

The condition (4.7), (A.4) or (A.6) is only a necessary condition for long time stability.

References

[1] Arnoy D.C. & Flaherty J.E., An adaptive local mesh refinement method for time-dependent partial differential equations, *Appl. Numer. Math.,* **5** (1989), 257-274.

[2] Berger M.J., Stability of interfaces with mesh refinement, *Math. Comput.,* **45** (1985), 301-318.

[3] Bell J., Berger M.J., Saltzman J. S., & Welcome M., Three dimensional adaptive mesh refinement for hyperbolic conservation laws, *SIAM J. Sci. Comput.,* **15** (1994), 127-138.

[4] Berger M.J. and Colella P., Local adaptive mesh refinement for shock hydrodynamics, *J. Comput. Phys.,* **82** (1989), 64-84.

[5] Berger M.J. & Oliger J., Adaptive mesh refinement for hyperbolic partial differential equations, *J. Comput. Phys.,* **53** (1984), 484-512.

[6] Brandt A., Multi-level adaptive solutions to boundary value problems, *Math. Comp.,* **31** (1997), 333-390.

[7] Browning G., Kreiss H.-O., & Oliger J., Mesh refinement, *Math. Comp.,* **27** (1973), 29-39.

[8] Ciment M., Stable difference schemes with uneven mesh spacings,*Math. Comp.,* **114** (1971), 219-226.

[9] Coirier W.J. & Powell K.G., An accuracy assessment of Cartesian-mesh approaches for the Euler equations, *J. Comput. Phys.,* **117** (1995), 121-131.

[10] Coirier W.J. & Powell K.G., Solution adaptive Cartesian cell approach for viscous and inviscid flows, *AIAA J.,* **34** (1996), 938-945.

[11] D. De Zeeuw and K. G. Powell, An adaptively refined Cartesian mesh solver for the Euler equations,*J. Comput. Phys.,* **104** (1993), 56-68.

[12] Foreman M.G., A two-dimensional dispersion analysis of selected methods for solving the linearized shallow-water equations, *J. Comput. Phys.,* **56** (1984), 287-323.

[13] Gropp W.D., Local uniform mesh refinement with moving grids, *SIAM J. Sci. Stat. Comput.,* **8** (1987), 292-304.

[14] Gustafsson B., Kreiss H.-O., & Sundström A., Stability theory of difference approximations for initial boundary value problems II, *Math. Comp.,* **26** (1972), 649-686.

[15] Lax P. & Wendroff B., System of conservation laws,*Comm. Pure Appl. Math., XIII,* (1960), 217-237.

[16] Lerat A. & Wu Z.N., Stable conservative multidomain treatments for implicit Euler solvers,*J. Comput. Phys,* **123** (1996), 45-64.

[17] Liu C.Q., Multigrid method for steady and time-dependent flow, *Computational Fluid Dynamics Review (World Scientifc Publ.),* **1** (1998), 512-535.

[18] LeVeque R.J., Cartesian grid methods for flow in irregular regions, In Num. Meth. Fl. Dyn. III, K. W. Morton and M.J. Baines, Eds., Clarendon Press, (1988), 375-382.

[19] McCormick S.F., Multilevel adaptive methods for partial differential equations, SIAM, Philadelphia 1989.

[20] Quirk J., An alternative to unstructured grids for computing gas dynamic flows around arbitrarily complex two-dimensional bodies, *Computers Fluids,* **23** (1994), 125-142.

[21] Song Y. and Tang T., Dispersion and group velocity in numerical schemes for three-dimensional hydrodynamic equations, *J. Comput. Phys.,* **105** (1993), 72-82.

[22] Tadmor E., Numerical viscosity and the entropy condition for conservative difference schemes, *Math. Comp.,* **43** (1984), 369-381.

[23] Trefethen L.N., Group velocity in finite difference schemes, *SIAM Review,* **24** (1982), 113-136.

[24] Trefethen L.N., Stability of finite-difference models containing two boundaries or interfaces, *Math. Comp.,* **45** (1985), 279-300.

[25] Wu Z.N., Anisotropic Cartesian grid method for viscous flow computations, *Computational Fluid Dynamics Review (World Scientifc Publ.),* **1** (1998), 93-113.